

Two kinds of constructions of generalized Kac-Moody algebras as subalgebras of Kac-Moody algebras

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Introduction

In [1], R. Borcherds introduced the notion of generalized Kac-Moody algebras (= GKM algebras for short), which is a good generalization of Kac-Moody algebras. The aim of this paper is to exhibit two kinds of constructions of GKM algebras as subalgebras of a symmetrizable Kac-Moody algebra.

In the first construction (§2), we get GKM algebras as what we call *regular subalgebras*. The regular subalgebra is a natural infinite dimensional analogue of a regular semi-simple subalgebra in the sense of Dynkin of a finite dimensional complex semi-simple Lie algebra. The latter plays an important role in the classification of semi-simple subalgebras (cf. [2]).

In the second construction (§3), we get GKM algebras as what we call *folding subalgebras*. This folding subalgebra is generated by certain sums, corresponding to a diagram automorphism, of the Chevalley generators of the Kac-Moody algebra. In the finite dimensional case, a folding subalgebra coincides with a fixed point subalgebra of a certain automorphism (cf. [4]). In the general case, folding subalgebras are subalgebras of fixed point subalgebras, but not necessarily coincide with them (cf. [5]).

Folding subalgebras have some good properties:

one is the inheritance of a standard invariant form, and another is the complete reducibility when representations are regarded as those of folding subalgebras (§3.4 and 3.5). In the affine case, a certain class of folding subalgebras and their branching rules are studied in [3].

§1. Generalized Kac-Moody algebras

In this section, we explain the notion of generalized Kac-Moody algebras for later use. Here we adopt the definition in [4, Chap. 11] of generalized Kac-Moody algebras, which is a little different from the original one in [1].

1.1. Definitions and notations.

Definition 1.1 ([4]). A real $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ is called a GGCM (= generalized GCM) if it satisfies the following:

- (C1) either $a_{ii} = 2$ or $a_{ii} \leq 0$;
- (C2) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$;
- (C3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Note that when $a_{ii} = 2$ for every i , A is a generalized Cartan matrix (= GCM).

Definition 1.2 ([4]). A triple $(b, \Pi = \{\alpha_i\}_{i=1}^n, \Pi^\vee = \{\alpha_i^\vee\}_{i=1}^n)$ is called a *realization* of the GGCM $A = (a_{ij})_{i,j=1}^n$ if it satisfies the following:

(R1) $\mathfrak{h}$ is a finite dimensional complex vector space, and $\dim_{\mathbb{C}} \mathfrak{h} = 2n - \text{rank } A$;

(R2) $\Pi^{\vee} = \{\alpha_i^{\vee}\}_{i=1}^n$ is a linearly independent subset of $\mathfrak{h}$, and $\Pi = \{\alpha_i\}_{i=1}^n$ is a linearly independent subset of $\mathfrak{h}^*$ (the algebraic dual of $\mathfrak{h}$);

(R3) $\langle \alpha_j, \alpha_i^{\vee} \rangle = a_{ij} \quad (1 \leq i, j \leq n)$.

Let $\tilde{\mathfrak{g}}(A)$ be a Lie algebra with generators $e_i, f_i \quad (1 \leq i \leq n)$, and $\mathfrak{h}$, and the following defining relations:

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} \alpha_i^{\vee} \quad (1 \leq i, j \leq n), \\ (F1) \quad [h, h'] &= 0 \quad (h, h' \in \mathfrak{h}), \\ [h, e_i] &= \langle \alpha_i, h \rangle e_i, \quad [h, f_i] = -\langle \alpha_i, h \rangle f_i \quad (1 \leq i \leq n, h \in \mathfrak{h}). \end{aligned}$$

We define $\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{r}$, where $\mathfrak{r}$ is a unique maximal ideal among the ideals of $\tilde{\mathfrak{g}}(A)$ intersecting $\mathfrak{h}$ trivially. This Lie algebra $\mathfrak{g}(A)$ is called a *generalized Kac-Moody algebra* (= GKM algebra) associated to A . (Especially when A is a GCM, $\mathfrak{g}(A)$ is called a *Kac-Moody algebra*.) The subalgebra $\mathfrak{h}$ is called the *Cartan subalgebra* of $\mathfrak{g}(A)$, and the elements $e_i, f_i \quad (1 \leq i \leq n)$ are called the *Chevalley generators* of $\mathfrak{g}(A)$.

It is shown in [4, Chap. 11] that, when the GGCM A is symmerizable, the GKM algebra $\mathfrak{g}(A)$ is a Lie algrbra with the generators $e_i, f_i \quad (1 \leq i \leq n)$, and $\mathfrak{h}$, and the defining relations (F1) and the following:

$$\begin{aligned}
& (\text{ad } e_i)^{1-a_{ij}} e_j = 0, \quad (\text{ad } f_i)^{1-a_{ij}} f_j = 0, \quad \text{if } a_{ii} = 2, \quad j \neq i, \\
& [e_i, e_j] = 0, \quad [f_i, f_j] = 0 \quad \text{if } a_{ij} = 0.
\end{aligned}
\tag{F2}$$

Here the GGCM A is called *symmetrizable* if there exist an invertible diagonal matrix D such that $D^{-1}A$ is symmetric. (In this case, we put $B := D^{-1}A$ and call it a *symmetrization* of A . And $\mathfrak{g}(A)$ is called a *symmetrizable GKM algebra*.)

We often consider the derived subalgebra $[\mathfrak{g}(A), \mathfrak{g}(A)]$ of $\mathfrak{g}(A)$ instead of a GKM algebra $\mathfrak{g}(A)$, and also call it a GKM algebra. Note that $\mathfrak{g}(A) = [\mathfrak{g}(A), \mathfrak{g}(A)] + \mathfrak{h}$, and that $[\mathfrak{g}(A), \mathfrak{g}(A)] \cap \mathfrak{h} = \sum_{i=1}^n \mathbb{C} \alpha_i^\vee$. Further, a GKM algebra $[\mathfrak{g}(A), \mathfrak{g}(A)]$ is a Lie algebra with the generators $e_i, f_i, \alpha_i^\vee$ ($1 \leq i \leq n$), and the defining relations (F2) and the following:

$$\begin{aligned}
& [e_i, f_j] = \delta_{ij} \alpha_i^\vee \quad (1 \leq i, j \leq n), \\
& (\text{F'1}) \quad [\alpha_i^\vee, \alpha_j^\vee] = 0 \quad (1 \leq i, j \leq n), \\
& [\alpha_i^\vee, e_j] = a_{ij} e_j, \quad [\alpha_i^\vee, f_j] = -a_{ij} f_j \quad (1 \leq i, j \leq n).
\end{aligned}$$

1.2. Roots and invariant bilinear forms. Let $A = (a_{ij})_{i,j=1}^n$ be a GGCM, $\mathfrak{g}(A)$ be a GKM algebra associated to A , and $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}(A)$. Then, we have the root space decomposition of $\mathfrak{g}(A)$:

$\mathfrak{g}(A) = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A); [h, x] = \langle \alpha, h \rangle x, \text{ for all } h \in \mathfrak{h}\}$ ($\alpha \in \mathfrak{h}^*$), and $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\}; \mathfrak{g}_\alpha \neq \{0\}\}$. We call $\mathfrak{g}_\alpha$ the root space attached to α , and Δ the root system of $\mathfrak{g}(A)$.

Note that $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$ ($1 \leq i \leq m$). Moreover, every root $\alpha \in \Delta$ is of the form $\alpha = \sum_{i=1}^n k_i \alpha_i$, $k_i \in \mathbb{Z}_{\geq 0}$ ($1 \leq i \leq n$) or $k_i \in \mathbb{Z}_{\leq 0}$ ($1 \leq i \leq n$). So we call $\Pi = \{\alpha_i\}_{i=1}^n \subset \Delta$ the simple root system, and call $\alpha \in \Delta$ positive root (resp. negative root) if the coefficients of α_i in the above expression are all non-negative (resp. non-positive). Denote by Δ_+ (resp. Δ_-) the set of all positive (resp. negative) roots, and by $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) the subalgebra of $\mathfrak{g}(A)$ generated by e_i , $1 \leq i \leq n$ (resp. f_i , $1 \leq i \leq n$). Note that $\mathfrak{n}_+ = \sum_{\alpha \in \Delta_+}^{\oplus} \mathfrak{g}_{\alpha}$ and $\mathfrak{n}_- = \sum_{\alpha \in \Delta_-}^{\oplus} \mathfrak{g}_{-\alpha}$.

Now, suppose that $A = (a_{ij})_{i,j=1}^n$ is a symmetrizable GGCM. Then, we can take a real diagonal matrix $D = \text{diag}(\varepsilon_1, \dots, \varepsilon_n)$ and a real symmetric matrix $B = (b_{ij})_{i,j=1}^n$ such that $A = DB$ and $\varepsilon_i > 0$ ($1 \leq i \leq n$). So, we fix such a decomposition of A and a complementary subspace $\mathfrak{h}''$ to $\mathfrak{h}' := \sum_{i=1}^n \mathbb{C}\alpha_i^{\vee}$ in $\mathfrak{h}$. Then, there exists uniquely a non-degenerate symmetric invariant bilinear form $(\cdot | \cdot)$ on $\mathfrak{g}(A)$ such that:

$$(B1) \quad (\alpha_i^{\vee} | h) = \langle \alpha_i, h \rangle \varepsilon_i \quad (1 \leq i \leq n, h \in \mathfrak{h}),$$

$$(B2) \quad (h | h') = 0 \quad (h, h' \in \mathfrak{h}).$$

This bilinear form is called a *standard invariant form* on $\mathfrak{g}(A)$. Note that the restriction $(\cdot | \cdot)|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate. So, we can define an isomorphism $\nu; \mathfrak{h} \rightarrow \mathfrak{h}^*$, determined by $\langle \nu(h), h' \rangle = (h | h')$ ($h, h' \in \mathfrak{h}$) as well as an induced bilinear form on $\mathfrak{h}^*$. We know the following equalities (cf. [4]):

$$(1.2.1) \quad v(\alpha_i^\vee) = \varepsilon_i \alpha_i \quad (1 \leq i \leq n),$$

$$(1.2.2) \quad (\alpha_i | \alpha_j) = b_{ij} = a_{ij} / \varepsilon_i \quad (1 \leq i, j \leq n),$$

$$(1.2.3) \quad \alpha_i^\vee = 2 / (\alpha_i | \alpha_i) \cdot v^{-1}(\alpha_i) \quad (1 \leq i \leq n),$$

$$(1.2.4) \quad (\alpha_i^\vee | \alpha_j^\vee) = b_{ij} \varepsilon_i \varepsilon_j \quad (1 \leq i, j \leq n),$$

$$(1.2.5) \quad [x, y] = (x|y) \cdot v^{-1}(\alpha) \text{ for } x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha} \quad (\alpha \in \Delta).$$

§2. Regular subalgebras of a symmetrizable Kac-Moody algebra

For the detailed accounts of this section, see [7].

2.1. Construction of regular subalgebras. In this subsection, we assume that $A = (a_{ij})_{i,j=1}^n$ is a symmetrizable GCM. Other notations are the same as in §1. For each i ($1 \leq i \leq n$), we define a simple reflection r_i of the space $\mathfrak{h}^*$ by $r_i(\lambda) := \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ ($\lambda \in \mathfrak{h}^*$). The subgroup W of $GL(\mathfrak{h}^*)$ generated by r_i ($1 \leq i \leq n$) is called the *Weyl group* of $\mathfrak{g}(A)$.

Definition 2.1. A subset $\bar{\Pi} = \{\beta_1, \dots, \beta_p, \beta_{p+1}, \dots, \beta_{p+q}\}$ of the root system Δ of $\mathfrak{g}(A)$ is called *fundamental* if it satisfies the following:

- (1) $\bar{\Pi} = \{\beta_i\}_{i=1}^{p+q}$ is a linearly independent subset of $\mathfrak{h}^*$;
- (2) $\beta_i - \beta_j \notin \Delta \cup \{0\}$ ($1 \leq i \neq j \leq p+q$);
- (3) $\beta_i \in \Delta^{\text{re}}$ ($1 \leq i \leq p$) and $\beta_j \in \Delta_+ \cap \Delta^{\text{im}}$ ($p+1 \leq j \leq p+q$).

Here $\Delta^{\text{re}} := W \cdot \bar{\Pi}$ is the set of all *real roots* of $\mathfrak{g}(A)$ and $\Delta^{\text{im}} := \Delta \setminus \Delta^{\text{re}}$ is the set of all *imaginary roots* of $\mathfrak{g}(A)$.

Remark 2.1. The above definition is a generalization of that (which is the case $q = 0$) by Morita [6].

Now, for each imaginary root β_j ($p+1 \leq j \leq p+q$), we define $\beta_j^\vee := \nu^{-1}(\beta_j) \in \mathfrak{h}$. For real root $\beta_i = w(\alpha_{k_i})$ ($w \in W$), $1 \leq i \leq p$, $\beta_i^\vee := w(\alpha_{k_i}^\vee) \in \mathfrak{h}$ has been defined as a dual real root of β_i , and we know $\beta_i^\vee = 2/(\beta_i | \beta_i) \cdot \nu^{-1}(\beta_i)$.

Proposition 2.1. Let $\bar{\Pi} = \{\beta_i\}_{i=1}^{p+q}$ be a fundamental subset of Δ , and put $\bar{A} = (\bar{a}_{ij})_{i,j=1}^{p+q}$, where $\bar{a}_{ij} = \langle \beta_j, \beta_i^\vee \rangle$. Then, $\bar{A}$ is a symmetrizable GGCM. Further, $\bar{A}$ is a GCM if and only if every β_i is a real root.

Remark 2.2. The symmetrizability of $\bar{A}$ is shown as follows:

Put $\bar{B} := ((\beta_i | \beta_j))_{i,j=1}^{p+q}$ and $\bar{D} := \text{diag}(2/(\beta_1 | \beta_1), \dots, 2/(\beta_p | \beta_p), 1, \dots, 1)$. Then we have $\bar{A} = \bar{D}\bar{B}$, ${}^t(\bar{A}) = \bar{A}$, and $\det \bar{D} \neq 0$. Note that the j -th diagonal element is 1, while $\bar{a}_{jj} = (\beta_j | \beta_j) \leq 0$ ($p+1 \leq j \leq p+q$).

Proposition 2.2. There exists a vector subspace $\mathfrak{h}_0$ of $\mathfrak{h}$, such that the triple $(\mathfrak{h}_0, \{\beta_i|_{\mathfrak{h}_0}\}_{i=1}^{p+q}, \{\beta_i^\vee\}_{i=1}^{p+q})$ is a realization of the GGCM $\bar{A}$.

We take and fix non-zero vectors $E_i \in \mathfrak{g}_{\beta_i}$ and $F_i \in \mathfrak{g}_{-\beta_i}$ such that $[E_i, F_i] = \beta_i^\vee$ ($1 \leq i \leq p+q$). Note that such vectors always exist (see (1.2.5)). Let $\bar{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by

E_i, F_i ($1 \leq i \leq p+q$), and a vector subspace h_0 of h such that the triple $(h_0, \{\beta_i|_{h_0}\}_{i=1}^{p+q}, \{\beta_i^\vee\}_{i=1}^{p+q})$ is a realization of $\bar{A}$. We call this kind of subalgebra a *regular subalgebra* of $g(A)$.

Theorem 2.1. Any regular subalgebra of $g(A)$ is canonically isomorphic to a GKM algebra. In fact, the above regular subalgebra $\bar{g}$ is isomorphic to a GKM algebra $g(\bar{A})$.

2.2. An embedding of GKM algebras into a symmetrizable Kac-Moody algebra. In §2.1, we constructed a regular subalgebra $\bar{g}$ of a symmetrizable Kac-Moody algebra $g(A)$, and $\bar{g}$ is canonically isomorphic to a GKM algebra $g(\bar{A})$. This GGCM $\bar{A}$ has the following *strong symmetrizability* ($\bar{S}\bar{S}$) (cf. Remark 2.2) and *integrality* ($\bar{I}$), if we normalize a standard invariant form $(\cdot|\cdot)$ on $g(A)$ in such a way that $(\alpha_i|\alpha_j) \in \mathbb{Z}$ ($1 \leq i, j \leq n$).

($\bar{S}\bar{S}$) There exist an invertible rational diagonal matrix $\bar{D}$ and a rational symmetric matrix $\bar{B} = (\bar{b}_{ij})_{i,j=1}^{p+q}$, such that $\bar{A} = \bar{D}\bar{B}$ and $\bar{D} = \text{diag}(\bar{e}_1, \dots, \bar{e}_p, 1, \dots, 1)$,

($\bar{I}$) $\bar{a}_{ij} \in \mathbb{Z}$ ($1 \leq i, j \leq p+q$).

Recall that the j -th diagonal element of $\bar{D}$ is 1, if $\bar{a}_{jj} \leq 0$.

Conversely, we have the following theorem.

Theorem 2.2. Let $A = (-a_{ij})_{i,j=1}^{p+q}$ be a GGCM, such that

$$(G1) \quad -a_{ii} = 2 \quad (1 \leq i \leq p),$$

$$(G2) \quad -a_{jj} \leq 0 \quad (p+1 \leq j \leq p+q).$$

And assume that A satisfies the following *strong symmetrizability condition* (SS) and *integrality condition* (I):

(SS) There exist an invertible rational diagonal matrix D and a rational symmetric matrix $B = (b_{ij})_{i,j=1}^{p+q}$, such that $A = DB$ and $D = \text{diag}(\varepsilon_1, \dots, \varepsilon_p, 1, \dots, 1)$;

(I) $a_{ij} \in \mathbb{Z} \quad (1 \leq i, j \leq p+q)$.

Then, there exists a symmetrizable GCM $\bar{A} = (\bar{a}_{ij})_{i,j=1}^{2(p+q)}$ such that the GKM algebra $\mathfrak{g}(A)$ is canonically isomorphic to a regular subalgebra $\bar{\mathfrak{g}}$ of the Kac-Moody algebra $\mathfrak{g}(\bar{A})$.

PROOF. Note that we can assume that $\varepsilon_i > 0 \quad (1 \leq i \leq p)$ without any loss of generality.

STEP 1. First, we put for $j \quad (p+1 \leq j \leq p+q)$ as follows:

$$u_j := \begin{cases} a_{jj} & (a_{jj} \neq 0) \\ 1 & (a_{jj} = 0) \end{cases}, \quad v_j := -(a_{jj} + 2),$$

$$X_j := \begin{cases} -2^{-1}(a_{jj} + \frac{2}{a_{jj}}) & (a_{jj} \neq 0) \\ -1 & (a_{jj} = 0) \end{cases},$$

$$Y_j := \begin{cases} a_{jj}^{-1} & (a_{jj} \neq 0) \\ 1 & (a_{jj} = 0) \end{cases}.$$

And for $i \quad (1 \leq i \leq p)$, $Z_i := -\varepsilon_i^{-1}$.

Second, we define $2(p+q) \times 2(p+q)$ matrix $\bar{D}$ and $\bar{B}$ as follows:

$$\bar{D} := \text{diag}(-Z_1^{-1}, -Z_1^{-1}, -Z_2^{-1}, -Z_2^{-1}, \dots, -Z_p^{-1}, -Z_p^{-1}, 2u_{p+1}, 2u_{p+1}, 2u_{p+2}, 2u_{p+2}, \dots, 2u_{p+q}, 2u_{p+q}),$$

$\bar{B} := (\bar{b}_{ij})_{i,j=1}^{2(p+q)}$, where

$$\begin{bmatrix} \bar{b}_{2k-1, 2\ell-1} & \bar{b}_{2k-1, 2\ell} \\ \bar{b}_{2k, 2\ell-1} & \bar{b}_{2k, 2\ell} \end{bmatrix} := \begin{bmatrix} Z_k \cdot a_{k\ell} & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} (1 \leq k \leq p, \\ 1 \leq \ell \leq p+q, \ell \neq k), \end{matrix}$$

$$\begin{bmatrix} \bar{b}_{2k-1, 2k-1} & \bar{b}_{2k-1, 2k} \\ \bar{b}_{2k, 2k-1} & \bar{b}_{2k, 2k} \end{bmatrix} := \begin{bmatrix} -2Z_k & Z_k \\ Z_k & -2Z_k \end{bmatrix} \quad (1 \leq k \leq p),$$

$$\begin{bmatrix} \bar{b}_{2k-1, 2\ell-1} & \bar{b}_{2k-1, 2\ell} \\ \bar{b}_{2k, 2\ell-1} & \bar{b}_{2k, 2\ell} \end{bmatrix} := \begin{bmatrix} -a_{k\ell} & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} (p+1 \leq k \leq p+q \\ 1 \leq \ell \leq p+q, \ell \neq k), \end{matrix}$$

and $\begin{bmatrix} \bar{b}_{2k-1, 2k-1} & \bar{b}_{2k-1, 2k} \\ \bar{b}_{2k, 2k-1} & \bar{b}_{2k, 2k} \end{bmatrix} := \begin{bmatrix} Y_k & X_k \\ X_k & Y_k \end{bmatrix} \quad (p+1 \leq k \leq p+q).$

As $(b_{ij})_{i,j=1}^{p+q} = B = D^{-1}A = \text{diag}(-Z_1, \dots, -Z_p, 1, \dots, 1) \cdot (-a_{ij})_{i,j=1}^{p+q}$, we have

$$b_{ij} = \begin{cases} Z_i \cdot a_{ij} & (1 \leq i \leq p, 1 \leq j \leq p+q) \\ -a_{ij} & (p+1 \leq i \leq p+q, 1 \leq j \leq p+q). \end{cases}$$

Therefore, $\bar{B}$ is clearly a symmetric matrix (see also Figure 1).

Finally, we put $\bar{A} = \bar{D}\bar{B}$. Then, $\bar{A} = (\bar{a}_{ij})_{i,j=1}^{2(p+q)}$, where

$$\begin{bmatrix} \bar{a}_{2k-1, 2\ell-1} & \bar{a}_{2k-1, 2\ell} \\ \bar{a}_{2k, 2\ell-1} & \bar{a}_{2k, 2\ell} \end{bmatrix} := \begin{bmatrix} -a_{k\ell} & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} (1 \leq k \leq p \\ 1 \leq \ell \leq p+q, \ell \neq k), \end{matrix}$$

$$\begin{bmatrix} \bar{a}_{2k-1, 2k-1} & \bar{a}_{2k-1, 2k} \\ \bar{a}_{2k, 2k-1} & \bar{a}_{2k, 2k} \end{bmatrix} := \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad (1 \leq k \leq p),$$

$$\begin{bmatrix} \bar{a}_{2k-1, 2\ell-1} & \bar{a}_{2k-1, 2\ell} \\ \bar{a}_{2k, 2\ell-1} & \bar{a}_{2k, 2\ell} \end{bmatrix} := \begin{bmatrix} -2a_{k\ell} \cdot u_k & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} (p+1 \leq k \leq p+q \\ 1 \leq \ell \leq p+q, \ell \neq k) \end{matrix},$$

and

$$\begin{bmatrix} \bar{a}_{2k-1, 2k-1} & \bar{a}_{2k-1, 2k} \\ \bar{a}_{2k, 2k-1} & \bar{a}_{2k, 2k} \end{bmatrix} := \begin{bmatrix} 2 & v_k \\ v_k & 2 \end{bmatrix} \quad (p+1 \leq k \leq p+q).$$

Hence $\bar{A}$ is clearly a symmetrizable GCM (see also Figure 2).

STEP 2. Let $\mathfrak{g}(\bar{A})$ be a Kac-Moody algebra associated to the above GCM $\bar{A}$, $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}(\bar{A})$, and $\Pi = \{\alpha_i\}_{i=1}^{2(p+q)} \subset \mathfrak{h}^*$ be the simple root system of $\mathfrak{g}(\bar{A})$. And let $(\cdot | \cdot)$ be a standard invariant form on $\mathfrak{g}(\bar{A})$ corresponding to the decomposition: $\bar{A} = \bar{D}\bar{B}$. Now, we put $\beta_k := \alpha_{2k-1} + \alpha_{2k}$ ($1 \leq k \leq p+q$). Then, $\beta_k \in \Delta_+$ since $\bar{a}_{2k-1, 2k} \leq -1$ ($1 \leq k \leq p+q$). Obviously, $\beta_i - \beta_j \notin \Delta \cup \{0\}$ ($1 \leq i \neq j \leq p+q$) and $\bar{\Pi} := \{\beta_i\}_{i=1}^{p+q} \subset \mathfrak{h}^*$ is linearly independent. So, $\bar{\Pi}$ is a fundamental subset of the root system Δ of $\mathfrak{g}(\bar{A})$. Therefore, we see from Theorem 2.1 that there exists a regular subalgebra $\bar{\mathfrak{g}}$ of $\mathfrak{g}(\bar{A})$, which is canonically isomorphic to a GKM algebra $\mathfrak{g}(\tilde{A})$ associated to $\tilde{A} := (\langle \beta_j, \beta_i^\vee \rangle)_{i,j=1}^{p+q}$. Theorem 2.2 now follows from the following claim:

CLAIM. $\tilde{A} = A$.

PROOF of the claim. Since $\beta_k = \alpha_{2k-1} + \alpha_{2k}$ ($1 \leq k \leq p+q$), we have

$$(\beta_k | \beta_\ell) = \bar{b}_{2k-1, 2\ell-1} + \bar{b}_{2k-1, 2\ell} + \bar{b}_{2k, 2\ell-1} + \bar{b}_{2k, 2\ell} =$$

$$\begin{cases} Z_k \cdot a_{k\ell} & (1 \leq k \leq p, 1 \leq \ell \leq p+q) \\ -a_{k\ell} & (p+1 \leq k \leq p+q, 1 \leq \ell \leq p+q) \end{cases}$$

Recall that for $\alpha \in \Delta$, $\alpha \in \Delta^{\text{re}}$ if and only if $(\alpha|\alpha) > 0$, and $\alpha \in \Delta^{\text{im}}$ if and only if $(\alpha|\alpha) \leq 0$, where $(\cdot|\cdot)$ is a standard invariant form on $\mathfrak{g}(\bar{A})$. Hence $\beta_k \in \Delta^{\text{re}} \cap \Delta_+$ ($1 \leq k \leq p$) and $\beta_k \in \Delta^{\text{im}} \cap \Delta_+$ ($p+1 \leq k \leq p+q$) from the above equalities. Therefore, for k ($1 \leq k \leq p$) we have $\tilde{a}_{k\ell} = \langle \beta_\ell, \beta_k^\vee \rangle = 2(\beta_k|\beta_\ell)/(\beta_k|\beta_k) = -a_{k\ell}$. And for k ($p+1 \leq k \leq p+q$), we have $\tilde{a}_{k\ell} = \langle \beta_\ell, \beta_k^\vee \rangle = (\beta_k|\beta_\ell) = -a_{k\ell}$. In conclusion, $\tilde{A} = (\tilde{a}_{k\ell})_{k,\ell=1}^{p+q} = (-a_{k\ell})_{k,\ell=1}^{p+q} = A$. Thus, the claim has been proved. $\square$

2.3. Sufficient conditions for the strong symmetrizability of GGCM. Let $A = (-a_{ij})_{i,j=1}^{p+q}$ be a GGCM satisfying the integrality condition (I), and reordered so that (G1) and (G2) hold. Here, we give sufficient conditions for the strong symmetrizability (SS) of the above GGCM A . Obviously, if A is symmetric, then A satisfies the condition (SS). For other sufficient conditions, we have the following.

Proposition 2.3. If the above GGCM A is symmetrizable, and satisfies the following two conditions, then A satisfies (SS).

(a) For every i ($1 \leq i \leq p$), there exists j ($p+1 \leq j \leq p+q$) such that $a_{ij} \neq 0$.

(b) The principal submatrix $A^{\text{im}} := (-a_{ij})_{i,j=p+1}^{p+q}$ is indecomposable and symmetric.

PROOF. Note that if A satisfies (SS), then A^{im} is necessarily symmetric.

For each i ($1 \leq i \leq p$), we put $Z_i := -a_{ji} \cdot a_{ij}^{-1}$ if $a_{ij} \neq 0$ ($p+1 \leq j \leq p+q$). Such a j exists from the condition (a), and Z_i does not depend on the choice of j from the condition (b). Then, we put $D := \text{diag}(-Z_1^{-1}, -Z_2^{-1}, \dots, -Z_p^{-1}, 1, 1, \dots, 1)$ and $B := (b_{ij})_{i,j=1}^{p+q}$, where

$$b_{ij} := \begin{cases} Z_i \cdot a_{ij} & (1 \leq i \leq p, 1 \leq j \leq p+q) \\ -a_{ij} & (p+1 \leq i \leq p+q, 1 \leq j \leq p+q) \end{cases}$$

Then, $A = DB$, and B is symmetric from the condition (b). Hence A satisfies (SS). $\square$

Corollary 2.1. Let $A = (-a_{ij})_{i,j=1}^{p+q}$ be a symmetrizable GGCM with (G1), (G2), and satisfying the integrality condition (I). If $a_{ij} \neq 0$ ($1 \leq i \neq j \leq p+q$) and $A^{im} := (-a_{ij})_{i,j=p+1}^{p+q}$ is symmetric, then A satisfies the strong symmetrizability condition (SS).

§3. Folding subalgebras of a symmetrizable GKM algebra

3.1. Diagram automorphisms of a GGCM. Let $A = (a_{ij})_{i,j=1}^n$ be an indecomposable, symmetrizable GGCM, $\mathfrak{g}(A)$ be a GKM algebra associated to A , and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}(A)$. Fix a decomposition of A : $A = DB$, where $D = \text{diag}(\varepsilon_1, \dots, \varepsilon_n)$ ($\varepsilon_i > 0$, $1 \leq i \leq n$), $B = (b_{ij})_{i,j=1}^n$ ($b_{ij} = b_{ji} \in \mathbb{R}$, $1 \leq i, j \leq n$). And let $(\cdot | \cdot)$ be a standard invariant form on $\mathfrak{g}(A)$ corresponding to the above

decomposition of A .

Definition 3.1. A permutation π on $I := \{1, 2, \dots, n\}$ is called a *diagram automorphism* of a GGCM A if

$$(D1) \quad a_{\pi(i), \pi(j)} = a_{ij} \text{ for every } i, j \ (1 \leq i, j \leq n).$$

Lemma 3.1. We have $\varepsilon_{\pi(i)} = \varepsilon_i$ for every $i \ (1 \leq i \leq n)$.

PROOF. Since $A = DB$, we have $a_{ij} = \varepsilon_i b_{ij}$ and $a_{\pi(i), \pi(j)} = \varepsilon_{\pi(i)} b_{\pi(i), \pi(j)}$ ($1 \leq i, j \leq n$). So, we have $\varepsilon_i b_{ij} = \varepsilon_{\pi(i)} b_{\pi(i), \pi(j)}$ and $\varepsilon_j b_{ji} = \varepsilon_{\pi(j)} b_{\pi(j), \pi(i)}$ ($1 \leq i, j \leq n$) from (D1). Therefore, we get $\varepsilon_i \varepsilon_{\pi(j)} b_{ij} b_{\pi(j), \pi(i)} = \varepsilon_j \varepsilon_{\pi(i)} b_{ji} b_{\pi(i), \pi(j)}$ ($1 \leq i, j \leq n$). Note that if $a_{ij} = a_{\pi(i), \pi(j)} \neq 0$, then $b_{ij} \neq 0$ and $b_{\pi(i), \pi(j)} \neq 0$. Hence, we obtain that $\varepsilon_i \varepsilon_{\pi(j)} = \varepsilon_j \varepsilon_{\pi(i)}$ or $\varepsilon_i^{-1} \cdot \varepsilon_{\pi(i)} = \varepsilon_j^{-1} \cdot \varepsilon_{\pi(j)}$ if $a_{ij} \neq 0$. Therefore, there exists a positive constant M such that $\varepsilon_i^{-1} \cdot \varepsilon_{\pi(i)} = M$ for every $i \ (1 \leq i \leq n)$, since the GGCM A is indecomposable. So, we have $\prod_{i=1}^n \frac{\varepsilon_{\pi(i)}}{\varepsilon_i} = M^n$. On the other hand, the left hand side equals to $\frac{\prod_{i=1}^n \varepsilon_{\pi(i)}}{\prod_{i=1}^n \varepsilon_i} = 1$. Therefore, we have $M^n = 1$, so that $M = 1$. Thus the assertion has been proved. $\square$

Now, since a diagram automorphism π is a permutation on I , we can express it uniquely as a commuting product of cyclic permutations: There exists uniquely a decomposition of I into its disjoint subsets $I_j \ (1 \leq j \leq m)$, such that the restriction $\pi_j := \pi|_{I_j}$ of π to I_j is a cyclic permutation ($1 \leq j \leq m$).

Lemma 3.2. For every j_1, j_2 ($1 \leq j_1, j_2 \leq m$) and $i_1, i_2 \in I_{j_1}$, we have $\sum_{k \in I_{j_2}} a_{k, i_1} = \sum_{k \in I_{j_2}} a_{k, i_2}$.

PROOF. Since $\pi_{j_1} = \pi|_{I_{j_1}}$ is a cyclic permutation, it is enough to assume that $i_2 = \pi(i_1)$. Then, we have $\sum_{k \in I_{j_2}} a_{k, i_2} = \sum_{k \in I_{j_2}} a_{k, \pi(i_1)} = \sum_{k \in I_{j_2}} a_{\pi^{-1}(k), i_1}$ (by (D1)) = $\sum_{k \in I_{j_2}} a_{k, i_1}$ $\square$

Now, we set $\bar{a}_{ij} := \sum_{k \in I_i} a_{k\ell}$ for $\ell \in I_j$ ($1 \leq i, j \leq m$). From Lemma 3.2, the right hand side does not depend on the choice of $\ell \in I_j$.

Lemma 3.3. If $\bar{a}_{ii} = \sum_{k \in I_i} a_{k, \ell}$ ($\ell \in I_i$) is a positive real number, then we have the following two cases:

CASE (A) $a_{k\ell} = 0$ for every $k, \ell \in I_i$ ($k \neq \ell$), and $a_{kk} = 2$ for every $k \in I_i$.

CASE (B) $a_{kk} = 2$ for every $k \in I_i$, and there exists a decomposition of I_i into its disjoint subsets $I_i^{(p)}$ ($1 \leq p \leq t_i$) such that $|I_i^{(p)}| = 2$ for every p ($1 \leq p \leq t_i$), $a_{k\ell} = 0$ for every $k \in I_i^{(p)}$, $\ell \in I_i^{(q)}$ ($1 \leq p \neq q \leq t_i$), and $a_{k\ell} = a_{\ell k} = -1$ for every $k, \ell \in I_i^{(p)}$ ($k \neq \ell$), $1 \leq p \leq t_i$. Here $|S|$ denotes the number of elements of a set S .

PROOF. First, recall that $\sum_{k \in I_i} a_{k\ell}$ ($\ell \in I_i$) does not depend on the choice of $\ell \in I_i$. Therefore, we have the following for

every $\ell \in I_1$:

$$\bar{a}_{11} = \sum_{k \in I_1} a_{k\ell} > 0 \implies a_{\ell\ell} > 0 \text{ (since } a_{k\ell} \leq 0, \text{ for } k \neq \ell)$$

$$\implies a_{\ell\ell} = 2 \text{ (since } A = (a_{ij})_{i,j=1}^n \text{ is a GGCM)}$$

$$\implies a_{\ell k} \in \mathbb{Z}_{\leq 0} \text{ for } k \in I_1 \text{ (} k \neq \ell \text{) (since } A \text{ satisfies (C2)).}$$

Therefore, we have $a_{\ell\ell} = 2$ ($\ell \in I_1$), and $a_{k\ell} \in \mathbb{Z}_{\leq 0}$ ($k \neq \ell$, $k, \ell \in I_1$). Hence we deduce that $\sum_{k \in I_1 \setminus \{\ell\}} a_{k\ell} = 0$ or -1 ($\ell \in I_1$) from the assumption. Moreover, $\sum_{k \in I_1 \setminus \{\ell\}} a_{k\ell}$ does not depend on $\ell \in I_1$ since $a_{\ell\ell} = 2$ ($\ell \in I_1$). Therefore, we have the following two cases:

$$\text{CASE (A)} \quad \sum_{k \in I_1 \setminus \{\ell\}} a_{k\ell} = 0, \text{ for every } \ell \in I_1,$$

$$\text{CASE (B)} \quad \sum_{k \in I_1 \setminus \{\ell\}} a_{k\ell} = -1, \text{ for every } \ell \in I_1.$$

In case (A), we have $a_{k\ell} = 0$ ($k \neq \ell$, $k, \ell \in I_1$), since $a_{k\ell} \leq 0$ ($k \neq \ell$). In case (B), for every $\ell \in I_1$, there exists exactly one $k_\ell \in I_1 \setminus \{\ell\}$ such that $a_{k_\ell, \ell} = -1$ and $a_{k\ell} = 0$ ($k \in I_1 \setminus \{\ell, k_\ell\}$), since $a_{k\ell} \in \mathbb{Z}_{\leq 0}$ ($k \neq \ell$, $k, \ell \in I_1$). Therefore, for every $\ell \in I_1$, $a_{\ell, k_\ell} = -1$ since $a_{k\ell} = 0$ implies $a_{\ell k} = 0$. Thus the assertion is now proved. $\square$

3.2. Construction of folding subalgebras. Notations are the same as in 3.1. We put for j ($1 \leq j \leq m$)

$$E'_j := \sum_{k \in I_j} e_k, \quad F'_j := \sum_{k \in I_j} f_k, \quad H'_j := \sum_{k \in I_j} \alpha_k^\vee, \quad \text{and } \beta_j := \sum_{k \in I_j} \alpha_k,$$

where e_i, f_i ($1 \leq i \leq n$) are the Chevalley generators, $\{\alpha_i\}_{i=1}^n$ is the set of all simple roots, and $\{\alpha_i^\vee\}_{i=1}^n$ is the set of all

simple coroots of the GKM algebra $\mathfrak{g}(A)$.

Proposition 3.1. We have the following equations:

$$(3.2.1) \quad [H'_i, E'_j] = \bar{a}_{ij} E'_j \quad (1 \leq i, j \leq m),$$

$$(3.2.2) \quad [H'_i, F'_j] = -\bar{a}_{ij} F'_j \quad (1 \leq i, j \leq m),$$

$$(3.2.3) \quad [E'_i, F'_j] = \delta_{ij} H'_i \quad (1 \leq i, j \leq m).$$

PROOF. PROOF for (3.2.1).

$$\begin{aligned} [H'_i, E'_j] &= [\sum_{k \in I_i} \alpha_k^\vee, \sum_{\ell \in I_j} e_\ell] = \sum_{k, \ell} [\alpha_k^\vee, e_\ell] = \sum_{k, \ell} \langle \alpha_k^\vee, e_\ell \rangle e_\ell = \\ &= \sum_{k, \ell} a_{k\ell} e_\ell = \sum_{\ell \in I_j} (\sum_{k \in I_i} a_{k\ell}) e_\ell = \sum_{\ell \in I_j} \bar{a}_{ij} e_\ell = \bar{a}_{ij} (\sum_{\ell \in I_j} e_\ell) = \\ &= \bar{a}_{ij} E'_j. \end{aligned}$$

The proof is similar for (3.2.2), and (3.2.3) is obvious. $\square$

Now, we say, for i ($1 \leq i \leq m$), "CASE X(i)" if $\bar{a}_{ii} \leq 0$, or if $\bar{a}_{ii} > 0$ and case (A) happens, and "CASE Y(i)" if $\bar{a}_{ii} > 0$ and case (B) happens. And we put $\hat{A} := (\hat{a}_{ij})_{i,j=1}^m$, where

$$\hat{a}_{ij} := \begin{cases} \bar{a}_{ij} & \text{if X(i)} \\ 2\bar{a}_{ij} & \text{if Y(i)} \end{cases}.$$

Moreover, we put for i ($1 \leq i \leq m$)

$$H_i := \begin{cases} H'_i & \text{if X(i)} \\ 2H'_i & \text{if Y(i)} \end{cases},$$

$$E_1 := \begin{cases} E'_1 & \text{if } X(1) \\ \sqrt{2} \cdot E'_1 & \text{if } Y(1) \end{cases}, \quad F_1 := \begin{cases} F'_1 & \text{if } X(1) \\ \sqrt{2} \cdot F'_1 & \text{if } Y(1) \end{cases}.$$

Then, we have the following propositions.

Proposition 3.2. $\hat{A}$ is a GGCM.

PROOF. We have to check (C1)-(C3) in Definition 1.1.

PROOF for (C1). In the case $X(1)$, we have

$$\hat{a}_{11} = \bar{a}_{11} = \begin{cases} \leq 0 & \text{if } \bar{a}_{11} \leq 0, \\ 2 & \text{if } \bar{a}_{11} > 0 \text{ and in the case (A).} \end{cases}$$

In the case $Y(1)$ (i.e., if $\bar{a}_{11} > 0$ and in the case (B)), we have

$$\hat{a}_{11} = 2\bar{a}_{11} = 2 \times (2 - 1) = 2.$$

PROOF for (C2). We have $\bar{a}_{1j} = \sum_{k \in I_1} a_{k\ell} \ (\ell \in I_j) \leq 0 \ (1 \leq j \leq m)$,

since $I_1 \cap I_j = \emptyset$. So, we have $\hat{a}_{1j} \leq 0$. Further, if $\hat{a}_{11} = 2$,

then $\bar{a}_{11} > 0$, and so $a_{kk} = 2$ for every $k \in I_1$. Therefore, $a_{k\ell} \in \mathbb{Z}_{\leq 0} \ (k \in I_1, \ell \neq k)$ since $A = (a_{ij})_{i,j=1}^n$ is a GGCM. Hence, $\bar{a}_{1j} = \sum_{k \in I_1} a_{k\ell} \in \mathbb{Z}_{\leq 0}$.

PROOF for (C3). For $i, j \ (1 \leq i \neq j \leq m)$, we have $\bar{a}_{1j} = \sum_{k \in I_1} a_{k\ell} \ (\ell \in I_j)$ and $a_{k\ell} \leq 0 \ (k \in I_1)$. So, $\bar{a}_{1j} = 0$ if and only if $a_{k\ell} = 0$ for every $k \in I_1$. Note that $\sum_{k \in I_1} a_{k\ell} \ (\ell \in I_j)$ does not depend on $\ell \in I_j$.

Therefore, $\bar{a}_{1j} = 0$ if and only if $a_{k\ell} = 0$ for every $k \in I_1$ and every $\ell \in I_j$. Now, recall that $a_{k\ell} = 0$ implies $a_{\ell k} = 0 \ (k \in I_1, \ell \in I_j)$ since A is a GGCM. Hence, $\bar{a}_{1j} = 0$ implies $\bar{a}_{ji} = 0$.

In conclusion, $\hat{A} = (\hat{a}_{ij})_{i,j=1}^m$ is a GGCM. $\square$

Remark 3.1. From the proof for (C3), we can deduce that the GGCM $\hat{A}$ is indecomposable.

Remark 3.2. Even if A is a GCM, $\hat{A}$ is not a GCM except for the case that for every i ($1 \leq i \leq m$), case (A) or case (B) in Lemma 3.3 happens.

Proposition 3.3. The GGCM $\hat{A}$ is symmetrizable.

PROOF. First, note that $(\alpha_{\pi(i)} | \alpha_{\pi(j)}) = b_{\pi(i), \pi(j)} = \varepsilon_{\pi(i)}^{-1} \cdot a_{\pi(i), \pi(j)} = \varepsilon_i^{-1} \cdot a_{ij} = b_{ij} = (\alpha_i | \alpha_j)$ from Lemma 3.1. So, we get

$$(3.2.4) \quad (\alpha_{i_1} | \sum_{k \in I_{j_2}} \alpha_k) = (\alpha_{i_2} | \sum_{k \in I_{j_2}} \alpha_k)$$

for every j_1, j_2 ($1 \leq j_1, j_2 \leq m$) and $i_1, i_2 \in I_{j_1}$, by the same way as

Lemma 3.2. Now, we put $\varepsilon^{(j)} := \varepsilon_k$ ($k \in I_j$) for j ($1 \leq j \leq m$) (not depend on the choice of $k \in I_j$ by Lemma 3.1). Then, we have

$\bar{a}_{ij} = \sum_{k \in I_i} a_{k\ell}$ ($\ell \in I_j$) $= \sum_{k \in I_i} \varepsilon_k b_{k\ell} = \varepsilon^{(i)} \cdot \sum_{k \in I_i} (\alpha_k | \alpha_\ell) = \varepsilon^{(i)} \cdot (\sum_{k \in I_i} \alpha_k | \alpha_\ell) = \varepsilon^{(i)} \cdot |I_j|^{-1} (\sum_{k \in I_i} \alpha_k | \sum_{\ell \in I_j} \alpha_\ell)$ (by (3.2.4)) $= \varepsilon^{(i)} |I_j|^{-1} (\beta_i | \beta_j)$. We define $\hat{B} := (|I_i|^{-1} |I_j|^{-1} (\beta_i | \beta_j))_{i,j=1}^m$, and $\hat{D} := \text{diag}(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_m)$, where

$$\hat{\varepsilon}_i := \begin{cases} \varepsilon^{(i)} |I_i| & \text{in case X(i)} \\ 2\varepsilon^{(i)} |I_i| & \text{in case Y(i)} \end{cases}.$$

Then, $\hat{A} = \hat{D}\hat{B}$, ${}^t(\hat{B}) = \hat{B}$, and $\det \hat{D} \neq 0$. Hence the GGCM $\hat{A}$ is symmetrizable. $\square$

Let $\hat{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by E_i , F_i , and H_i ($1 \leq i \leq m$).

Definition 3.2. We call the above subalgebra $\hat{\mathfrak{g}}$ of $\mathfrak{g}(A)$ the *folding subalgebra* (of $\mathfrak{g}(A)$) corresponding to a diagram automorphism π of A .

3.3. Main result. In this subsection, we obtain the following theorem, which is the main result of §3.

Theorem 3.1. Any folding subalgebra of $\mathfrak{g}(A)$ is canonically isomorphic to the derived algebra of a GKM algebra. Let $\hat{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by E_i , F_i , and H_i ($1 \leq i \leq m$). Then, the canonical isomorphism Φ of the derived algebra $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$ onto the folding subalgebra $\hat{\mathfrak{g}}$ is given as:

$$\Phi(\hat{e}_i) = E_i, \quad \Phi(\hat{f}_i) = F_i, \quad \text{and} \quad \Phi(\hat{\alpha}_i^\vee) = H_i \quad (1 \leq i \leq m).$$

Here $\hat{e}_i$, $\hat{f}_i$ ($1 \leq i \leq m$) are the Chevalley generators, and $\{\hat{\alpha}_i^\vee\}_{i=1}^m$ is the set of all simple coroots of the GKM algebra $\mathfrak{g}(\hat{A})$.

PROOF. We have to check that E_i , F_i , and H_i ($1 \leq i \leq m$) satisfy all the defining relations for the symmetrizable GKM algebra $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$. However, the relations (F'1) in §1.1 are clear from Proposition 3.1. So, we have only to check relations (F2).

STEP 1. We first check $[E_i, E_j] = 0$ and $[F_i, F_j] = 0$ if $\hat{a}_{ii} \leq 0$ and $\hat{a}_{ij} = 0$ ($1 \leq i \neq j \leq m$). As shown in the proof of Proposition 3.2, $\hat{a}_{ij} = 0$ if and only if $a_{k\ell} = 0$ for every $k \in I_i$

and $\ell \in I_j$. And note that $a_{k\ell} = 0$ implies $[e_k, e_\ell] = 0$ since $\mathfrak{g}(A)$ is a GKM algebra (see §1). So, we have

$[E'_i, E'_j] = [\sum_{k \in I_i} e_k, \sum_{\ell \in I_j} e_\ell] = \sum_{k, \ell} [e_k, e_\ell] = 0$. Hence, we have $[E_i, E_j] = 0$. The proof is similar for the relation $[F_i, F_j] = 0$.

STEP 2. Next, we check that $(\text{ad } E_i)^{1-\hat{a}_{ij}} E_j = 0$ and $(\text{ad } F_i)^{1-\hat{a}_{ij}} F_j = 0$ if $\hat{a}_{ii} = 2$ and $j \neq i$. We only check the relation for E_i and E_j , since the proof is similar for F_i and F_j .

Obviously, $\mathbb{C}E_i + \mathbb{C}H_i + \mathbb{C}F_i$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, with standard basis $\{E_i, H_i, F_i\}$. Further, we have $[H_i, E_j] = \hat{a}_{ij} E_j$ and $[F_i, E_j] = 0$, as shown above. Therefore, the relation $(\text{ad } E_i)^{1-\hat{a}_{ij}} E_j = 0$ follows from the local nilpotency of $\text{ad } E_i$ on $\mathfrak{g}(A)$ and the following relation in the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ ($\cong \mathbb{C}E_i + \mathbb{C}H_i + \mathbb{C}F_i$):

$$[F_i, E_i^k] = -k(k-1)E_i^{k-1} - kE_i^{k-1}H_i \quad (k \geq 1).$$

Therefore, we have only to show the local nilpotency of $\text{ad } E_i$ on $\mathfrak{g}(A)$.

STEP 3. Now, $\hat{a}_{ii} = 2$ implies $\bar{a}_{ii} > 0$, and so either case (A) or case (B) happens by Lemma 3.3.

In CASE (A). We have $a_{k\ell} = 0$ for $k, \ell \in I_i$ ($k \neq \ell$) and $a_{kk} = 2$ for $k \in I_i$. In this case, $E_i = E'_i = \sum_{k \in I_i} e_k$. Note that $\text{ad } e_k$ ($k \in I_i$) is locally nilpotent on $\mathfrak{g}(A)$ since $a_{kk} = 2$. Further, we have $[e_k, e_\ell] = 0$ ($k, \ell \in I_i$, $k \neq \ell$) since $a_{k\ell} = 0$. Therefore,

$\text{ad } E_1 = \sum_{k \in I_1} (\text{ad } e_k)$ is locally nilpotent on $\mathfrak{g}(A)$.

In CASE (B). Also in this case, $\text{ad } e_k$ ($k \in I_1$) is locally nilpotent on $\mathfrak{g}(A)$ since $a_{kk} = 2$. Recall that $|I_1^{(p)}| = 2$, and $a_{k\ell} = a_{\ell k} = -1$ ($k, \ell \in I_1^{(p)}$, $k \neq \ell$) for every p ($1 \leq p \leq t_1$). Say $I_1^{(p)} = \{k, \ell\}$, and put $z := [e_k, e_\ell]$. Then, $[e_k, z] = [e_k, [e_k, e_\ell]] = (\text{ad } e_k)^2 e_\ell = 0$, since $a_{k\ell} = -1$ (see §1). Similarly, $[e_\ell, z] = 0$ since $a_{\ell k} = -1$. On the other hand, $r_\ell(\alpha_k) = \alpha_k - \langle \alpha_k, \alpha_\ell^\vee \rangle \alpha_\ell = \alpha_k - a_{\ell k} \alpha_k = \alpha_k + \alpha_\ell$. So, $\alpha_k + \alpha_\ell$ is a real root of the GKM algebra $\mathfrak{g}(A)$ since $a_{kk} = a_{\ell\ell} = 2$ (see [4, Chap. 11]). Therefore, $\text{ad } z = \text{ad}([e_k, e_\ell])$ is locally nilpotent on $\mathfrak{g}(A)$, since $[e_k, e_\ell] \in \mathfrak{g}_{\alpha_k + \alpha_\ell}$ (cf. [4, Chap. 3]). Now, we can easily deduce that $\text{ad}(e_k + e_\ell)$ is locally nilpotent on $\mathfrak{g}(A)$, from the local nilpotency of $\text{ad } e_k$, $\text{ad } e_\ell$, and $\text{ad } z$, and the following commutation relations:

$$[e_k, z] = 0, [e_\ell, z] = 0, [e_k, e_\ell] = z.$$

Hence, if we put $e_i^{(p)} := \sum_{k \in I_1^{(p)}} e_k$ ($1 \leq p \leq t_1$), then $\text{ad } e_i^{(p)}$ is

locally nilpotent on $\mathfrak{g}(A)$. Further, we have $[e_i^{(p)}, e_i^{(q)}] =$

$$[\sum_{k \in I_1^{(p)}} e_k, \sum_{\ell \in I_1^{(q)}} e_\ell] = \sum_{k, \ell} [e_k, e_\ell] = 0 \quad (1 \leq p \neq q \leq t_1),$$

since $a_{k\ell} = 0$ for $k \in I_1^{(p)}$, $\ell \in I_1^{(q)}$. Therefore, $\text{ad } E'_1 = \sum_{k \in I_1} (\text{ad } e_k)$

$$= \sum_{p=1}^{t_1} (\text{ad } e_i^{(p)})$$

is locally nilpotent on $\mathfrak{g}(A)$, and so is $\text{ad } E_1$.

STEP 4. Thus we have checked all the defining relations for the GKM algebra $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$. Therefore, we get the surjective homomorphism $\Phi: [\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})] \rightarrow \hat{\mathfrak{g}}$, such that $\Phi(\hat{e}_1) = E_1$, $\Phi(\hat{f}_1) = F_1$, and $\Phi(\hat{\alpha}_i^\vee) = H_i$ ($1 \leq i \leq m$). Because H_i ($1 \leq i \leq m$) $\in \mathfrak{h}$ are linearly

independent, we have $(\text{Ker } \Phi) \cap \sum_{i=1}^m \mathbb{C}\hat{\alpha}_i^\vee = \{0\}$. On the other hand, it is easy to see that $\text{Ker } \Phi$ is a graded ideal of $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})] = (\sum_{i=1}^m \mathbb{C}\hat{\alpha}_i^\vee) \oplus \sum_{\hat{\alpha} \in \hat{\Delta}}^{\oplus} \hat{\mathfrak{g}}_{\hat{\alpha}}$, where $\hat{\Delta}$ is the root system of the GKM algebra $\mathfrak{g}(\hat{A})$ and $\hat{\mathfrak{g}}_{\hat{\alpha}}$ is the root space attached to $\hat{\alpha} \in \hat{\Delta}$. Hence $\text{Ker } \Phi = \{0\}$ (see [4, Chap. 1]). This completes the proof of the Theorem. $\square$

Remark 3.3. From the above theorem, we see that, with respect to the operation of making folding subalgebras, the category of Kac-Moody algebras is not closed (see Remark 3.2), but the category of GKM algebras is closed.

3.4. The inheritance of a standard invariant form. Here we prove a certain inheritance of a standard invariant form to a folding subalgebra, as was proved in the case of a regular subalgebra in [7]. The notations are the same as in §3.1 - 3.3.

Proposition 3.4. Let $(\cdot|\cdot)_1$ be a standard invariant form on $\mathfrak{g}(\hat{A})$ corresponding to the decomposition of $\hat{A}$: $\hat{A} = \hat{D}\hat{B}$ in Proposition 3.3. Then, the restriction of the standard invariant form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$ to the folding subalgebra $\hat{\mathfrak{g}}$ of $\mathfrak{g}(A)$ can be identified with the restriction of $(\cdot|\cdot)_1$ to the derived algebra $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$ of $\mathfrak{g}(\hat{A})$ through the canonical isomorphism $\Phi: [\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})] \rightarrow \hat{\mathfrak{g}}$, except for the following case:
 A is of type $A_{n-1}^{(1)}$ ($n \geq 2$), π is a cyclic permutation on $\{1, 2, \dots, n\}$, and $\hat{A} = 0_1$ (1×1 zero-matrix).

PROOF. First, recall that the GGCM $\hat{A}$ is indecomposable (see Remark 3.1). Second, it is easy to check that $(H_i | H_j) = (\hat{\alpha}_i^\vee | \hat{\alpha}_j^\vee)_1$ for i, j ($1 \leq i, j \leq m$). Then, the proposition follows from the following fact (see [4, Chap. 2]):

FACT. If the GGCM $\hat{A}$ is indecomposable, any two invariant bilinear forms on the derived algebra $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$ of $\mathfrak{g}(\hat{A})$ are proportional. $\square$

Remark 3.4. If A is of type $A_{n-1}^{(1)}$ ($n \geq 2$) and π is a cyclic permutation on $\{1, 2, \dots, n\}$, then $\hat{A} = 0_1$ and the proportional constant is n .

3.5. The complete reducibility. For integrable highest weight modules of a Kac-Moody algebra, we have the following complete reducibility with respect to its folding subalgebra.

Proposition 3.5. Let $A = (a_{ij})_{i,j=1}^n$ be an indecomposable, symmetrizable GCM, $\Lambda \in \mathfrak{h}^*$ be a dominant integral weight, and $L(\Lambda)$ be an integrable highest weight module with highest weight Λ over the Kac-Moody algebra $\mathfrak{g}(A)$. Assume that $\hat{A}$ is again a GCM. Then, as $\hat{\mathfrak{g}}$ -modules, $L(\Lambda)$ is isomorphic to a direct sum of $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$ -modules $L(\lambda)$ such that $\lambda \in (\sum_{i=1}^m \mathbb{C} \hat{\alpha}_i^\vee)^*$, $\langle \lambda, \hat{\alpha}_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ ($1 \leq i \leq m$), under the identification: $\hat{\mathfrak{g}} \cong [\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$.

PROOF. We use the Kac's complete reducibility theorem (see [4, Chap. 10]). And we can show the conditions of Kac's theorem in exactly the same way as in the step 3 of the proof of Theorem 3.1. $\square$

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$$\overline{B} = \begin{array}{c} \left[\begin{array}{c} \overbrace{\begin{array}{cccccccc} -2Z_1 & Z_1 & Z_1 a_{12} & 0 & \dots & Z_1 a_{1p} & 0 & Z_1 a_{1,p+1} & 0 & Z_1 a_{1,p+2} & 0 & \dots & Z_1 a_{1,p+q} & 0 \end{array}}^{2p} \\ \underbrace{\begin{array}{cccccccc} Z_1 & -2Z_1 & 0 & 0 & \dots & 0 & 0 & 0 \end{array}}^{2p} \\ Z_2 a_{21} & 0 & -2Z_2 & Z_2 & \dots & Z_2 a_{2p} & 0 & Z_2 a_{2,p+1} & 0 & Z_2 a_{2,p+2} & 0 & \dots & Z_2 a_{2,p+q} & 0 \\ 0 & 0 & Z_2 & -2Z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_p a_{p1} & 0 & Z_p a_{p2} & 0 & -2Z_p & Z_p & Z_p a_{p,p+1} & 0 & Z_p a_{p,p+2} & 0 & \dots & Z_p a_{p,p+q} & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_p & -2Z_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \\ \\ \left[\begin{array}{c} \underbrace{\begin{array}{cccccccc} -a_{p+1,1} & 0 & -a_{p+1,2} & 0 & \dots & -a_{p+1,p} & 0 & Y_{p+1} & X_{p+1} & -a_{p+1,p+2} & 0 & \dots & -a_{p+1,p+q} & 0 \end{array}}^{2q} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{p+1} & Y_{p+1} & 0 & 0 & 0 & 0 & 0 \\ -a_{p+2,1} & 0 & -a_{p+2,2} & 0 & \dots & -a_{p+2,p} & 0 & -a_{p+2,p+1} & 0 & Y_{p+2} & X_{p+2} & \dots & -a_{p+2,p+q} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{p+2} & Y_{p+2} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{p+q,1} & 0 & -a_{p+q,2} & 0 & \dots & -a_{p+q,p} & 0 & -a_{p+q,p+1} & 0 & -a_{p+q,p+2} & 0 & \dots & Y_{p+q} & X_{p+q} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{p+q} & Y_{p+q} & 0 \end{array} \right] \end{array}$$

Figure 1

$$\overline{A} = \begin{array}{c} \left[\begin{array}{cccccccc} 2 & -1 & -a_{12} & 0 & -a_{1p} & 0 & -a_{1,p+1} & 0 & -a_{1,p+2} & 0 & \dots & -a_{1,p+q} & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{21} & 0 & 2 & -1 & -a_{2p} & 0 & -a_{2,p+1} & 0 & -a_{2,p+2} & 0 & \dots & -a_{2,p+q} & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ -a_{p1} & 0 & -a_{p2} & 0 & \dots & 2 & -1 & -a_{p,p+1} & 0 & -a_{p,p+2} & 0 & \dots & -a_{p,p+q} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] & \left[\begin{array}{cccccccc} -2a_{p+1,1} \cdot U_{p+1} & 0 & -2a_{p+1,2} \cdot U_{p+1} & 0 & \dots & -2a_{p+1,p} \cdot U_{p+1} & 0 & 2 & V_{p+1} & -2a_{p+1,p+2} \cdot U_{p+1} & 0 & \dots & -2a_{p+1,p+q} \cdot U_{p+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{p+1} & 2 & 0 & 0 & 0 & 0 \\ -2a_{p+2,1} \cdot U_{p+2} & 0 & -2a_{p+2,2} \cdot U_{p+2} & 0 & \dots & -2a_{p+2,p} \cdot U_{p+2} & 0 & -2a_{p+2,p+1} \cdot U_{p+2} & 0 & 2 & V_{p+2} & \dots & -2a_{p+2,p+q} \cdot U_{p+2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{p+2} & 2 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ -2a_{p+q,1} \cdot U_{p+q} & 0 & -2a_{p+q,2} \cdot U_{p+q} & 0 & \dots & -2a_{p+q,p} \cdot U_{p+q} & 0 & -2a_{p+q,p+1} \cdot U_{p+q} & 0 & -2a_{p+q,p+2} \cdot U_{p+q} & 0 & \dots & 2 & V_{p+q} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{p+q} & 2 \end{array} \right] \end{array}$$

Figure 2