

DOUBLE TRANSFERS AT THE PRIME 2

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The S^1 -transfer map $\tau : CP_0^n \rightarrow S^{-1}$ is a stable map defined using the transfer construction for the principal S^1 -bundle over CP^n , where $CP_0^n = CP^n \cup *$ is the disjoint union of the complex projective space CP^n and a base point. Then, $\tau_2 = \tau \wedge \tau : CP_0^\infty \wedge CP_0^\infty \rightarrow S^{-2}$ is called the double transfer map, and gives information of filtration 2 in the stable homotopy groups of spheres. Knapp [Kn] has given the first great step toward understanding the double transfer map, and the authors in [BC] have extended his results.

In [Hi], Hilditch has found a factorization $\bar{u}_2 : CP_0^\infty \wedge CP_0^\infty \rightarrow \Sigma^{-4}N_2$ of the double transfer map τ_2 through N_2 , the realization of the second stage of the chromatic filtration given by Ravenel [Ra], under the condition that spectra are localized at an odd prime p . His result extends that of Miller in [Mi], and enables the transfer images to be compared with the chromatic filtration.

In the case that spectra are localized at 2, just the same factorization of τ_2 as above does not exist (cf. [Hi; Remark 3.20]). In this note, I show that the restriction of τ_2 on $CP^\infty \wedge CP^\infty$ is factored through a map $\bar{u}_2 : CP^\infty \wedge CP^\infty \rightarrow \Sigma^{-4}N_2$ even if it is localized at $p = 2$. Although we limit the content to this point, we notice that such factorization enables us to calculate the image of $(\tau_2)_* : \pi_*^s(CP^\infty \wedge CP^\infty)_{(2)} \rightarrow \pi_*^s(S^0)_{(2)}$ to some extent, using the result of Shimomura [Sh] concerning a calculation of the Novikov-Adams spectral sequence at prime 2.

§1 PRELIMINARIES.

First, we prepare some properties about the K and KO -cohomology of relevant spaces.

Let ξ be the canonical complex line bundle over CP^m for $0 \leq m \leq \infty$, and $(CP^m)^{k\xi}$ the Thom space of $k\xi$ for an integer k . We denote the Thom space by $CP_k^{m+k} = (CP^m)^{k\xi}$, and $CP_k = (CP^\infty)^{k\xi}$. Also, we put $CP = CP_1$ as in the introduction. $k\xi$ is K -orientable for any integer k , and thus we have a K -Thom class $U_k^K \in K^{2k}(CP_k)$ ([AB]). Here and hereafter, we only discuss the cases in which Lim^1 of K and KO -cohomology groups are 0, and thus we may regard $K^i(CP_k)$ as $K^i(CP_k^{2N+k})$ for some large N .

We remark that $2k\xi$ is KO -orientable but $(2k+1)\xi$ is not KO -orientable. Since we need the information of $KO^*(CP)$ and $KO^*(CP_{-1})$ later, we recall the structure of $KO^*(CP_{2k+1})$ first. Let $r: K \rightarrow KO$ and $c: KO \rightarrow K$ be the realification and complexification respectively, and $t \in K_2$ the generator. We put $X = [\xi - 1] \in K^0(CP)$ and $Y = r(X) \in KO^0(CP)$. Then, the following is shown by Fujii [Fu] in the case of $n \geq 0$, and we can prove it even in the case of $n < 0$.

Lemma 1.1. *Let $n = 2k + 1$ and $m \geq 1$. Then there is an element $\bar{u}_n \in KO^{2n}(CP_n)$ which satisfies the following:*

- (1) $KO^{2n}(CP_n^{2m+n-1}) = \mathbb{Z}\{\bar{v}_n Y^i \mid 0 \leq i \leq m-1\}$.
- (2) $c(\bar{v}_n) = U_n^K(2 + X)/(1 + X)^{k+1}$.

Corollary 1.2.

- (1) For any odd integer n , $i^*(\bar{v}_n) = 2\iota$.
- (2) $c(\bar{v}_{-1}) = U_{-1}^K(2 + X)$.

Let $\sinh^{-1}(T)$ be the inverse of the formal power expansion on T of the function $\sinh(T)$, and put $S(T) = (\sqrt{T}/2)/(\sinh^{-1}(\sqrt{T}/2))$. Then, we define an

element $G_n(Y) \in KO^{2n}(CP_n; Q)$ for odd n as follows:

$$(1.3) \quad G_n(Y) = \frac{1}{2} S(Y)^{-n} \left(1 + \frac{Y}{4}\right)^{-1/2} \bar{v}_n.$$

Let $ch : K^*(-; Q) \rightarrow H^{2*}(-; Q)$ be the Chern character, and $ph = ch \circ c : KO^*(-; Q) \rightarrow H^{2*}(-; Q)$ the Pontrjagin character. We denote by $U_n^H \in H^{2n}(CP_n; Z)$ the Thom class of $n\xi$ in the ordinary cohomology group. Then we have the following lemma, which is implicit in [CK].

Lemma 1.4. *For any odd n , we have $ph(G_n(Y)) = U_n^H$.*

Consider the following commutative diagram:

$$(1.5) \quad \begin{array}{ccccc} \pi_s^{2n}(CP_n; Q) & \xrightarrow{h^{KO}} & KO^{2n}(CP_n; Q) & \xrightarrow{c} & K^{2n}(CP_n; Q) \\ \downarrow h^H & & \downarrow ph & & \downarrow ch \\ H^{2n}(CP_n; Q) & \longrightarrow & \bigoplus_{i \geq 0} H^{2n+4i}(CP_n; Q) & \longrightarrow & \bigoplus_{j \geq 0} H^{2n+2j}(CP_n; Q), \end{array}$$

where $\pi_s^{2n}(-)$ denotes the stable cohomotopy group and h^{KO} is the KO -Hurewicz map. In this diagram, the vertical homomorphisms are all isomorphisms and the horizontal two maps are inclusions. We put

$$(1.6) \quad u_1 = (h^H)^{-1}(U_n^H) \in \pi_s^{2n}(CP_n; Q).$$

Then we can show the following, in which $h^K = c \circ h^{KO}$ is the K -Hurewicz map.

Lemma 1.7.

- (1) $h^{KO}(u_1) = G_n(Y)$ for odd n .
- (2) $h^K(u_1) = U_n^K(\log(1+X)/X)^n$ for any n .

Corollary 1.8. *We have $c(G_n(Y)) = U_n^K(\log(1+X)/X)^n$. In particular, $c(G_1(Y)) = t^{-1} \log(1+X) \in K^2(CP; Q)$.*

We note that Corollary 1.8 can be also proved directly. We need in §3 the following corollary of Lemma 1.7 and Corollary 1.8.

Corollary 1.9. For odd $n = 2k - 1$, there is an element $V \in KO^{2n}(CP_{n+1}; Q)$ which is uniquely defined by the equation $j^*(V) = h^{KO}(u_1) - (1/2)\bar{v}_n$ and satisfies

$$c(V) = U_{n+1}^K t \left(\frac{(\log(1+X))^n}{X^{n+1}} - \frac{1 + \frac{X}{2}}{X(1+X)^k} \right).$$

§2 THE COFIBER OF THE TRANSFER MAP

Stably, we can consider CP as a subspace of CP_0 , and we denote by $\tilde{\tau} : CP \rightarrow S^{-1}$ the restriction of the S^1 -transfer map $\tau : CP_0 \rightarrow S^{-1}$. Let $W = S^{-2} \bigcup_{\tilde{\tau}} C(\Sigma^{-1}CP)$ be the cofiber of $\tilde{\tau} : \Sigma^{-1}CP \rightarrow S^{-2}$. Since the cofiber of $\tau : \Sigma^{-1}CP_0 \rightarrow S^{-2}$ is stably homotopy equivalent to CP_{-1} (cf. [Mi],[Kn]), we have inclusion maps $i' : CP \rightarrow CP_0$ and $i' : W \rightarrow CP_{-1}$, and the following homotopy commutative diagram up to sign:

$$(2.1) \quad \begin{array}{ccccccc} S^{-2} & \xrightarrow{i} & W & \xrightarrow{j} & CP & \xrightarrow{\tilde{\tau}} & S^{-1} \\ \parallel & & \downarrow i' & & \downarrow i' & & \parallel \\ S^{-2} & \xrightarrow{i} & CP_{-1} & \xrightarrow{j} & CP_0 & \xrightarrow{\tau} & S^{-1}. \end{array}$$

Then the following is obvious.

Lemma 2.2.

- (1) $0 \rightarrow H^k(CP; Z) \xrightarrow{j^*} H^k(W; Z) \xrightarrow{i^*} H^k(S^{-2}; Z) \rightarrow 0$ is a split exact sequence for any k , and $(i')^* : H^k(CP_{-1}; Z) \rightarrow H^k(W; Z)$ is an epimorphism with the kernel $H^0(CP_{-1}; Z) = Z\{U_{-1}^H x\}$.
- (2) $0 \rightarrow K^k(CP) \xrightarrow{j^*} K^k(W) \xrightarrow{i^*} K^k(S^{-2}) \rightarrow 0$ is a split exact sequence for any k , and $(i')^* : K^*(CP_{-1}) \rightarrow K^*(W)$ is an epimorphism with the kernel $K_*\{U_{-1}^K X\}$.

Concerning $KO^{-2}(W)$, we have the following:

Proposition 2.3. There is an element $w \in KO^{-2}(W)$ which satisfies the following:

- (i) $KO^{-2}(W) = Z\{w\} \oplus j^*(KO^{-2}(CP))$ and $j^* : KO^{-2}(CP) \rightarrow KO^{-2}(W)$ is a monomorphism.
- (ii) $(i')^*(\bar{v}_{-1}) = 2w$ for $(i')^* : KO^{-2}(CP_{-1}) \rightarrow KO^{-2}(W)$.
- (iii) $i^*(w)$ is a generator of $KO^{-2}(S^{-2})$ for the inclusion $i : S^{-2} \rightarrow W$.
- (iv) $c(w) = (i')^*(U_{-1}^K)$ in $K^{-2}(W)$.

§3 FACTORIZATION

In this section, we show a factorization of $\tilde{\tau} \wedge \tilde{\tau}$, which is one of our main results. We will use the following notations: $S(G)$ denotes the Moore spectrum of a group G , and $S^i G = \Sigma^i S(G)$; $\psi = \psi^3 - 1 : KO_{(2)} \rightarrow KO_{(2)}$ is the stable Adams operation, and Ad the fiber spectrum of ψ . Thus we have a cofibering $Ad \xrightarrow{j} KO_{(2)} \xrightarrow{\psi} KO_{(2)}$, and we put $Ad^i G = Ad \wedge S^i G$.

Let $N_i \xrightarrow{l_i} M_i \xrightarrow{j_i} N_{i+1} \xrightarrow{\delta_{i+1}} \Sigma N_i$ be the cofiber sequence such that

$$(3.1) \quad \cdots \rightarrow \Sigma^{-2} N_2 \xrightarrow{\delta_2} \Sigma^{-1} N_1 \xrightarrow{\delta_1} S^0$$

is the geometrical realization of the chromatic filtration by Ravenel [Ra], where l_i is the Bousfield localization [Bo] with respect to the $v_i^{-1}BP_*$ -homology. Then $N_0 \xrightarrow{l_0} M_0 \xrightarrow{j_0} N_1$ is identified with $S^0 \xrightarrow{i} S^0 Q \xrightarrow{\rho_Z} S^0 Q/Z$ by definition, where ρ_Z is the mod Z reduction. By [Bo], the second cofiber sequence $N_1 \xrightarrow{l_1} M_1 \xrightarrow{j_1} N_2$ is canonically identified with

$$(3.2) \quad S^0 Q/Z \xrightarrow{h^{Ad}} Ad^0 Q/Z \xrightarrow{j} \overline{Ad}^0 Q/Z,$$

where $\overline{Ad} = Ad/S^0$. Hereafter, we denote $KO_{(2)}$ simply by KO , and put $KO^i G = KO \wedge S^i G$.

Let $u_1 \in \pi_s^{-2}(CP_{-1}; Q)$ be the element in (1.6) for $n = -1$. Then we have an element $\bar{u}_1 \in \pi_s^{-2}(CP; Q/Z)$ which makes the following diagram homotopy

commutative up to sign:

$$(3.3) \quad \begin{array}{ccccccc} S^{-2} & \xrightarrow{i} & W & \xrightarrow{j} & CP & \xrightarrow{\tilde{r}} & S^{-1} \\ \parallel & & \downarrow u_1 \circ i' & & \downarrow \bar{u}_1 & & \parallel \\ S^{-2} & \xrightarrow{i} & S^{-2}Q & \xrightarrow{\rho_Z} & S^{-2}Q/Z & \xrightarrow{\delta_1} & S^{-1}, \end{array}$$

where the upper cofiber sequence is that of (2.1).

For $V \in KO^{-2}(CP_0; Q)$ in Lemma 1.9 for $n = -1$, we put $\tilde{V} = (i')^*(V) \in KO^{-2}(CP; Q)$. Then from Lemma 1.9, the following is clear.

Lemma 3.4. $j^*(\tilde{V}) = h^{KO}(u_1 \circ i') - w$, $\rho_Z \tilde{V} = h^{KO} \bar{u}_1$ and $c(\tilde{V}) = t(1/\log(1 + X) - 1/X)$, where w is the element in Proposition 2.3.

Let $g_i \in KO_{4i}$ be the generator and $a(i) = 1$ (resp. 2) if i is even (resp. odd). For the Bernoulli number $B_i \in Q$ defined by the equation $z/(e^z - 1) = \sum_{i \geq 0} (B_i/i!)z^i$, we consider the following K -theoretical Bernoulli numbers:

$$(3.5) \quad \tilde{B}_i^{KO} = (B_{2i}/(2i!))(g_i/a(i)) \in KO_{4i} \otimes Q \quad \text{and} \quad \tilde{B}_i^K = (B_i/i!)t^i \in K_{2i} \otimes Q.$$

For $CP_0 \wedge CP_0$, we will denote its K -cohomology groups by $KO^*(CP_0 \wedge CP_0) = KO_*[[Y_1, Y_2]]$ and $K^*(CP_0 \wedge CP_0) = K_*[[X_1, X_2]]$, where Y_i and X_i denote the respective Euler classes of ξ . We can consider as $KO^{-4}(CP \wedge CP; Q) \subset KO^{-4}(CP \wedge W; Q) \subset KO^{-4}(CP_0 \wedge CP_0; Q)$, and define an element $h(Y_1, Y_2) \in KO^{-4}(CP \wedge CP; Q) \subset KO^{-4}(CP \wedge W; Q)$ by

$$h(Y_1, Y_2) = \sum_{i, j \geq 0} \frac{9^j - 1}{9^{i+j} - 1} \tilde{B}_i^{KO} \tilde{B}_j^{KO} G_1(Y_1)^{2i-1} \otimes G_1(Y_2)^{2j-1},$$

where $G_1(Y)$ is the element of (1.3) for $n = 1$. Using this element, we define

$$(3.6) \quad \tilde{u} = \tilde{V} \otimes w + h(Y_1, Y_2) \in KO^{-4}(CP \wedge W; Q).$$

Similarly as $h(Y_1, Y_2)$, we can define $h_C(X_1, X_2) = \sum_{i, j > 0} (3^j - 1)/(3^{i+j} - 1) \tilde{B}_i^K \tilde{B}_j^K (t^{-1} \log(1 + X_1))^{i-1} \otimes (t^{-1} \log(1 + X_2))^{j-1}$. Then, by using Corollary 1.9 and Proposition 2.3, we have the following lemma, in which we denote by $1/X$ the element $t^{-1}U_{-1}^K \in K^0(CP_{-1})$.

Lemma 3.7.

- (1) $c(h(Y_1, Y_2)) = h_C(X_1, X_2) \in K^{-4}(CP \wedge CP; Q).$
- (2) $c(\tilde{V} \otimes w) = (i' \wedge i')^*(t^2(1/\log(1+X_1) - 1/X_1) \otimes (1/X_2)),$
as an element of $K^{-4}(CP \wedge W; Q).$

Let $U(X_1, X_2) = t^2(1/\log(1+X_1) - 1/X_1) \otimes (1/X_2) \in K^{-4}(CP_0 \wedge CP_{-1}; Q).$

Then we have the following corollary.

Corollary 3.8. $c(\tilde{u}) = (i' \wedge i')^*(U(X_1, X_2) + h_C(X_1, X_2)) \in K^{-4}(CP \wedge W; Q).$

This corollary shows that, through $(i' \wedge i')^*$, $c(\tilde{u})$ has just the same formula with that in the case of an odd prime p in [BC]. The following is crucial.

Proposition 3.9. *The element $\tilde{u} \in KO^{-4}(CP \wedge W; Q)$ satisfies the following:*

- (1) $(1 \wedge i)^*(\tilde{u}) = \tilde{V} \in KO^{-4}(CP \wedge S^{-2}; Q)$ and
- (2) $\psi(\tilde{u}) \in \text{Im}[I : KO^{-4}(CP \wedge W) \rightarrow KO^{-4}(CP \wedge W; Q)],$ where
 $\psi = \psi^3 - 1 : KO^0 Q \rightarrow KO^0 Q$ is the stable Adams operation.

Proof. (1) follows immediately from the definition of $\tilde{u}$, because $(i)^*(w) = 1$ by Proposition 2.3 (iii) and $(1 \wedge i)^*h(Y_1, Y_2) = h(Y_1, 0) = 0$. Also, we have (2) by a direct calculation in KO -theory, but it is better to apply the complexification c once and consider it in the K -theory. Then, by Corollary 3.8 the calculation is just the same as that done in [Hi] or [BC] for the case of an odd prime, and we have $c\psi(\tilde{u}) = c((\psi^3(w) - w) \otimes \psi^3(w)) \in K^{-4}(CP \wedge W) \subset K^{-4}(W \wedge W).$ Since $c : KO^{-4}(CP \wedge W) \rightarrow K^{-4}(CP \wedge W)$ is a monomorphism, we have $\psi(\tilde{u}) = (\psi^3(w) - w) \otimes \psi^3(w) \in KO^{-4}(CP \wedge W)$, and thus we have (2).

Let $\bar{u}_1$ be the element in (3.3), $\bar{j} : \overline{Ad} \rightarrow \overline{KO}$ the map induced from $j : Ad \rightarrow KO$ and $\bar{\rho} : KO \rightarrow \overline{KO}$ the reduction.

Theorem 3.10. *We have an element $\bar{u}_2 \in \overline{Ad}^{-4}(CP \wedge CP; Q/Z)$ which satisfies*

$$\delta_1(\bar{u}_2) = [\bar{u}_1 \circ (1 \wedge \tilde{\tau})] \quad \text{and} \quad (1 \wedge j)^* \bar{j}_*(\bar{u}_2) = \bar{\rho} \rho_Z(\tilde{u}).$$

Proof. Proposition 3.9 (2) means $\psi \circ \rho_Z \circ \tilde{u} \simeq 0$, and thus there is an element $u_2 \in Ad^{-4}(CP \wedge W; Q/Z)$ with $j_*(u_2) = \rho_Z(\tilde{u})$. Proposition 3.9 (1) and Corollary 1.9 yield $(1 \wedge i)^* \rho_Z(\tilde{u}) = \rho_Z(\tilde{V}) = h^{KO}(\tilde{u}) = j_* h^{Ad}(\tilde{u})$. Then These two equations give $[\tilde{u}_1 \circ h^{Ad}] = [u_2 \circ (1 \wedge i)]$, since $j_* : Ad^{-2}(CP; Q/Z) \rightarrow KO^{-2}(CP; Q/Z)$ is a monomorphism. Then it derives a map from the cofiber sequence $CP \wedge W \xrightarrow{1 \wedge j} CP \wedge CP \xrightarrow{1 \wedge \tilde{r}} CP \wedge S^{-1}$ to the cofiber sequence $Ad^{-4}Q/Z \xrightarrow{\tilde{p}} \overline{Ad}^{-4}Q/Z \xrightarrow{\delta_2} S^{-3}Q/Z$. Thus, we have $\tilde{u}_2 : CP \wedge CP \rightarrow \overline{Ad}^{-4}Q/Z$ with the required properties, and it completes the proof.

Since the chromatic filtration $\Sigma^{-2}N_2 \xrightarrow{\delta_2} \Sigma^{-1}N_1 \xrightarrow{\delta_1} S^0$ is equal to $\overline{Ad}^{-2}Q/Z \xrightarrow{\delta_2} S^{-1}Q/Z \xrightarrow{\delta_1} S^0$, we have the desired factorization of $\tilde{r} \wedge \tilde{r}$ as follows:

Corollary 3.11. $\tilde{r} \wedge \tilde{r} \simeq \delta_2 \delta_1 \tilde{u}_2$.

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