

A relative version of Kawamata-Viehweg's vanishing theorem

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Introduction

In this note we present a relative version of Kawamata-Viehweg's vanishing theorem (cf. [K1], [V]), which can be viewed as a generalization of Grauert-Riemenschneider's result [GR]. As a consequence it follows that every (numerically) log-terminal singularity is rational. Thus our result is an improvement of Elkik[E]. However, there is no new idea in our proof. Everything can be found in Elkik's paper. We just erase what is unnecessary to our purpose. Then we need less assumption and the result become stronger.

§1. Preliminaries

(1.1) Notation and Convention. Basically we use the customary notation in algebraic geometry. "Variety" means an irreducible reduced projective scheme over the complex number field $\mathbb{C}$. "Manifold" means a non-singular variety. Tensor products of line bundles are denoted additively. Given a coherent sheaf F and a line bundle L on a scheme S , by $F[L]$ we denote $F \otimes L$, where L is the invertible sheaf of sections of L . The pull-back f^*L by a given morphism $f: T \rightarrow S$ is denoted by L_T .

(1.2) A coherent sheaf F is said to be finite if the support $\text{Supp}(F)$ of F is a finite set. In this case $h^0(F)$ is equal to the length $l(F)$ of F .

(1.3) Given a variety V , by $\text{Div}(V)$ we denote the group of Cartier divisors on V . An element of $\text{Div}(V) \otimes \mathbb{Q}$ (resp. $\text{Pic}(V) \otimes \mathbb{Q}$) is called a $\mathbb{Q}$ -Cartier divisor (resp. $\mathbb{Q}$ -bundle). The $\mathbb{Q}$ -bundle defined by a $\mathbb{Q}$ -Cartier divisor D is denoted by $[D]$, or simply by D when there is no danger of confusion.

(1.4) We define intersection numbers of $\mathbb{Q}$ -bundles in the natural way. They are rational numbers. A $\mathbb{Q}$ -bundle L is said to be numerically semipositive (or nef) if $LC \geq 0$ for any curve C in V . L is said to be big if there exist a positive integer m , an effective Cartier divisor E and an ample divisor A such that $mL = [E + A]$. It is well-known that a nef $\mathbb{Q}$ -bundle L is big if and only if $L^n > 0$, where $n = \dim V$.

(1.5) A $\mathbb{Q}$ -bundle is said to be effective if $L = [D]$ for some effective $\mathbb{Q}$ -Cartier divisor D . L is said to be pseudo-effective if $L + A$ is effective for any ample $\mathbb{Q}$ -bundle A .

(1.6) Proposition. Let L be a pseudo-effective $\mathbb{Q}$ -bundle on a variety V and let Λ be a linear system on V free from base points. Then L_H is pseudo-effective for any generic member H of Λ .

Proof. Take an ample line bundle A on V . For every positive integer k , we have an effective $\mathbb{Q}$ -divisor D_k such that $kL + A = [D_k]$. Clearly $S_k = \{H \in \Lambda \mid D_k \subset H\}$ is a proper closed subset of Λ . So $U = \Lambda - \bigcup_k S_k \neq \emptyset$. We claim that L_H is pseudo-effective for any $H \in U$. Indeed, for any ample $\mathbb{Q}$ -bundle B on H , $kB - A_H$ is effective for some $k \gg 0$. Since $H \notin S_k$, $[kL + A]_H = [D_k]_H$ is effective. Hence

so is $L_H + B = k^{-1}([kL + A]_H + kB - A_H)$. Thus we complete the proof.

(1.7) Let $D = \sum \delta_i D_i$ be the prime decomposition of a $\mathbb{Q}$ -divisor D on a manifold M . We say that D is negligible if $0 \leq \delta_i < 1$ for the coefficient of each component of D and if the support $\cup D_i$ of D has no singularity other than normal crossings.

(1.8) The starting point of this paper is the following theorem of Kawamata-Viehweg (cf. [K1], [V]).

Theorem. Let L be a line bundle on a manifold M . Suppose that there exists a negligible $\mathbb{Q}$ -divisor D such that $L - K - D$ is nef and big, where K denotes the canonical bundle of M . Then $H^i(M, L) = 0$ for any $i > 0$.

§2. Main results

(2.1) Definition. Let $f: M \rightarrow V$ be a surjective morphism from a manifold M onto a variety V . Then, a $\mathbb{Q}$ -bundle L on M is said to be f -nef (resp. f -pseudo-effective) if $L + f^*A$ is nef (resp. pseudo-effective) for some $\mathbb{Q}$ -bundle A on V . A divisor E on M is said to be f -exceptional if $\dim(f(\text{Supp}(E))) \leq \dim V - 2$.

(2.2) Lemma. Let f, M, V be as above and let E be an f -exceptional effective divisor on M . Let L be a line bundle on M such that $-L - D$ is f -nef for some negligible $\mathbb{Q}$ -divisor D on M . Then $f_*(\bigoplus_{\mathbb{Q}} [L + E]) = 0$.

Proof. We use the induction on the number r of prime components of E . The assertion is trivial if $r = 0$. So we

assume $r > 0$. We claim that there is a component E_j of E such that the restriction of $L + E$ to E_j is not f -pseudo-effective.

Assuming the claim, we prove the lemma as follows. Since $[L + E + f^*A]_{E_j}$ is not pseudo-effective for any ample line bundle A on V , we infer $f_*(\underline{Q}_{E_j}[L + E]) = 0$. On the other hand, we have $\text{Ker}(\underline{Q}_E \rightarrow \underline{Q}_{E_j}) \simeq \text{Coker}(\underline{I} \rightarrow \underline{I}_j) \simeq \underline{Q}_{E'}[-E_j]$, where $E' = E - E_j$ and $\underline{I}, \underline{I}_j$ are the $\underline{Q}_M$ -ideals defining E and E_j respectively. Hence we get an exact sequence $0 \rightarrow f_*(\underline{Q}_{E'}[L + E']) \rightarrow f_*(\underline{Q}_E[L + E]) \rightarrow f_*(\underline{Q}_{E_j}[L + E]) = 0$. Applying the induction hypothesis to E' , we complete the proof.

In order to prove the claim, we use the induction on $m = \dim M$. We will derive a contradiction assuming that $[L + E]_{E_j}$ is f -pseudo-effective for every component E_j of E . Taking an ample line bundle A on V and replacing L by $L + f^*tA$ with $t \gg 0$, we may assume that $[L + E]_{E_j}$ is pseudo-effective for any E_j .

When $m = 2$, write $E - D = X - Y$, where X and Y are effective $\mathbb{Q}$ -divisors without common components. Then $\text{Supp}(X) \subset \text{Supp}(E)$ and X is f -exceptional. Hence, by the index theorem, $XE_j < 0$ for some component E_j of X . Then $(E + L)E_j \leq (E - D)E_j \leq XE_j < 0$. Therefore $[L + E]_{E_j}$ is not pseudo-effective, a contradiction.

When $m > 2$ and $\dim f(E) = 0$, take a generic hyperplane

section N of M . By (1.6), the restriction of $L + E$ to $E_j \cap N$ is pseudo-effective for every j . Since E_N is f_N -exceptional, this is impossible by the induction hypothesis. When $\dim f(E) > 0$, take a generic hyperplane section H on V and set $N = f^*H$. Then E_N is f_N -exceptional. So, by (1.6) and the induction hypothesis, we obtain a contradiction as above. Thus we prove the claim in either case.

(2.3) Lemma. Let f, M, V be as in (2.1) and let L be a line bundle on M . Suppose that $\dim M = \dim V = n$ and that $L - K - D$ is f -nef for some negligible $\mathbb{Q}$ -divisor D on M , where K denotes the canonical bundle of M . Then $R^q f_*(\underline{\underline{O}}_M[L]) = 0$ for any $q > 0$.

Proof. Take an ample line bundle A on V . We use the Leray spectral sequence with $E_2^{p,q}(t) = H^p(V, R^q f_*(\underline{\underline{O}}_M[L]) \otimes [tA])$ converging to $H^{p+q}(M, L + tA_M)$. By Serre's vanishing theorem we have $E_2^{p,q}(t) = 0$ for $p > 0$, $t \gg 0$. On the other hand, $H^i(M, L + tA_M) = 0$ for $i > 0$, $t \gg 0$ by (1.8). Hence $E_2^{0,q}(t) = 0$ for $q > 0$, $t \gg 0$. This implies $R^q f_*(\underline{\underline{O}}_M[L]) = 0$ because A is ample.

(2.4) Corollary. Let f, M, V, K be as above and let L be a line bundle on M such that $-L - D$ is f -nef for some negligible $\mathbb{Q}$ -divisor D on M . Then $R^q f_*(\underline{\underline{O}}_M[K - L]) = 0$ for any $q > 0$.

(2.5) Main Theorem. Let $f: M \rightarrow V$ be a surjective morphism from a manifold M onto a variety V with $\dim M = \dim V = n$. Let L be a line bundle on M . Suppose that

there exist negligible $\mathbb{Q}$ -divisors D, D^* and an f -exceptional effective divisor E on M such that $L + E - K - D$ and $-L - D^*$ are f -nef, where K is the canonical bundle of M . Then $R^q f_*(\underline{\mathcal{O}}_M[L]) = 0$ for any $q > 0$.

Our proof consists of several steps. We use the induction on n . The assertion is obvious if $n = 1$. So we assume $n \geq 2$ in the sequel. We take an ample line bundle A on V .

(2.6) Claim. $\underline{R}^q = R^q f_*(\underline{\mathcal{O}}_M[L])$ is a finite sheaf for any $q > 0$.

Proof. Take a general hyperplane section H of V and set $N = f^*H$. By Bertini's theorem, N is a manifold and D_N, D_N^* are negligible $\mathbb{Q}$ -divisors on N . Since $[K + f^*H]_N$ is the canonical bundle of N , we can apply the induction hypothesis to L_N to obtain $R^q f_{*N}(\underline{\mathcal{O}}_N[L]) = 0$ for $q > 0$. On the other hand, we have an exact sequence $\underline{R}^q[-H] \rightarrow \underline{R}^q \rightarrow R^q f_*(\underline{\mathcal{O}}_N[L])$ derived from $0 \rightarrow \underline{\mathcal{O}}_M[-L] \rightarrow \underline{\mathcal{O}}_M \rightarrow \underline{\mathcal{O}}_N \rightarrow 0$. So, we infer $\text{Supp}(\underline{R}^q) \cap H = \emptyset$ by Nakayama's lemma. Hence $\underline{R}^q$ is finite because H is ample.

(2.7) Claim. $R^q f_*(\underline{\mathcal{O}}_M[K - L - E])$ is finite for any $q > 0$ and $f_*(\underline{\mathcal{O}}_E[K - L]) = 0$.

Proof. Set $L^* = K - L - E$. Exchanging the role of D and D^* , L^* satisfies the same condition as L . So the argument in (2.6) proves the first assertion. The second follows from (2.2).

(2.8) Using the long exact sequence of $R^q f_*$'s of the exact sequence $0 \rightarrow \underline{\mathcal{O}}_M[K - L - E] \rightarrow \underline{\mathcal{O}}_M[K - L] \rightarrow \underline{\mathcal{O}}_E[K - L]$

$\rightarrow 0$, we infer from (2.4) that $R^q f_*(\underline{Q}_M[K - L - E]) \simeq R^{q-1} f_*(\underline{Q}_E[K - L])$ for any $q \geq 1$. Moreover, by the second assertion of (2.7), we have $f_*(\underline{Q}_M[K - L - E]) \simeq f_*(\underline{Q}_M[K - L])$. This sheaf will be denoted by F in the sequel.

(2.9) Claim. $h^i(V, F[-tA]) = 1(R^{i-1} f_*(\underline{Q}_M[K - L - E]))$ for any $n > i \geq 2$, $t \gg 0$. When $i \leq 1$, $h^i(V, F[-tA]) = 0$.

Proof. We use the Leray spectral sequence with $E_2^{p,q} = H^p(V, R^q f_*(\underline{Q}_M[K - L - E])[-tA])$ converging to $H^{p+q}(M, K - L - E - tA)$. By (2.7), $E_2^{p,q} = 0$ if $p > 0$ and $q > 0$. On the other hand, $h^j(M, K - L - E - tA) = h^{n-j}(M, L + E + tA) = 0$ for $j < n$ by (1.8). Therefore $E_2^{p,0} \simeq E_2^{0,p-1}$ for each $p < n$. In particular $E_2^{1,0} = E_2^{0,0} = 0$. This implies the claim.

(2.10) Claim. $h^i(E, K - L) = 1(R^i f_*(\underline{Q}_E[K - L])) = 1(R^{i+1} f_*(\underline{Q}_M[K - L - E]))$ for any $i \geq 0$.

For a proof, we use the Leray spectral sequence converging to $h^i(E, K - L)$ and apply (2.7) and (2.8).

(2.11) Claim. $R^q f_*(\underline{Q}_E[L + E]) \simeq R^{q+1}$ for any $q \geq 0$.

Proof. Applying (2.3) we obtain $R^q f_*(\underline{Q}_M[L + E]) = 0$ for any $q > 0$. In view of the long exact sequence of $R^q f_*$'s of the exact sequence $0 \rightarrow \underline{Q}_M[L] \rightarrow \underline{Q}_M[L + E] \rightarrow \underline{Q}_E[L + E] \rightarrow 0$, we infer $R^q f_*(\underline{Q}_E[L + E]) \simeq \underline{R}^{q+1}$ for any $q > 0$. Moreover, using (2.2), we obtain $f_*(\underline{Q}_E[L + E]) = \underline{R}^1 = 0$.

(2.12) Claim. $h^i(E, L + E) = 1(\underline{R}^{i+1})$ for any $i \geq 0$.

For a proof, use (2.11) and the Leray spectral sequence.

(2.13) End of the proof of (2.5). For any $q \geq 1$ and $t \gg 0$ we have $1(\underline{R}^q) = h^q(M, L + tA)$ (similarly as in (2.3))

$$= h^{n-q}(M, K - L - tA) \quad (\text{by Serre duality})$$

$$= h^{n-q}(V, F[-tA]) \quad (\text{by (2.4)}).$$

If $q \geq n - 1$, the last term vanishes by (2.9) and we are done.

If $q \leq n - 2$, we continue:

$$\begin{aligned}
 &= 1(R^{n-q-1} f_*(\underline{O}_M[K - L - E])) \quad (\text{by (2.9)}) \\
 &= h^{n-q-2}(E, K - L) \quad (\text{by (2.10)}) \\
 &= h^{q+1}(E, L + E) \quad (\text{by Serre duality on } E) \\
 &= 1(\underline{R}^{q+2}) \quad (\text{by (2.12)}).
 \end{aligned}$$

Thus, by the descending induction on q , we prove $\underline{R}^q = 0$ for any $q > 0$.

(2.14) Theorem. Let $f: M \rightarrow V$ be a resolution of singularities of a normal variety V . Suppose that there exist a negligible $\mathbb{Q}$ -divisor D and an f -exceptional effective $\mathbb{Q}$ -divisor E on M such that $E - K - D$ is f -nef, where K is the canonical bundle of M . Then V has only rational singularities.

Proof. Take a birational morphism $\pi: M' \rightarrow M$ such that $\text{Supp}(R) \cup \pi^{-1}(D \cup E)$ is a divisor having no other singularity than normal crossings, where R is the ramification divisor of π . Let $\sum \delta_i D_i$ be the prime decomposition of $\pi^*(E - D) + R$. Then $\delta_i > -1$ for each i since D is negligible. Hence $\pi^*(E - D) + R = E' - D'$ for some negligible $\mathbb{Q}$ -divisor D' and an effective divisor E' on M' . Moreover E' is f' -exceptional, where $f' = f \circ \pi$. Let K' be the canonical

bundle of M' . Then $K' = \pi^* K + R$ and hence $E' - D' - K' = \pi^*(E - D - K)$ is f' -nef. Now, applying (2.5) by setting $L = \underline{O}_{M'}$ and $D^* = 0$, we get $R^q f'_* \underline{O}_{M'} = 0$ for any $q > 0$. This implies that V has only rational singularities.

(2.15) Corollary (compare [K2]). Every log-terminal singularity in the sense of Kawamata [K3] is rational.

In fact, in case of log-terminal singularities, we have $\mathbb{Q}$ -divisors D, E as above such that $E - K - D$ is the pull-back of a $\mathbb{Q}$ -bundle on V .

(2.16) Corollary (compare [E]). Every canonical singularity is rational.

§3. Comments

(3.1) By Lefschetz' principle, the results in §2 are valid over any field of characteristic zero.

(3.2) Let D be a $\mathbb{Q}$ -Weil divisor on a normal variety V . Suppose that there is a desingularization $f: M \rightarrow V$ together with a negligible $\mathbb{Q}$ -divisor D' and an f -exceptional effective $\mathbb{Q}$ -divisor E on M such that $D = f_* D'$ and $K + D - E$ is numerically equivalent to $f^* L$ for some $\mathbb{Q}$ -bundle L on V . In such a case the pair (V, D) will be said to have only negligible singularities. By (2.14), V has only rational singularities in this case.

Apparently, this is a generalization of (2.15). However, as a matter of fact, every negligible singularity is log-terminal (in the wider sense in [K3]).

In order to see this, let f, M, D' be as above and take an ample line bundle A on V . Then $H = K + D' - E + tA_M$ is nef for $t \gg 0$. Moreover $L = H + E$ is a Zariski decomposition (cf. [F]). Since $L - K - D' = tA_M$ is nef and big, H is semiample by [F; A7]. Take a positive integer m such that both mD' and mE are usual divisor and that $Bs|mH| = \emptyset$. Let $\rho: M \rightarrow W$ be the rational mapping defined by $|mH|$. Then, for any curve C in M such that $f(C)$ is a point, we have $HC = 0$ and hence $\rho(C)$ is a point. From this we infer that ρ factors through V and mH comes from $\text{Pic}(V)$. Thus $K + D' - E$ is the pull-back of a $\mathbb{Q}$ -bundle on V . Obviously this $\mathbb{Q}$ -bundle corresponds to the $\mathbb{Q}$ -Weil divisor $K_V + D$. Thus V has log-terminal singularities only.

(3.3) Let D be a negligible $\mathbb{Q}$ -divisor on a manifold M with canonical bundle K and suppose that $K + D$ is big and that the graded algebra G associated to $K + D$ is finitely generated. Then, as we saw in [F; A3], $K + D$ admits a Zariski decomposition on a birational model M' of M and the semipositive part H is semiample. So we have a birational morphism $f: M' \rightarrow W = \text{Proj}(G)$. We claim that (W, f_*D) has only negligible singularities.

Indeed, if N is the negative part of the Zariski decomposition, we have $H^0(M', sN + f^* \underline{O}_W(t)) \simeq H^0(M', f^* \underline{O}_W(t))$ for any $s > 0, t > 0$. From this we infer that N is f -exceptional. Now the assertion is easy to prove.

Thus, roughly speaking, negligible singularities are the

singularities which can appear as the singularities of $(K + D)$ -models with D being a negligible $\mathbb{Q}$ -divisor.

(3.4) A negligible singularity is not always log-terminal in the strong sense in [K2], namely, not always $\mathbb{Q}$ -Gorenstein.

For example, let W be the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$ and let $S = \mathbb{P}^1 \times l \subset W$, where l is a line on the second factor $\mathbb{P}^2$. Let (V, D) be the affine cone over (W, S) . Then the singularity of (V, D) at the vertex is negligible, although it is not $\mathbb{Q}$ -Gorenstein. It is also easy to see that such a singularity does really appear on a certain $(K + D)$ -model as in (3.3).

Finally, we remark that the results in [K2] are valid for varieties having only negligible singularities.

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Note added in proof.

(A1) Very recently, Kawamata and Nakayama have proved our result over non-complete variety too. To be precise, we make the following

(A2) Definition. Let $f: X \rightarrow V$ be proper morphism of analytic varieties and let L be a $\mathbb{Q}$ -bundle on X . Then L is said to be f -nef if $LC \geq 0$ for any curve C in X such that $f(C)$ is a point. L is said to be f -big if the restriction of L to a general fiber of f is big.

Remark. If L is f -nef in the sense (2.1), then L is obviously f -nef in the above sense too. However, the converse is not true in general (even if f is birational).

(A3) Theorem (Nakayama). Let $f: M \rightarrow V$ be a proper morphism from a complex manifold M onto an analytic variety V . Let L be a line bundle on M such that $L - K - D$ is f -nef and f -big for some negligible $\mathbb{Q}$ -divisor D on M , where

K is the canonical bundle of M . Then $R^q f_* (\underline{\mathcal{O}}_M[L]) = 0$ for any $q > 0$.

Outline of proof. We may assume that V is Stein and then M is weakly 1-complete in the sense of Nakano. Starting from his vanishing theorem, we proceed in the same way as we obtained Kawamata-Viehweg's vanishing theorem from Kodaira's. Then we get $H^q(M, L) = 0$, which implies the assertion.

(A4) Theorem (Kawamata). Let $f: M \rightarrow V$ be as in (A3) and let L be a line bundle on M . Suppose that $\dim M = \dim V$ and that there is an f -exceptional effective divisor E and negligible $\mathbb{Q}$ -divisors D, D^* such that $L + E - K - D$ and $-L - D^*$ are f -nef. Then $R^q f_* (\underline{\mathcal{O}}_M[L]) = 0$ for any $q > 0$.

This is proved similarly as (2.5). We replace (2.3) by (A3), and use Grothendieck duality in place of Serre duality. To do so, some technical reformulations are necessary; for example, in (2.9), we should replace $h^*(M, K - L - E - tA)$ by $\text{Ext}^*(Rf_* (\underline{\mathcal{O}}_M[K - L - E]), \omega_V^*)$ and consider the spectral sequence with $E_2^{p,q} = \text{Ext}^p(R^{-q}f_* (\underline{\mathcal{O}}_M[K - L - E]), \omega_V^*)$. Details will be published elsewhere.