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Solutions to a Nearly Simple Harmonic Vibration Equation by Means of N- Fractional Calculus

デカルト出版・西本勝之 (Katsuyuki Nishimoto)
Research Institute for Applied Mathematics
Descartes Press Co.

Abstract

In this paper, solutions to a nearly simple harmonic vibration equation are discussed by means of N- fractional calculus, and some investigation of the solutions are reported.

Keywords : N- Fractional Calculus, Simple Harmonic Vibration Equation.

Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)
Let $D=\{D_-, D_+\}$, $C=\{C_-, C_+\}$,
C_- be a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$,
C_+ be a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$,
D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .
(Here D contains the points over the curve C).
Moreover, let $f=f(z)$ be a regular function in $D(z\in D)$,

$$f_\nu=(f)_\nu={}_C(f)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \qquad (\nu\notin Z^-), \tag{1}$$

$$(f)_{-m}=\lim_{\nu\to -m}(f)_\nu \qquad (m\in Z^+), \tag{2}$$

where $\quad -\pi\le\arg(\zeta-z)\le\pi$ for C_- , $\quad 0\le\arg(\zeta-z)\le 2\pi$ for C_+ ,

$\zeta\ne z$, $z\in C$, $\nu\in R$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$), with respect to z , of the function f , if $|(f)_\nu|<\infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^ν be*

$$N^\nu=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{d\zeta}{(\zeta-z)^{\nu+1}}\right) \qquad (\nu\notin Z^-) \quad [\textit{Refer to } (1)\,], \tag{3}$$

with

$$N^{-m}=\lim_{\nu\to -m}N^\nu \qquad (m\in Z^+), \tag{4}$$

and define the binary operation $\circ$ as

$$N^\beta\circ N^\alpha f=N^\beta N^\alpha f=N^\beta(N^\alpha f) \qquad (\alpha,\beta\in R), \tag{5}$$

then the set

$$\left\{N^\nu\right\}=\left\{N^\nu \mid \nu\in R\right\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1}=N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that

$f\in F=\left\{f;\ 0\ne|f_\nu|<\infty, \nu\in R\right\}$, *where $f=f(z)$ and $z\in C$.* (*vis.* $-\infty<\nu<\infty$).

(For our convenience, we call $N^{\beta} \circ N^{\alpha}$ as product of N^{β} and N^{α} .)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an* " *Action product group which has continuous index* ν " *for the set of* F . (*F.O.G.* ; Fractional calculus operator group) [3].

§ 1. General solution to nearly simple harmonic vibration equations

Theorem 1. *Let* $\varphi \in \wp^{\circ} = \{\varphi : 0 \neq |\varphi_{\nu}| < \infty, \nu \in R\}$, *then the homogeneous fractional order differintegral equation (nearly simple harmonic vibration equation for* $|\varepsilon| << 1$)

$$\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0 \qquad \begin{pmatrix} \omega \neq 0, & \varphi = \varphi(t), \\ t, \varepsilon \in R. & \varepsilon \neq -2 \end{pmatrix} \tag{0}$$

has the following general solutions .

(i)
$$\varphi = \sum_{n=0}^{m} a_n e^{A(n,\varepsilon)\omega^{2/(2+\varepsilon)} t} \times \left[\cos B(n,\varepsilon)\,\omega^{2/(2+\varepsilon)} t + i \sin B(n,\varepsilon)\,\omega^{2/(2+\varepsilon)} t\right] , \tag{1}$$

where

$A(n, \varepsilon) = \cos \beta(n,\varepsilon)$ (2), $B(n, \varepsilon) = \sin \beta(n,\varepsilon)$ (3)

and

$$\beta(n,\varepsilon) = \pi(1+2n) / (2+\varepsilon) \tag{4}$$

for $\varepsilon \in R$.

(ii)
$$\varphi = \sum_{n=0}^{m} a_n e^{G(n,\varepsilon)\omega^{r(\varepsilon)} t} \times \left[\cos H(n,\varepsilon)\,\omega^{r(\varepsilon)} t + i \sin H(n,\varepsilon)\,\omega^{r(\varepsilon)} t\right] , \tag{5}$$

where

$G(n, \varepsilon) = \cos \pi\left(\frac{1}{2}+n\right) r(\varepsilon)$ (6), $H(n, \varepsilon) = \sin \pi\left(\frac{1}{2}+n\right) r(\varepsilon)$ (7)

and

$$r(\varepsilon) = \sum_{k=0}^{\infty} (-\varepsilon/2)^k \tag{8}$$

for $|\varepsilon| < 2$.

(iii)
$$\varphi \approx \sum_{n=0}^{m} a_n e^{P(n,\varepsilon)\omega^{1-(\varepsilon/2)} t} \times \left[\cos Q(n,\varepsilon)\,\omega^{1-(\varepsilon/2)} t + i \sin Q(n,\varepsilon)\,\omega^{1-(\varepsilon/2)} t\right], \tag{9}$$

where

$$P(n, \varepsilon) = \cos \pi\left(\tfrac{1}{2}+n\right)\left(1-\tfrac{\varepsilon}{2}\right) , \tag{10}$$

$$Q(n,\varepsilon) = \sin \pi\left(\tfrac{1}{2}+n\right)\left(1-\tfrac{\varepsilon}{2}\right) , \tag{11}$$

for $|\varepsilon| << 1$.

Where a_n is an arbitrary constant correspond to $\beta(n,\varepsilon)$ and

m is finite when ε is a rational number, and

m is infinite when ε is an irrational number.

Note 1. We must call (9) as approximate (or almost) general solution to equation (0), because it is not general solution in the strict sense.

Proof of (i)

Set
$$\varphi = e^{\lambda t} , \tag{12}$$
then operate $N^{2+\varepsilon}$ to the both sides of (12), we have then
$$N^{2+\varepsilon}\varphi = \varphi_{2+\varepsilon} = \lambda^{2+\varepsilon} e^{\lambda t} . \qquad [1] \tag{13}$$

Therefore, we have
$$\lambda^{2+\varepsilon} + \omega^2 = 0 , \tag{14}$$
from (13), (12) and (0).

Hence
$$\lambda = (-\omega^2)^{1/(2+\varepsilon)} = e^{i\pi(1+2n)/(2+\varepsilon)}\omega^{2/(2+\varepsilon)} \tag{15}$$
$$\equiv \gamma(n,\varepsilon) \qquad (n = 0,1,2,\cdots,m) . \tag{16}$$

Then letting
$$\beta(n,\varepsilon) = \pi(1+2n) / (2+\varepsilon) \tag{4}$$
we have
$$\gamma(n,\varepsilon) = e^{i\beta(n,\varepsilon)}\omega^{2/(2+\varepsilon)} \tag{17}$$
$$= \{A(n,\varepsilon) + iB(n,\varepsilon)\}\omega^{2/(2+\varepsilon)} \tag{18}$$
where $A(n,\varepsilon)$ and $B(n,\varepsilon)$ are the ones shown by (2) and (3), respectively.

We have then a particular solution
$$\varphi = e^{\gamma(n,\varepsilon)t} \tag{19}$$
$$= e^{\{A(n,\varepsilon) + iB(n,\varepsilon)\}\omega^{2/(2+\varepsilon)}t} \equiv \varphi|_{(n)} \qquad \text{(denote)} \tag{20}$$
to equation (0).

Inversely (20) satisfies equation (0) clearly. Therefore, we have (1) from
$$\varphi = \sum_{n=0}^{m} a_n \cdot \varphi|_{(n)} \tag{21}$$
as the general solution to equation (0), where a_n is an arbitrary constant correspond to $\beta(n,\varepsilon)$.

Proof of (i i)

For $|\varepsilon| < 2$, we have
$$\frac{1}{2+\varepsilon} = \frac{1}{2} r(\varepsilon) , \tag{22}$$
where
$$r(\varepsilon) = \sum_{k=0}^{\infty}(-\varepsilon/2)^k . \tag{8}$$

We have then

$$\beta(n,\varepsilon) = \pi(\tfrac{1}{2}+n)\, r(\varepsilon) \qquad (23)$$

from (22) and (4).

Therefore, we have (5) from (23) and (1).

Proof of (i i i)

For $|\varepsilon| << 1$, we have

$$r(\varepsilon) = \sum_{k=0}^{\infty}(-\varepsilon/2)^k \approx 1-\tfrac{\varepsilon}{2}\,, \qquad (24)$$

then

$$\beta(n,\varepsilon) = \pi(\tfrac{1}{2}+n).(1-\tfrac{\varepsilon}{2})\ . \qquad (25)$$

Therefore, we have (9) from (25) and (5).

§ 2. Investigation for $\varphi|_{(n)}$

Here we investigate the solutions $\varphi|_{(n)}$ of the case (i i i) in § 1.

Theorem 2. *When* $\omega > 0$,

$$\varphi|_{(r)} \approx e^{(-1)^r\{(2r+1)\varepsilon\pi/4\}\omega t}\left[\cos Q t + (-1)^r i \sin Q t\right], \qquad (26)$$

is convergent for

$$\begin{pmatrix} 0 < -\varepsilon << 1 & \textit{for}\ \ r = 2k \\ 0 < \varepsilon << 1 & \textit{for}\ \ r = 2k+1 \end{pmatrix}, \qquad (27)$$

where

$$Q = \omega(1-\tfrac{\varepsilon}{2}\log\omega)\,, \qquad (28)$$

$$\varphi|_{(n)} \approx e^{P(n,\varepsilon)\omega^{1-(\varepsilon/2)}t}\left[\cos Q(n,\varepsilon)\omega^{1-(\varepsilon/2)}t + i\sin Q(n,\varepsilon)\omega^{1-(\varepsilon/2)}t\right], \qquad (29)$$

and $k = 0,1,2,\cdots$.

Proof. (I) Investigation for $\varphi|_{(0)}$

When $|\varepsilon| << 1$, we have (29) from § 1. (20) having § 1. (10) and (11), since

$$\varphi|_{(n)} \approx e^{\{P(n,\varepsilon)+i\,Q(n,\varepsilon)\}\omega^{1-(\varepsilon/2)}t}\ .$$

In the case of $n = 0$, we have

$$P(0,\varepsilon)\omega^{1-(\varepsilon/2)} \approx (\varepsilon\pi/4)\omega\,, \qquad (30)$$

and

$$Q(0,\varepsilon)\omega^{1-(\varepsilon/2)} \approx \omega(1-\tfrac{\varepsilon}{2}\log\omega) = Q\,, \qquad (31)$$

from (10) and (11) respectively, because we have

$$\cos\frac{\pi}{2}\left(1-\frac{\varepsilon}{2}\right)=\sin\frac{\varepsilon\pi}{4}=\sum_{k=0}^{\infty}(-1)^k\frac{(\varepsilon\pi/4)^{2k+1}}{(2k+1)!}\qquad(|\varepsilon\pi/4|<\infty), \tag{32}$$

$$\sin\frac{\pi}{2}\left(1-\frac{\varepsilon}{2}\right)=\cos\frac{\varepsilon\pi}{4}=\sum_{k=0}^{\infty}(-1)^k\frac{(\varepsilon\pi/4)^{2k}}{(2k)!}\qquad(|\varepsilon\pi/4|<\infty), \tag{33}$$

and

$$\omega^{1-(\varepsilon/2)}=\omega\cdot e^{\varepsilon\log\omega^{-1/2}}=\omega\sum_{k=0}^{\infty}\frac{(\varepsilon\log\omega^{-1/2})^k}{k!}\qquad(|\varepsilon\log\omega^{-1/2}|<\infty). \tag{34}$$

Therefore, we have

$$\varphi|_{(0)}\approx e^{(\varepsilon\pi/4)\omega t}\left[\cos Qt+i\sin Qt\right],\quad(|\varepsilon|<<1). \tag{35}$$

The solution (35) is divergent for $\varepsilon>0$ and is convergent (damping form) for $\varepsilon<0$ when $\omega>0$. And we have

$$Q=\omega(1-\tfrac{\varepsilon}{2}\log\omega)>\omega\qquad(\varepsilon<0,\omega>1). \tag{36}$$

Then, letting

$T_Q=2\pi/Q$; period of the function $\cos Qt$,

and

$T_\omega=2\pi/\omega$; period of the function $\cos\omega t$,

we have

$$T_Q<T_\omega\quad(Q>\omega)\quad\text{when}\quad\varepsilon<0.$$

That is, we have that " the period T_Q of $\cos Qt=\cos\omega(1-\frac{\varepsilon}{2}\log\omega)t$ is smaller than the one T_ω of $\cos\omega t$ " when $\varepsilon<0$, $\omega>1$.

(II) Investigation for $\varphi|_{(1)}$

In the case of $n=1$, we have

$$P(1,\varepsilon)\approx-(\varepsilon\pi/4)\omega, \tag{37}$$

and

$$Q(1,\varepsilon)\approx-\omega(1-\tfrac{\varepsilon}{2}\log\omega)=-Q, \tag{38}$$

from (10) and (11) respectively, because we have

$$\cos\frac{3\pi}{2}\left(1-\frac{\varepsilon}{2}\right)=-\sin\frac{\varepsilon3\pi}{4}=-\sum_{k=0}^{\infty}(-1)^k\frac{(\varepsilon3\pi/4)^{2k+1}}{(2k+1)!}\qquad(|\varepsilon3\pi/4|<\infty), \tag{39}$$

$$\sin\frac{3\pi}{2}\left(1-\frac{\varepsilon}{2}\right)=-\cos\frac{\varepsilon3\pi}{4}=-\sum_{k=0}^{\infty}(-1)^k\frac{(\varepsilon3\pi/4)^{2k}}{(2k)!}\qquad(|\varepsilon3\pi/4|<\infty), \tag{40}$$

and (34).

Therefore, we have

$$\varphi|_{(1)}\approx e^{-(\varepsilon3\pi/4)\omega t}\left[\cos Qt-i\sin Qt\right],\quad(|\varepsilon|<<1). \tag{41}$$

The solution (41) is convergent (damping form) for $\varepsilon>0$ and is divergent for $\varepsilon<0$ when $\omega>0$. And we have

$$Q=\omega(1-\tfrac{\varepsilon}{2}\log\omega)<\omega\qquad(\varepsilon>0,\omega>1). \tag{42}$$

Then, in this case we have

$$T_Q>T_\omega\quad(Q<\omega)\quad\text{when}\quad\varepsilon>0,\ \omega>1.$$

That is, we have that " the period T_Q of $\cos Qt = \cos\omega(1-\frac{\varepsilon}{2}\log\omega)t$ is larger than the one T_ω of $\cos\omega t$ " when $\varepsilon>0,\ \omega>1$.

(III) Investigation for $\varphi|_{(2)}$

In the case of $n=2$, we have

$$P(2,\varepsilon)\approx(\varepsilon 5\pi/4)\omega, \tag{43}$$

and

$$Q(2,\varepsilon)\approx\omega(1-\tfrac{\varepsilon}{2}\log\omega)=Q, \tag{44}$$

from (10) and (11) respectively, because we have

$$\cos\frac{5\pi}{2}\left(1-\frac{\varepsilon}{2}\right)=\sin\frac{\varepsilon 5\pi}{4}=\sum_{k=0}^{\infty}(-1)^k\frac{(\varepsilon 5\pi/4)^{2k+1}}{(2k+1)!}\quad(|\varepsilon 5\pi/4|<\infty), \tag{45}$$

$$\sin\frac{5\pi}{2}\left(1-\frac{\varepsilon}{2}\right)=\cos\frac{\varepsilon 5\pi}{4}=\sum_{k=0}^{\infty}(-1)^k\frac{(\varepsilon 5\pi/4)^{2k}}{(2k)!}\quad(|\varepsilon 5\pi/4|<\infty), \tag{46}$$

and (34).

Therefore, we have

$$\varphi|_{(2)}\approx e^{(\varepsilon 5\pi/4)\omega t}\left[\cos Qt+i\sin Qt\right],\quad(|\varepsilon|<<1). \tag{47}$$

The solution (47) is divergent for $\varepsilon>0$ and is convergent (damping form) for $\varepsilon<0,\ \omega>0$. And we have

$$Q=\omega(1-\tfrac{\varepsilon}{2}\log\omega)>\omega\quad(\varepsilon<0,\omega>1). \tag{48}$$

Therefore we have

$$T_Q<T_\omega\quad(Q>\omega)\quad\text{when}\quad\varepsilon<0,\ \omega>1.$$

That is, we have that " the period T_Q of $\cos Qt=\cos\omega(1-\frac{\varepsilon}{2}\log\omega)t$ is smaller than the one T_ω of $\cos\omega t$ " when $\varepsilon<0,\ \omega>1$.

(IV) Repeating the same procedure as (I) ~ (III), we have this theorem clearly.

Note. Notice that when $\omega>0$,the solutions

$\varphi|_{(2k)}$ $(k=0,1,2,\cdots)$ are convergent (damping form) for $\varepsilon<0$, and

$\varphi|_{(2k+1)}$ are convergent (damping form) for $\varepsilon>0$, respectively.

Theorem 3. *When $\omega>0$ the nearly simple harmonic vibration equation*

$$\varphi_{2+\varepsilon}+\varphi\cdot\omega^2=0\quad\begin{pmatrix}\omega\neq0, & \varphi=\varphi(t),\\ t,\varepsilon\in\boldsymbol{R}. & |\varepsilon|<<1\end{pmatrix} \tag{49}$$

has converging almost general solutions

$$\varphi\approx\sum_{k=0}^{p}a_{2k}\varphi|_{(2k)}\quad\text{when}\quad\varepsilon<0 \tag{50}$$

and

$$\varphi\approx\sum_{k=0}^{p}a_{2k+1}\varphi|_{(2k+1)}\quad\text{when}\quad\varepsilon>0, \tag{51}$$

where

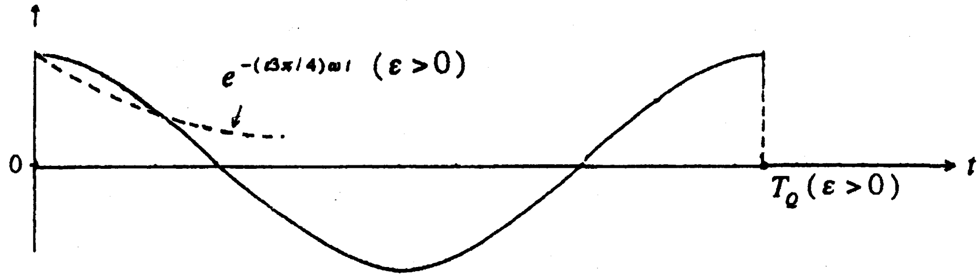
p is finite when ε is a rational number, and

p is infinite when ε is an irrational number.

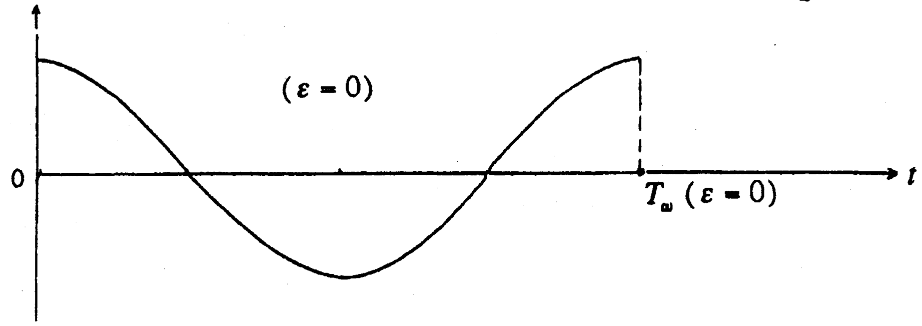
Proof. It is clear from the proof of Theoem 2, since we have (9) as solutions to equation (49)

§ 3. Some Graphs for $\varphi|_{(n)}$

To equation $\varphi_{2+\varepsilon} + \varphi\cdot\omega^2 = 0 \quad (0<\varepsilon<<1)$ we have a convergent (damping form) solution
$\varphi|_{(1)} \approx e^{-(\varepsilon 3\pi/4)\omega t}\left[\cos Qt - i\sin Qt\right], \quad (\omega>0).$
To this function we have $T_Q\,(>T_\omega)$ as its period for $\varepsilon>0$, since $Q=\omega(1-\frac{\varepsilon}{2}\log\omega)<\omega \quad (\varepsilon>0,\ \omega>1)$.



To equation $\varphi_2 + \varphi\cdot\omega^2 = 0 \quad (\varepsilon=0)$ we have the solution $\varphi = \cos\omega t + i\sin\omega t$. The period of which is $T_\omega = 2\pi/\omega$.



To equation $\varphi_{2+\varepsilon} + \varphi\cdot\omega^2 = 0 \quad (0<-\varepsilon<<1)$ we have a convergent (damping form) solution
$\varphi|_{(0)} \approx e^{(\varepsilon\pi/4)\omega t}\left[\cos Qt + i\sin Qt\right], \quad (\omega>0).$
To this function we have $T_Q\,(<T_\omega)$ as its period for $\varepsilon<0$, since $Q=\omega(1-\frac{\varepsilon}{2}\log\omega)>\omega \quad (\varepsilon<0,\ \omega>1)$

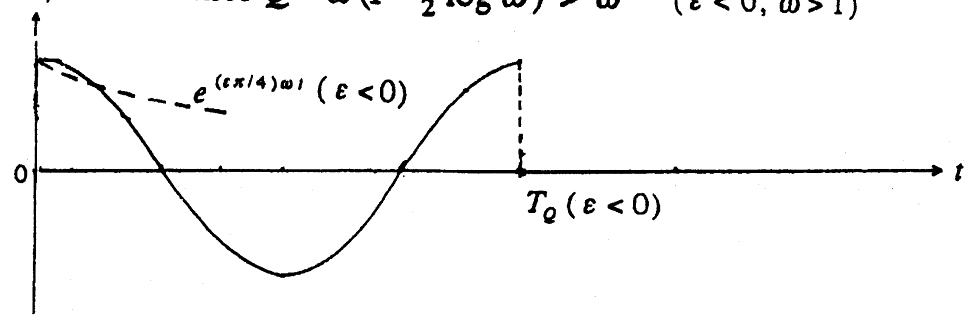


Fig. 1. Case of $\omega>1$.

In **Fig.1.**,the graphs of $\mathrm{Re}\,\varphi|_{(0)}$ and $\mathrm{Re}\,\varphi|_{(1)}$, for the case $\omega>1$,are shown in which the portion of amplitude and the one of vibration are separated . When $0<\omega<1$,we have $T_Q(\text{of } \mathrm{Re}\,\varphi|_{(1)}) < T_\omega$ $(\varepsilon>0)$ and $T_Q(\text{of } \mathrm{Re}\,\varphi|_{(0)}) > T_\omega$ $(\varepsilon<0)$. When $\omega=1$, we have $T_Q(\text{of } \mathrm{Re}\,\varphi|_{(1)}) = T_\omega$ $(\varepsilon>0)$ and $T_Q(\text{of } \mathrm{Re}\,\varphi|_{(0)}) = T_\omega$ $(\varepsilon<0)$ since we have $Q=\omega\,(\varepsilon>0, \varepsilon<0)$. (See **Fig.2** and **Fig.3**, respectively.)

(i) For example the nearly simple harmonic vibration equation

$$\varphi_{2+0.01}+\varphi\cdot\omega^2=\frac{d^{2.01}\varphi}{dt^{2.01}}+\varphi\cdot\omega^2=0 \quad (\varepsilon=0.01>0) \tag{52}$$

has solution

$$\varphi|_{(1)}\approx e^{-(0.01\times3\pi/4)\omega t}\left[\cos Qt-i\sin Qt\right] \quad (\omega>0), \tag{53}$$

whose amplitude is $e^{-(0.03\pi/4)\omega t}$ and the period is

$$T_Q=\frac{2\pi}{Q}=\frac{2\pi}{\omega(1-0.005\cdot\log\omega)}>\frac{2\pi}{\omega}=T_\omega \quad (\omega>1). \tag{54}$$

(ii) The equation

$$\varphi_{2-0.01}+\varphi\cdot\omega^2=\frac{d^{1.99}\varphi}{dt^{1.99}}+\varphi\cdot\omega^2=0 \quad (\varepsilon=-0.01<0) \tag{55}$$

has solution

$$\varphi|_{(0)}\approx e^{(-0.01\pi/4)\omega t}\left[\cos Qt+i\sin Qt\right], \quad (\omega>0), \tag{56}$$

whose amplitude is $e^{(-0.01\pi/4)\omega t}$ and the period is

$$T_Q=\frac{2\pi}{Q}=\frac{2\pi}{\omega(1+0.005\cdot\log\omega)}<\frac{2\pi}{\omega}=T_\omega \quad (\omega>1). \tag{57}$$

(iii) The equation

$$\varphi_2+\varphi\cdot\omega^2=0 \tag{58}$$

has solution

$$\varphi=\cos\omega t+i\sin\omega t\ . \tag{59}$$

This solution is produced in the process in which ε changes its sign in the equation

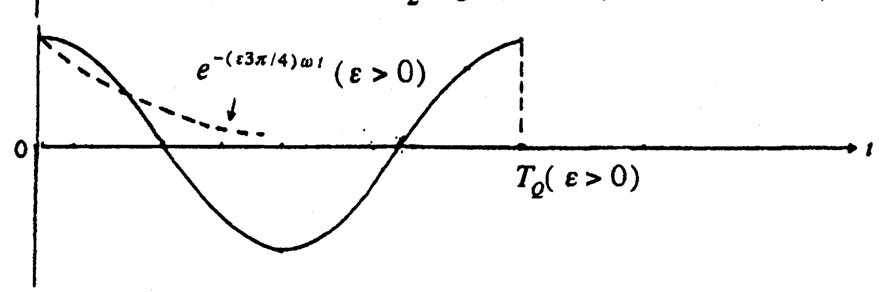
$$\varphi_{2+\varepsilon}+\varphi\cdot\omega^2=0 \tag{49}$$

Notice that; When $\omega>0$,

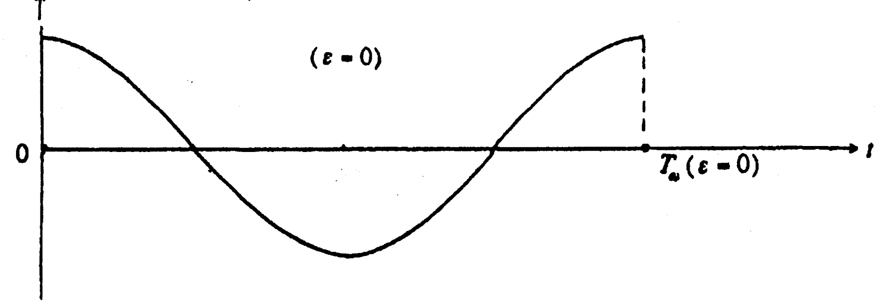
$\mathrm{Re}\,\varphi|_{(n)}$, having $n=$ even number, give the same form damping vibration curves as the one of $\mathrm{Re}\,\varphi|_{(0)}$ for $\varepsilon<0$, and

$\mathrm{Re}\,\varphi|_{(n)}$, having $n=$ odd number, give the same form damping vibration curves as the one of $\mathrm{Re}\,\varphi|_{(1)}$ for $\varepsilon>0$.

To equation $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$ $(0 < \varepsilon << 1)$ we have a convergent (damping form) solution
$\varphi|_{(1)} \approx e^{-(\varepsilon 3\pi/4)\omega t}[\cos Qt - i\sin Qt]$, $(\omega > 0)$.
To this function we have $T_Q(< T_\omega)$ as its period for $\varepsilon > 0$, since $Q = \omega(1 - \frac{\varepsilon}{2}\log\omega) > \omega$ $(\varepsilon > 0,\ 0 < \omega < 1)$.



To equation $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$ $(\varepsilon = 0)$ we have a solution $\varphi = \cos\omega t + i\sin\omega t$. The period of which is $T_\omega = 2\pi/\omega$.



To equation $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$ $(0 < -\varepsilon << 1)$ we have a convergent (damping form) solution
$\varphi|_{(0)} \approx e^{(\varepsilon\pi/4)\omega t}[\cos Qt + i\sin Qt]$, $(\omega > 0)$.
To this function we have $T_Q(> T_\omega)$ as its period for $\varepsilon < 0$, since $Q = \omega(1 - \frac{\varepsilon}{2}\log\omega) < \omega$ $(\varepsilon < 0,\ 0 < \omega < 1)$.

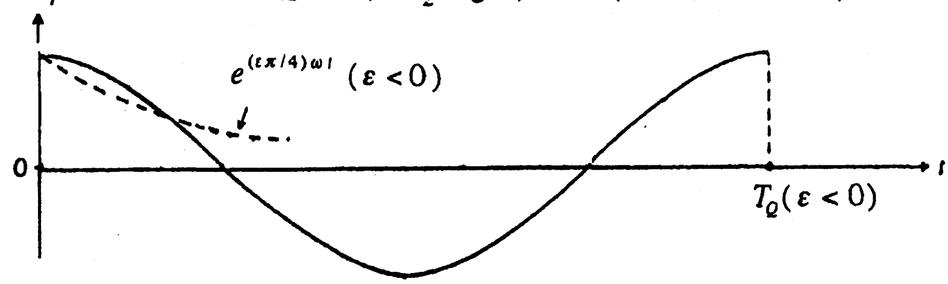
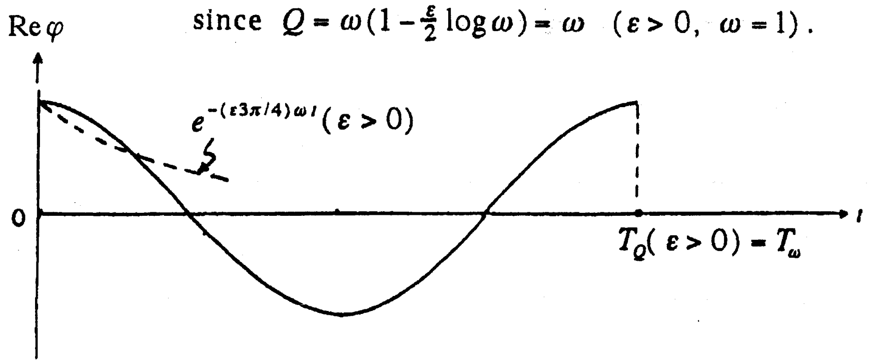
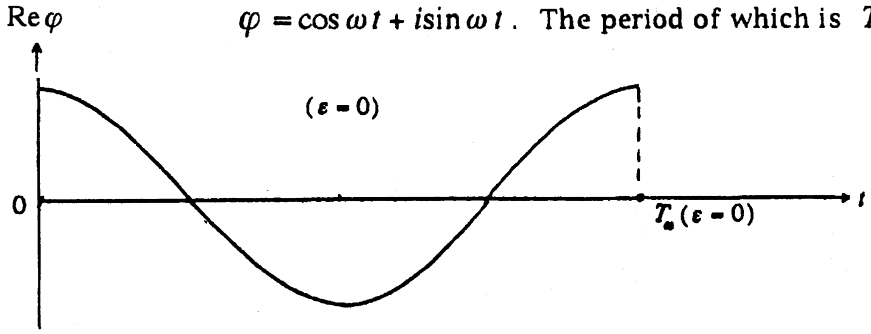


Fig. 2. Case of $0 < \omega < 1$.

To equation $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$ $(0 < \varepsilon << 1)$ we have a convergent (damping form) solution
$\varphi|_{(1)} \approx e^{-(\varepsilon 3\pi/4)\omega t}[\cos Qt - i\sin Qt]$, $(\omega > 0)$.
To this function we have $T_Q = T_\omega$ as its period for $\varepsilon > 0$, since $Q = \omega(1 - \frac{\varepsilon}{2}\log\omega) = \omega$ $(\varepsilon > 0,\ \omega = 1)$.



To equation $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$ $(\varepsilon = 0)$ we have a solution $\varphi = \cos\omega t + i\sin\omega t$. The period of which is $T_\omega = 2\pi/\omega$.



To equation $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$ $(0 < -\varepsilon << 1)$ we have a convergent (damping form) solution
$\varphi|_{(0)} \approx e^{(\varepsilon\pi/4)\omega t}[\cos Qt + i\sin Qt]$, $(\omega > 0)$.
To this function we have $T_Q = T_\omega$ as its period for $\varepsilon < 0$, since $Q = \omega(1 - \frac{\varepsilon}{2}\log\omega) = \omega$ $(\varepsilon < 0,\ \omega = 1)$.

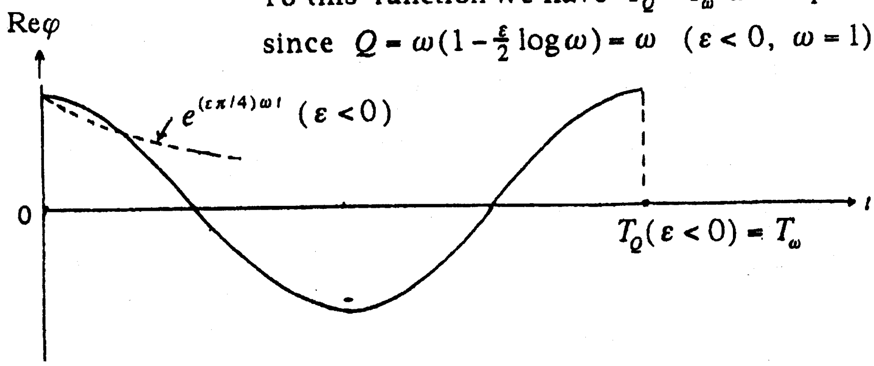


Fig. 3. Case of $\omega = 1$.

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Katsuyuki Nishimoto
Institute of Applied Mathematics
Descartes Press Co.
2 - 13 - 10 Kaguike, Koriyama
963 - 8833 Japan