

Sharp estimates for some integral operators of convex functions of order α

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INTRODUCTION

For $0 \leq \alpha < 1$, denote by $C(\alpha)$ the class of normalised univalent convex functions f of order α , defined in the open unit disc $D = \{z : |z| < 1\}$. Thus $f \in C(\alpha)$, if and only if, $f(0) = 0$, $f'(0) = 1$ and

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha$$

for $z \in D$. The class $C(\alpha)$ has been extensively studied. In [1] Bernardi gave a series of non-sharp lower bounds for the real part of certain weighted integral operators of $f \in C(0)$. The object of this paper is to give sharp versions of some of Bernardi's results for $f \in C(\alpha)$. We also extend a classical result of Stroh  cker [3] to obtain sharp estimates for the real part of some iterated integral operators in $C(\alpha)$. Our methods are quite elementary.

RESULTS

THEOREM 1. *Let $f \in C(\alpha)$ and $z = re^{i\theta} \in D$. For $n \geq 2$, set $n!A_n(\alpha) = \prod_{k=1}^{\infty} (k - 2\alpha)$ and $A_1(\alpha) = 1$. Then*

(i) *For a real and $a \neq -1, -2, \dots$,*

$$\operatorname{Re} \left(z^{-(1+a)} \int_0^z t^{a-1} f(t) dt \right) \geq \sum_{n=1}^{\infty} \frac{(-r)^{n-1} A_n(\alpha)}{(n+a)},$$

(ii) *For $c_1, c_2 \neq -1, -2, \dots$ and $c_2 > c_1$,*

$$\operatorname{Re} \left(z^{-2} \int_0^z f(t) [(t/z)^{c_1-1} - (t/z)^{c_2-1}] dt \right) \geq (c_2 - c_1) \sum_{n=1}^{\infty} \frac{(-r)^{n-1} A_n(\alpha)}{(n+c_1)(n+c_2)},$$

(iii) For a, c real and $a \neq 0, -1, -2, \dots, c \neq -1, -2, \dots$,

$$\operatorname{Re} \left(z^{-(1+c)} \int_0^z f(t) t^{c-1} (\log(z/t))^{a-1} dt \right) \geq \Gamma(a) \sum_{n=1}^{\infty} \frac{(-r)^{n-1} A_n(\alpha)}{(n+c)^a},$$

where Γ is the Gamma function.

(iv) For c real and $c \neq 0, -1, -2, \dots$,

$$\operatorname{Re} \left(z^{-(1+c)} \int_0^z f(t) (z-t)^{c-1} dt \right) \geq \sum_{n=1}^{\infty} (-r)^{n-1} B(c, n+1) A_n(\alpha),$$

where B is the Beta function.

In all cases, equality occurs for the function $f_0 \in C(\alpha)$, where

$$\begin{aligned} f_0(z) &= \sum_{n=1}^{\infty} (-1)^{n-1} A_n(\alpha) z^n \\ &= \begin{cases} \frac{1 - (1+z)^{2\alpha-1}}{(1-2\alpha)}, & \text{for } \alpha \neq 1/2 \\ \log(1+z), & \text{for } \alpha = 1/2. \end{cases} \end{aligned}$$

THEOREM 2. Let $f \in C(\alpha)$ and $z = re^{i\theta} \in D$. For $n = 1, 2, \dots$, define

$$I_n(z) = \frac{1}{z} \int_0^z I_{n-1}(t) dt,$$

where $I_0(z) = f(z)/z$. Then for $n \geq 0$,

$$\operatorname{Re} I_n(z) \geq \gamma_n(r),$$

where

$$\frac{1}{2} \leq \gamma_n(r) = \sum_{k=1}^{\infty} \frac{(-r)^{k-1} A_k(\alpha)}{k^n} < 1.$$

The result is sharp for f_0 as given in Theorem 1.

We note that when $n = 0$, we obtain the following result of Brickman et al. [2] which we shall use in the proofs of Theorem 1 and 2.

LEMMA. Let $f \in C(\alpha)$ and $z = re^{i\theta}$. Then for $0 \leq \alpha < 1$,

$$\operatorname{Re} \left(\frac{f(z)}{z} \right) \geq \begin{cases} \frac{1 - (1+r)^{2\alpha-1}}{(1-2\alpha)r}, & \text{for } \alpha \neq 1/2 \\ \frac{\log(1+r)}{r}, & \text{for } \alpha = 1/2. \end{cases}$$

The results are sharp for the function f_0 given above.

PROOF OF THEOREM 1:

In each case, we will give the proof when $\alpha \neq \frac{1}{2}$. When $\alpha = \frac{1}{2}$, the proofs are similar. Write $t = \rho e^{i\theta}$, then applying the Lemma in each of the following, we have

(i)

$$\begin{aligned} \operatorname{Re} \left(\frac{1}{z^{1+a}} \int_0^z f(t) t^{a-1} dt \right) &= r^{-(1+a)} \int_0^r \rho^a \operatorname{Re} (f(\rho e^{i\theta}) / \rho e^{i\theta}) d\rho \\ &\geq \frac{r^{-(1+a)}}{(1-2\alpha)} \int_0^r \rho^{a-1} (1 - (1+\rho)^{2\alpha-1}) d\rho \\ &= r^{-(1+a)} \sum_{n=1}^{\infty} (-1)^{n-1} A_n(\alpha) \int_0^r \rho^{n+a-1} d\rho, \end{aligned}$$

The result now follows at once.

(ii)

$$\begin{aligned}
& \operatorname{Re} \left(\frac{1}{z^2} \int_0^z f(t) [(t/z)^{c_1-1} - (t/z)^{c_2-1}] dt \right) \\
&= \frac{1}{r^2} \int_0^r \rho [(\rho/r)^{c_1-1} - (\rho/r)^{c_2-1}] \operatorname{Re} (f(\rho e^{i\theta})/\rho e^{i\theta}) d\rho \\
&\geq \frac{1}{r^2} \int_0^r [(\rho/r)^{c_1-1} - (\rho/r)^{c_2-1}] \sum_{n=1}^{\infty} (-1)^{n-1} A_n(\alpha) \rho^n d\rho \\
&= \sum_{n=1}^{\infty} (-r)^{n-1} A_n(\alpha) \int_0^1 x^n (x^{c_1-1} - x^{c_2-1}) dx \\
&= (c_2 - c_1) \sum_{n=1}^{\infty} \frac{(-r)^{n-1} A_n(\alpha)}{(n+c_1)(n+c_2)},
\end{aligned}$$

for $c_2 > c_1$ and $c_1, c_2 \neq -1, -2, \dots$

(iii)

$$\begin{aligned}
& \operatorname{Re} \left(\frac{1}{z^{1+c}} \int_0^z f(t) t^{c-1} (\log(z/t))^{a-1} dt \right) \\
&= \frac{1}{r^{1+c}} \int_0^r \rho^c (\log(r/\rho))^{a-1} \operatorname{Re} (f(\rho e^{i\theta})/\rho e^{i\theta}) d\rho \\
&\geq \frac{1}{r^2} \int_0^r (\rho/r)^{c-1} (\log(r/\rho))^{a-1} \sum_{n=1}^{\infty} (-1)^{n-1} A_n(\alpha) \rho^n d\rho \\
&= \sum_{n=1}^{\infty} (-r)^{n-1} A_n(\alpha) \int_0^1 x^{n+c-1} (\log(1/x))^{a-1} dx \\
&= \Gamma(a) \sum_{n=1}^{\infty} \frac{(-r)^{n-1} A_n(\alpha)}{(n+c)^a},
\end{aligned}$$

for $a \neq 0, -1, -2, \dots$, $c \neq -1, -2, \dots$

(iv)

$$\begin{aligned}
& \operatorname{Re} \left(\frac{1}{z^{1+c}} \int_0^z f(t)(z-t)^{c-1} dt \right) \\
&= \frac{1}{r^{1+c}} \int_0^r \rho(r-\rho)^{c-1} \operatorname{Re} (f(\rho e^{i\theta})/\rho e^{i\theta}) d\rho \\
&\geq \frac{1}{r^2} \int_0^r \left(1 - \frac{\rho}{r}\right)^{c-1} \sum_{n=1}^{\infty} (-1)^{n-1} A_n(\alpha) \rho^n d\rho \\
&= \sum_{n=1}^{\infty} (-r)^{n-1} A_n(\alpha) \int_0^1 (1-x)^{c-1} x^n dx \\
&= \sum_{n=1}^{\infty} (-r)^{n-1} B(c, n+1) A_n(\alpha), \quad \text{for } c \neq 0, -1, -2, \dots
\end{aligned}$$

This completes the proof of Theorem 1.

PROOF OF THEOREM 2:

It follows easily from the Lemma that for $0 \leq \alpha < 1$,

$$\operatorname{Re} I_0(z) \geq \sum_{k=1}^{\infty} (-r)^{k-1} A_k(\alpha) = \gamma_0(r).$$

Next, writing $t = \rho e^{i\theta}$ we have,

$$\begin{aligned}
\operatorname{Re} I_n(z) &= \operatorname{Re} \frac{1}{z} \int_0^z I_{n-1}(t) dt \\
&\geq \frac{1}{r} \int_0^r \sum_{k=1}^{\infty} \frac{(-\rho)^{k-1} A_k(\alpha)}{k^{n-1}} d\rho \\
&= \sum_{k=1}^{\infty} \frac{(-r)^{k-1} A_k(\alpha)}{k^n} = \gamma_n(r),
\end{aligned}$$

where we have used induction. For $n \geq 0$ and $0 \leq \alpha < 1$, $\gamma_n(r)$ is absolutely convergent for $0 \leq r < 1$ and hence rearranging the terms appropriately shows that $\frac{1}{2} < \gamma_n(r) < 1$.

REFERENCES

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