

Review of A. Hatcher & J. Wagoner's paper

'Pseudo-isotopies of compact manifolds' (Part II)

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Let M^n be a connected compact smooth manifold and $P = P(M, \partial M)$ be the pseudo-isotopy space of M .

The aim of the paper [1] by A. Hatcher & J. Wagoner is the computation of $\pi_0(P)$. As "Part II" of the introduction, we shall explain how the first obstruction $\Sigma: \pi_0(P) \longrightarrow \text{Wh}_2(\pi_1 M)$ is constructed.

$\pi_0(P)$ is replaced by $\pi_1(F, E:p)$, so we start from $\pi_1(F, E:p)$.

Theorem 1. There is a surjection $\Sigma: \pi_1(F, E:p) \longrightarrow \text{Wh}_2(\pi_1 M)$.

Our target group $\text{Wh}_2(\pi_1 M)$ has another presentation. For simplicity, let $\Lambda = Z[\pi_1 M]$ and $G = \pi_1 M$.

Proposition 0. $\text{Wh}_2(\pi_1 M) \cong U(\Lambda)/U(\pm G)$,

where $U(\Lambda) = \{x \in \text{St}(\Lambda) \mid \pi(x) = (a_{pq}), \pi: \text{St}(\Lambda) \longrightarrow E(\Lambda)$

$$(1) \quad a_{pq} = 0 \quad \text{if} \quad q < p$$

$$(2) \quad a_{pp} = \pm g_p \quad \text{for some} \quad g_p \in G,$$

(This is a subgroup of $\text{St}(\Lambda)$.)

$U(\pm G)$ = the subgroup of $U(\Lambda)$, generated

$$\text{by} \quad \begin{cases} w_{pq}(\pm g) \cdot w_{pq}(-1) & g \in G \\ x_{pq}(\lambda) & \text{with } p < q, \lambda \in \Lambda. \end{cases}$$

$[f_t] \in \pi_1(F, E:p)$ be a one parameter family of functions $f_t: M \times I \longrightarrow I$, $t \in [0, 1]$, where $f_0 = p$: the standard projection

and $f_1 = f$ are in E . We make the following deformations, keeping both ends fixed and without changing the homotopy class of $[f_t]$. The geometrical details are complicated so we give only a rough sketch.

1st step. By the stratification theory of the function space F , we can approximate the one parameter family by a generic family. Here the generic family consists of the Morse functions f_t except for finite t , and for finite t , f_t is a function with a birth (or a death) point and some non-degenerate critical points.

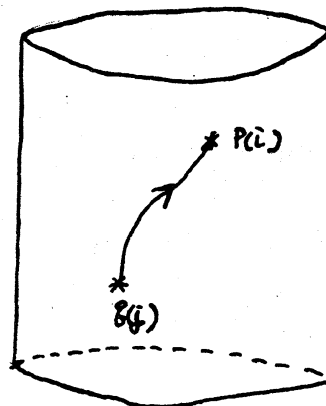
2nd step. Choose a one parameter family $\{\eta_t\}$ of gradient like vector fields for $\{f_t\}$.

Then we can consider the trajectories.

Let p and q be the two critical points of f_t . If there are no trajectories leading up from p to q (or q to p), we say p and q are independent.

Proposition 1. If p is a birth (or a death) point, then $\{\eta_t\}$ can be deformed, for which p is independent of all the other critical points.

Next, let p and q be the two non-degenerate critical points of index i and j . The trajectory from q to p is called the i/j -intersection. Using the general position methods, we deform the path $\{\eta_t, f_t\}$, then by the dimensional reason, there are no i/j -intersections for $i < j$, and there are only a finite number of i/i -intersections, they are important for us and they are also called "gradient crossings" or "handle additions".

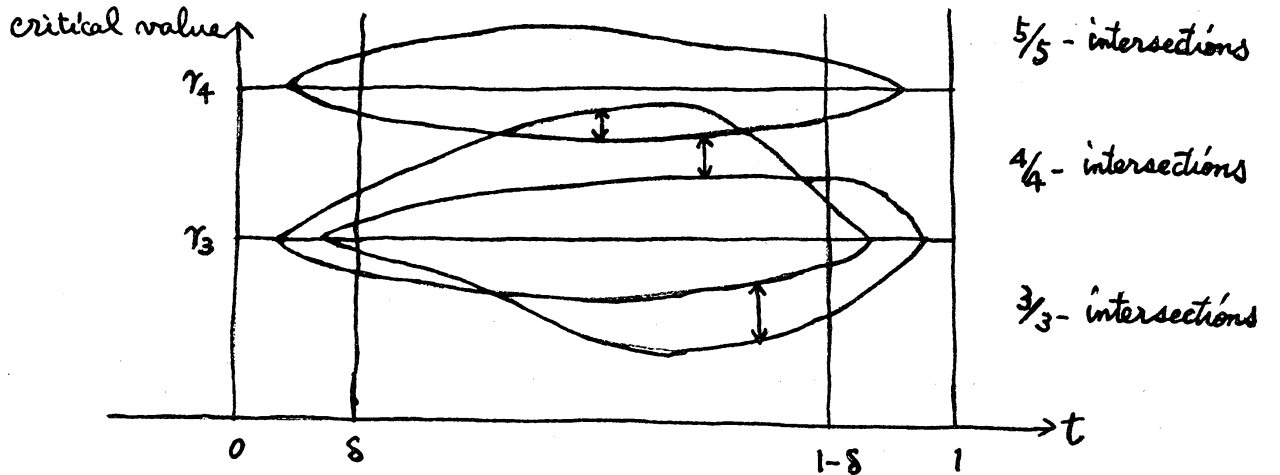


For fixed t , there are a finite number of $i/(i-1)$ -intersections, they are the incidence points.

3rd step. By the independent trajectory principle, we make $\{\eta_t, f_t\}$ to be an ordered family. That is, for $0 < r_0 < r_1 < \dots < r_n < 1$, if p is a degenerate critical point of f_t , of index i , then $f_t(p) = r_i$ and if p is a non-degenerate critical point of f_t , of index i , then $f_t(p) \in [r_{i-1}, r_i]$.

For small $\delta > 0$, we make more deformations, so that all the birth points occur in $[0, \delta]$, and all the death points occur in $[1-\delta, 1]$, and there are no i/i -intersections in $[0, \delta] \cup [1-\delta, 1]$.

For example, the graphic will be:



Let us explain more about the i/i -intersections.

For finite t , there are finitely many i/i -intersections from q_t to p_t . Let $S_t^{n-i} \cap S_t^{i-1} = \{Q_1, Q_2, \dots, Q_k\}$, where S_t^{n-i} is the unstable sphere of q_t and S_t^{i-1} is the stable sphere of p_t , the both are in the middle level surface $f_t^{-1}(c)$, for $f_t(q_t) < c < f_t(p_t)$. Let v be the base point of $M \times I$, and choose the base paths γ_p and γ_q , and γ_{Q_i} be the i/i -intersection. Then the composition $\gamma_p * \gamma_{Q_i}^{-1} * \gamma_q^{-1}$ decides an element in $G = \pi_1 M$.

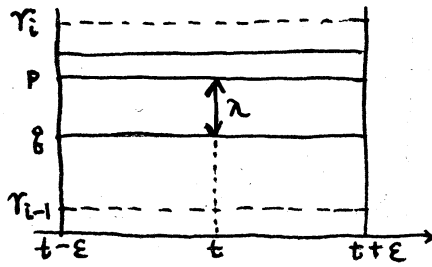
A diagram of a transmission line. It consists of two parallel horizontal lines representing the conductors. The top conductor is labeled P_t and the bottom conductor is labeled g_t . A vertical double-headed arrow between the two conductors is labeled l . The right end of the conductors is connected to a common ground line, which is labeled t at its right end.

If we give p_t & q_t , the indices of Steinberg group, for example, p and q , then a Steinberg symbol $x_{pq}(\lambda) \in \text{St}(\Lambda)$ corresponds to each i/i -intersection λ .

Next, we give the algebraic property of the i/i -intersections. For (η_t, f_t) , where f_t is a Morse function, there is a chain complex, which is defined in the following way.

Let $(V_i; \partial_- V_i, \partial_+ V_i) = (f_t^{-1}([r_{i-1}, r_i]), f_t^{-1}(r_{i-1}), f_t^{-1}(r_i))$. Let $p: \widetilde{M \times I} \rightarrow M \times I$ be the universal cover of $M \times I$ and for any subset $A \subset M \times I$, let $p^{-1}(A) = \overline{A}$. Choose paths from a fixed base point to each critical point and orient the stable manifold of each critical point as in the s-cobordism theory. We have (C_*, ∂_*) , where $C_i(f_t) = H_i(\overline{V_i}, \overline{\partial_- V_i})$ is a free Λ -module, whose basis are determined by the liftings of the stable disks of index i critical points of f_t .

Proposition 2. In the graphic, $\varepsilon > 0$ be small enough so that



there are no other i/i -intersections.

Let $\varepsilon_1, \dots, \varepsilon_p, \dots, \varepsilon_q, \dots$ be the basis of $C_i(f_{t-\varepsilon})$ determined by

the stable disks of index i critical points of $f_{t-\varepsilon}$, then

$\varepsilon_1, \dots, \varepsilon_p + \lambda \varepsilon_q, \dots, \varepsilon_q, \dots$ are the basis of $C_i(f_{t+\varepsilon})$ determined in a similar way.

So there is a transformation of basis $C_i(f_{t-\varepsilon}) \leftarrow C_i(f_{t+\varepsilon})$, expressed by the elementary matrix $\pi(x_{pq}(\lambda)) = e_{pq}(\lambda)$.

To treat everything at once, we introduce the standard complex (ω, σ) , which is defined in the following way.

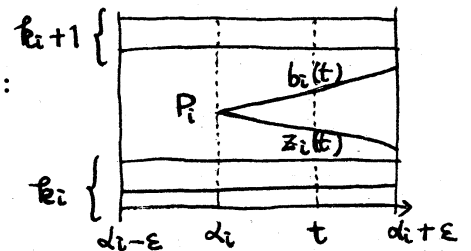
For $i > 0$, let C_i be the free left Λ -module over $\{b_i^\alpha, z_i^\beta\}_{\alpha, \beta \in \mathbb{Z}}$, and C_0 be the free left Λ -module over $\{z_0^\beta\}_{\beta \in \mathbb{Z}}$. We call b_i^α "the boundary indices" and z_i^β "the cycle indices".

Define the boundary operator and the contraction operator by $\omega = \{\omega_i: C_i \rightarrow C_{i-1}\}$ $\sigma = \{\sigma_i: C_i \rightarrow C_{i+1}\}$

$$\begin{cases} \omega_i(b_i^\alpha) = z_{i-1}^\alpha \\ \omega_i(z_i^\beta) = 0 \end{cases} \quad \begin{cases} \sigma_i(b_i^\alpha) = 0 \\ \sigma_i(z_i^\beta) = b_{i+1}^\beta \end{cases}$$

In the graphic, $\{p_1, p_2, \dots, p_m\}$ be the set of all birth points of $\{\eta_t, f_t\}$ such that p_i is a birth point of f_{α_i} , of index k_i , where $\alpha_1 < \alpha_2 < \dots < \alpha_m$.

The graphic near time $t = \alpha_i$ looks like:
for small $\varepsilon > 0$.



Here $b_i(t)$ and $z_i(t)$ are the couple of non-degenerate critical points, born at p_i .

Choose (a) a base path γ_i from v to p_i (v is the base point in $M \times I$.)

(b) the orientations of $W(b_i(t))$ and $W(z_i(t))$ so that $\partial_i(b_i(t)) = +z_i(t)$

(c) for $b_i(t)$, some boundary index $b_{k_i+1}^\alpha$ and for $z_i(t)$ the corresponding cycle index $z_{k_i}^\alpha$.

Then each i/i -intersection λ has a symbol $x_{pq}(\lambda) \in \text{St}_i(\Lambda)$, $p, q \in \{b_i^\alpha, z_i^\beta\}_{\alpha, \beta \in \mathbb{Z}}$ $\lambda \in \Lambda$.

For each i , in the graphic read the Steinberg symbols from left to right and multiply and write it down by $x_i \in \text{St}_i(\Lambda)$.

The multi-Steinberg word is defined by $x = (x_0, x_1, \dots, x_i, \dots) \in \bigoplus_i \text{St}_i(\Lambda)$.

f_δ and $f_{1-\delta}$ are the Morse functions, so we have the chain complexes $C_*(f_\delta)$ and $C_*(f_{1-\delta})$ and the chain transformation between them, because of Proposition 2.

$$\begin{array}{ccc}
 \downarrow & & \downarrow \\
 C_i(f_\delta) & \xleftarrow{\pi(x_i)} & C_i(f_{1-\delta}) \\
 \sigma_{i-1} \uparrow \downarrow \omega_i & & \delta_{i-1} \uparrow \downarrow \partial_i \\
 C_{i-1}(f_\delta) & \xleftarrow{\pi(x_{i-1})} & C_{i-1}(f_{1-\delta}) \\
 \downarrow & & \downarrow
 \end{array}$$

$$\{ \partial_i = \pi(x_i) \omega_i \pi(x_{i-1})^{-1} = x \omega_i \} = x \omega$$

$$\{ \delta_{i-1} = \pi(x_{i-1}) \sigma_{i-1} \pi(x_i)^{-1} = x \sigma_{i-1} \} = x \sigma.$$

After the above choice, we can regard $C_*(f_\delta) = (\omega, \sigma)$, and $C_*(f_{1-\delta}) = x(\omega, \sigma)$. Here is the operation of $x \in \bigoplus_i \text{St}_i(\Lambda)$ on (ω, σ) .

As $f \in E$ (i.e. f_1 has no critical points), all the critical points of $f_{1-\delta}$ will be cancelled in some death points.

Proposition 3. In the following graphic, p and q can be cancelled, if and only if, $\partial(\text{base}(p)) = \pm g(\text{base}(q))$ for some $g \in G$.



$$\text{Then, } (x\omega_i)(\text{basis element}) = \begin{cases} \pm g(\text{basis element}) \\ \text{or} \\ 0 \end{cases}$$

As $x(\omega, \sigma)$ is almost the standard complex, by the permutation of the indices of each Steinberg group $\text{St}_i(\Lambda)$, which is realized by some element $u = (u_0, \dots, u_i, \dots) \in \bigoplus_i W_i(\pm G) \subset \bigoplus_i \text{St}_i(\Lambda)$, we have the following formula

$$(*) \quad \begin{cases} (ux)\omega_i(b_i^\alpha) = \pm g z_{i-1}^\alpha \\ (ux)\omega_i(z_i^\beta) = 0. \end{cases}$$

Like the Whitehead torsion of the chain complex,

$(ux)\omega_{\text{ev}} + (ux)\sigma_{\text{ev}}: \bigoplus_{i \geq 0} C_{2i} \longrightarrow \bigoplus_{i \geq 0} C_{2i+1}$ is an isomorphism and expressed by the matrix

		C_1		C_3		C_5			
		b_1^α	z_1^β	b_3^α	z_3^β	b_5^α	z_5^β		
C_0	z_0^β	δ_0		0		0			
C_2	b_2^α								
	z_2^β	∂_2		δ_2		0			
C_4	b_4^α								
	z_4^β	0		∂_4		δ_4			
		0		0					

This matrix is the desired upper triangular matrix in $\pi(U(\Lambda))$ because of the formula (*) and the contraction formula induced by (*). This matrix is $\pi((\prod_{i \geq 0} u_{2i} \cdot x_{2i})(\prod_{i \geq 0} x_{2i+1}^{-1} \cdot u_{2i+1}^{-1}))$, where $\pi: \oplus St_1(\Lambda) \longrightarrow \oplus E_1(\Lambda)$.

We define $\Sigma: \Pi_1(F, E; p) \longrightarrow Wh_2(\pi_1 M)$ by $\Sigma([f_t]) = (\prod_{i \geq 0} u_{2i} \cdot x_{2i})(\prod_{i \geq 0} x_{2i+1}^{-1} \cdot u_{2i+1}^{-1}) \mod U(\pm G)$.

Our first obstruction $\Sigma: \pi_0(P) \longrightarrow Wh_2(\pi_1 M)$ is defined by the formula $\Sigma([g]) = \Sigma([\text{path from } p \text{ to } p \circ g])$ for $g \in P(M)$.

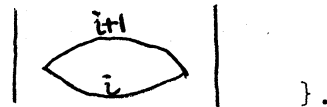
The proof of well-definedness is not easy. We have to consider the two-parameter families, and their generic families have some codimension 2 singularities (dovetail points).

To show the Σ is surjective, for an element z in $Wh_2(\pi_1 M)$, represented by $\prod x_{pq}(\lambda) \in K_2(\Lambda)$, we construct a path from p to f , where $f \in E$. The constructions are realized by the embeddings of the standard path models. This is done for $\dim M \geq 5$.

The first obstruction describes the i/i -intersections, then the kernel of Σ consists of the one parameter families without i/i -intersections.

Let $\mathcal{D} = \{[f] \in \pi_0(E) \cong \pi_0(P) \mid \text{path from } p \text{ to } f \text{ has}$

a graphic like:



Proposition 4. $\mathcal{D}$ is a subgroup of $\pi_0(E)$, for $\dim M^n \geq 4$.

Theorem 2. $\text{Ker } \Sigma = \mathcal{D}$, for $\dim M^n \geq 5$.

The birth and death points are crucial in this theory, but they are tame and easy to treat.

A. Hatcher has defined in [2], the 2nd obstruction for $\pi_0(P)$, $\theta: \pi_0(P) \longrightarrow \text{Wh}_1(\pi_1 M: \mathbb{Z}_2 \times \pi_2 M)$, which describes the kernel $\mathcal{O}$. But it has mistakes, those problems are solved by K. Igusa in [7].

References

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