

$1 < |t| < 2$ に対する Lawson-Lim-Pálfi による 作用素冪平均について

Operator Power means due to Lawson-Lim-Pálfi for $1 < |t| < 2$

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1. INTRODUCTION

This paper is based on [6].

For a weight vector $\omega = (\omega_1, \dots, \omega_n)$ such as $\omega_i \geq 0$ for $i = 1, \dots, n$ and $\sum_{i=1}^n \omega_i = 1$ and positive invertible operators $\mathbb{A} = (A_1, \dots, A_n)$, the power mean $P_t(\omega; \mathbb{A})$ for $t \in [-1, 1] \setminus \{0\}$ due to Lawson-Lim-Pálfi [4, 5] is defined by the unique positive invertible solution of the following non-linear equation:

$$\begin{aligned} X &= \sum_{i=1}^n \omega_i (X \sharp_t A_i) \quad \text{for } t \in (0, 1] \\ X &= \left[\sum_{i=1}^n \omega_i (X^{-1} \sharp_{-t} A_i^{-1}) \right]^{-1} \quad \text{for } t \in [-1, 0) \end{aligned}$$

where $A \sharp_t B = A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2}$ is the t -weighted geometric mean of A and B .

For a weight vector $\omega = (\omega_1, \dots, \omega_n)$ and positive invertible operators $\mathbb{A} = (A_1, \dots, A_n)$, the Karcher mean $G_K(\omega; \mathbb{A})$ of $A_1, \dots, A_n$ is defined by the unique positive invertible solution of the Karcher equation:

$$\sum_{i=1}^n \omega_i \log(X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}}) = 0.$$

The power mean $P_t(\omega; \mathbb{A})$ is monotone increasing for t :

$$P_t(\omega; \mathbb{A}) \leq P_s(\omega; \mathbb{A}) \quad \text{for } -1 \leq t \leq s \leq 1$$

and the Karcher mean is realized as the strong limit of the power means:

$$\text{s-}\lim_{t \rightarrow 0} P_t(\omega; \mathbb{A}) = G_K(\omega; \mathbb{A})$$

under the strong-operator topology.

Problem: The range in which the power means $P_t(\omega; \mathbb{A})$ are defined, is $[-1, 1] \setminus \{0\}$. However, the range in which the power arithmetic means $(\sum_{i=1}^n \omega_i A_i^t)^{1/t}$ are defined, is the set of all real numbers $\mathbb{R}$. It is then natural to ask the following question: Is it possible to extend the range in which the power means are defined?

The purpose of this paper is to extend the range of the definition of power means $P_t(\omega; \mathbb{A})$ defined by Lawson-Lim-Pálfi [4, 5].

2. PRELIMINARY

Let $B(H)$ be the C^* -algebra of all bounded linear operators on a Hilbert space H equipped with the operator norm, $S(H)$ the set of all bounded self-adjoint operators, and $\mathbb{P} = \mathbb{P}(H)$ be the open convex cone of all positive invertible operators. For $X, Y \in S(H)$, we write $X \leq Y$ if $Y - X$ is positive, and $X < Y$ if $Y - X$ is positive invertible.

For $A, B \in \mathbb{P}$ and $t \in [0, 1]$, the t -geometric operator mean is defined as

$$A \sharp_t B = A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2}.$$

For convenience, we use the notation $\natural_t$ for the binary operation

$$A \natural_t B = A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2} \quad \text{for } t \notin [0, 1],$$

whose formula is the same as $\sharp_t$. Though $A \sharp_t B$ for $t \in [0, 1]$ has the monotonicity, $A \natural_s B$ for $s \notin [0, 1]$ has not it.

Lemma 1. *Let $A, B, X, Y \in \mathbb{P}$ and $1 < t \leq 2$. Then*

(i) *If $X \leq Y$, then $Y \natural_t A \leq X \natural_t A$.*

(ii) *If $A \leq B$ with $m_1 \leq A \leq M_1$ and $m_2 \leq B \leq M_2$ and $m \leq X \leq M$, then*

$$X \natural_t A \leq K(m_i/M, M_i/m, t) X \natural_t B \quad \text{for } i = 1, 2,$$

where the generalized Kantorovich constant $K(m, M, t)$ is defined by

$$(1) \quad K(m, M, t) = \frac{mM^t - Mm^t}{(t-1)(M-m)} \left(\frac{t-1}{t} \frac{M^t - m^t}{mM^t - Mm^t} \right)^t$$

for any real number $t \in \mathbb{R}$, see [3, Theorem 2.53].

(iii) *If $m \leq A \leq M$, then*

$$\|X\|^{1-t} m^t \leq X \natural_t A \leq \|X^{-1}\|^{-(1-t)} M^t.$$

Proof. We only prove (i): For $1 < t \leq 2$

$$\begin{aligned} Y \natural_t A &= A \natural_{1-t} Y = A^{1/2} (A^{-1/2} Y A^{-1/2})^{1-t} A^{1/2} \\ &= A^{1/2} (A^{1/2} Y^{-1} A^{1/2})^{t-1} A^{1/2} \\ &\leq A^{1/2} (A^{1/2} X^{-1} A^{1/2})^{t-1} A^{1/2} \quad \text{by } 0 < t-1 < 1 \text{ and } Y^{-1} \leq X^{-1} \\ &= X \natural_t A. \end{aligned}$$

□

The Thompson metric on $\mathbb{P}$ is defined by

$$d(A, B) = \log \max\{M(A/B), M(B/A)\}$$

where

$$M(A/B) = \inf\{\lambda > 0 : A \leq \lambda B\} = \|B^{-1/2} A B^{-1/2}\| = r(B^{-1} A).$$

It is known that d is a complete metric on $\mathbb{P}$ and

$$d(A, B) = \|\log B^{-1/2} A B^{-1/2}\| = \|\log A^{-1/2} B A^{-1/2}\|,$$

see [7]. We list some basic properties of the Thompson metric:

Lemma 2 (see [1, 2]). *For $A, B, C, D \in \mathbb{P}$*

(i) $d(A, B) = d(A^{-1}, B^{-1}) = d(T^* A T, T^* B T)$ for invertible $T \in B(H)$;

- (ii) $d(A + B, C + D) \leq \max\{d(A, C), d(B, D)\}$;
- (iii) $d(A^t, B^t) \leq td(A, B)$ for $t \in [0, 1]$;
- (iv) $d(\alpha A, \alpha B) = d(A, B)$ for positive real number $\alpha > 0$;
- (v) $d(A \sharp_t B, C \sharp_t D) \leq (1 - t)d(A, C) + td(B, D)$ for $t \in [0, 1]$.

For $A, B \in \mathbb{P}$, a map $\gamma_{A,B} : \mathbb{R} \mapsto \mathbb{P}$ defined by $\gamma_{A,B}(t) = A \sharp_t B$ for $t \in \mathbb{R}$ is a path joining A and B . In particular, it is known that $\gamma_{A,B}(t)$ for $t \in [0, 1]$ is a path joining A to B in $\mathbb{P}$. Then we have the following:

Theorem 3. *Let $A, B \in \mathbb{P}$. Then*

$$d(A \sharp_s B, A \sharp_t B) = |s - t|d(A, B) \quad \text{for all } s, t \in \mathbb{R}.$$

Proof. By definition of the Thompson metric and Lemma 2

$$\begin{aligned} d(A \sharp_s B, A \sharp_t B) &= d((A^{-1/2}BA^{-1/2})^s, (A^{-1/2}BA^{-1/2})^t) \\ &= d((A^{-1/2}BA^{-1/2})^{s-t}, I) = \|\log(A^{-1/2}BA^{-1/2})^{s-t}\| \\ &= |s - t| \|\log A^{-1/2}BA^{-1/2}\| = |s - t|d(A, B). \end{aligned}$$

□

We have the following estimate in the case of $1 < t < 2$, which corresponds to (v) of Lemma 2:

Theorem 4. *Let $A, B, C, D \in \mathbb{P}$ such that $m_1A \leq C \leq M_1A$ and $m_2B \leq D \leq M_2B$ for some scalars $0 < m_1 \leq M_1$ and $0 < m_2 \leq M_2$. For each $1 < t < 2$*

$$d(A \sharp_t B, C \sharp_t D) \leq (t - 1)d(A, C) + td(B, D) + \log K(t)$$

where $K(t) = \max\{K(m_1, M_1, t), K(m_2, M_2, t)\}$ and the generalized Kantorovich constant $K(m, M, t)$ is defined by (1).

3. MAIN RESULT

In this section, we extend to the range in which the power means due to Lawson-Lim-Pálfi are defined. For this, we need the following Lemma:

Lemma 5. *Let $X, Y, A \in \mathbb{P}$ and $1 < t \leq 2$. Then*

$$d(X \sharp_t A, Y \sharp_t A) \leq (t - 1)d(X, Y).$$

Proof. For $1 < t \leq 2$,

$$\begin{aligned} d(X \sharp_t A, Y \sharp_t A) &= d(A \sharp_{1-t} X, A \sharp_{1-t} Y) \\ &= d((A^{1/2}X^{-1}A^{1/2})^{t-1}, (A^{1/2}Y^{-1}A^{1/2})^{t-1}) \quad \text{by (i) of Lemma 2} \\ &\leq (t - 1)d(A^{1/2}XA^{1/2}, A^{1/2}YA^{1/2}) \quad \text{by (iii) of Lemma 2} \\ &= (t - 1)d(X, Y) \quad \text{by (i) of Lemma 2.} \end{aligned}$$

□

Theorem 6. *Let $A_1, A_2, \dots, A_n \in \mathbb{P}$ and a weight vector $\omega = (\omega_1, \dots, \omega_n)$. Then for each $1 < t < 2$, the following equation has a unique positive invertible solution:*

$$X = \sum_{i=1}^n \omega_i (X \sharp_t A_i).$$

Proof. We will show that the map $f : \mathbb{P} \mapsto \mathbb{P}$ defined by $f(X) = \sum_{i=1}^n \omega_i(X \natural_t A_i)$ is a strict contraction with respect to the Thompson metric. Let $X, Y > 0$.

$$\begin{aligned} d(f(X), f(Y)) &\leq \max_{1 \leq i \leq n} \{d(\omega_i(X \natural_t A_i), \omega_i(Y \natural_t A_i))\} \quad \text{by (ii) of Lemma 2} \\ &= \max_{1 \leq i \leq n} \{d(X \natural_t A_i, Y \natural_t A_i)\} \quad \text{by (iv) of Lemma 2} \\ &\leq (t-1)d(X, Y) \quad \text{by Lemma 5.} \end{aligned}$$

Since $1 < t < 2$, it follows that f is a strict contraction and hence f has a unique fixed point. $\square$

Definition 7. Let $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{P}^n$ and a weight vector $\omega = (\omega_1, \dots, \omega_n)$. For $t \in (1, 2)$, we denote by $P_t(\omega; \mathbb{A})$ the unique positive invertible solution of

$$X = \sum_{i=1}^n \omega_i(X \natural_t A_i).$$

For $t \in (-2, -1)$, we define $P_t(\omega; \mathbb{A}) = P_{-t}(\omega; \mathbb{A}^{-1})^{-1}$, where $\mathbb{A}^{-1} = (A_1^{-1}, \dots, A_n^{-1})$. In fact, $X = P_t(\omega; \mathbb{A})$ is the unique positive invertible solution of $X = (\sum_{i=1}^n \omega_i(X \natural_{-t} A_i)^{-1})^{-1}$ and $X^{-1} = \sum_{i=1}^n \omega_i(X^{-1} \natural_{-t} A_i^{-1})$ if and only if $X^{-1} = P_{-t}(\omega; \mathbb{A}^{-1})$.

Remark 8. Let $t \in (1, 2)$. Put $f : \mathbb{P} \mapsto \mathbb{P}$ defined by $f(X) = \sum_{i=1}^n \omega_i(X \natural_t A_i)$. By Theorem 6, f is a strict contraction for the Thompson metric and by the Banach fixed point theorem

$$\lim_{k \rightarrow \infty} f^k(X) = P_t(\omega; \mathbb{A}) \quad \text{for any } X \in \mathbb{P}.$$

Similarly, the map $g(X) = (\sum_{i=1}^n \omega_i(X \natural_{-t} A_i)^{-1})^{-1}$ is a strict contraction for the Thompson metric and $\lim_{k \rightarrow \infty} g^k(X) = P_{-t}(\omega; \mathbb{A})$ for any $X \in \mathbb{P}$.

For $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{P}^n$, $M \in B(H)$, $\omega = (\omega_1, \dots, \omega_n)$ and for a permutation σ on n -letters, we set

$$\begin{aligned} M\mathbb{A}M^* &= (MA_1M^*, \dots, MA_nM^*), \quad A_\sigma = (A_{\sigma(1)}, \dots, A_{\sigma(n)}) \\ \hat{\omega} &= \frac{1}{1 - \omega_n}(\omega_1, \dots, \omega_{n-1}). \end{aligned}$$

We list some basic properties of $P_t(\omega; \mathbb{A})$ for $t \in (-2, 2) \setminus [-1, 1]$.

Proposition 9. Let $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{P}^n$, a weight vector $\omega = (\omega_1, \dots, \omega_n)$ and let $t \in (-2, 2) \setminus [-1, 1]$.

- (i) $P_t(\omega; \mathbb{A}) = (\sum_{i=1}^n \omega_i A_i^t)^{1/t}$ if the A_i 's commute;
- (ii) $P_t(\omega_\sigma; \mathbb{A}_\sigma) = P_t(\omega; \mathbb{A})$ for any permutation σ ;
- (iii) $P_t(\omega; M\mathbb{A}M^*) = MP_t(\omega; \mathbb{A})M^*$ for any invertible M ;
- (iv) $P_t(\omega; \mathbb{A}^{-1})^{-1} = P_{-t}(\omega; \mathbb{A})$;
- (v) $\sum_{i=1}^n \omega_i A_i \leq P_t(\omega; \mathbb{A})$ for $t \in (1, 2)$;
- (vi) $P_t(\omega; \mathbb{A}) \leq (\sum_{i=1}^n \omega_i A_i^{-1})^{-1}$ for $t \in (-2, -1)$;
- (vii) If $m \leq A_i \leq M$, then $m \leq P_t(\omega; \mathbb{A}) \leq m^{1-t} M^t$ for $t \in (1, 2)$ and $m^{-t} M^{1+t} \leq P_t(\omega; \mathbb{A}) \leq M$ for $t \in (-2, -1)$;
- (viii) For $t \in (1, 2)$, $P_t(\omega; A_1, \dots, A_{n-1}, X) = X$ if and only if $X = P_t(\hat{\omega}; A_1, \dots, A_{n-1})$.

Proof. Proofs from (i) to (iv) and (vii) are similar to those of [5].

(v): Put $X = P_t(\omega; \mathbb{A})$ for $t \in (1, 2)$. Since $(1-t)A + tB \leq A \natural_t B$ for $1 < t < 2$, we have

$$\begin{aligned} X &= \sum_{i=1}^n \omega_i(X \natural_t A_i) \geq \sum_{i=1}^n \omega_i((1-t)X + tA_i) \\ &= (1-t)X + t \sum_{i=1}^n \omega_i A_i \end{aligned}$$

and hence $X \geq \sum_{i=1}^n \omega_i A_i$.

(vi): Put $X = P_t(\omega; \mathbb{A})$ for $t \in (-2, -1)$. Since $X = (\sum_{i=1}^n \omega_i(X^{-1} \natural_{-t} A_i^{-1}))^{-1}$, it follows that

$$\begin{aligned} X^{-1} &= \sum_{i=1}^n \omega_i(X^{-1} \natural_{-t} A_i^{-1}) \geq \sum_{i=1}^n \omega_i((1+t)X^{-1} + (-t)A_i^{-1}) \\ &= (1+t)X^{-1} - t \sum_{i=1}^n \omega_i A_i^{-1} \end{aligned}$$

and hence $X \leq (\sum_{i=1}^n \omega_i A_i^{-1})^{-1}$ for $t \in (-2, -1)$. $\square$

Theorem 10. Let $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{P}^n$ such that $0 < m \leq A_i \leq M$ for some scalars $0 < m \leq M$ and a weight vector $\omega = (\omega_1, \dots, \omega_n)$. Let $1 < t \leq s < 2$. Then

$$\begin{aligned} &d(P_t(\omega; \mathbb{A}), P_s(\omega; \mathbb{A})) \\ &\leq \frac{s-t}{(2-s)(2-t)} [t\Delta(\mathbb{A}) + \log K(m/M, (M/m)^t, t)], \end{aligned}$$

where the generalized Kantorovich constant $K(m, M, t)$ is defined by (1) and $\Delta(\mathbb{A}) = \max_{1 \leq i, j \leq n} \{d(A_i, A_j)\}$ denotes the d -diameter of $\mathbb{A} = (A_1, \dots, A_n)$.

Proof. Put $X = P_t(\omega; \mathbb{A})$ and $Y = P_s(\omega; \mathbb{A})$, then by definition it follows that $X = \sum_{i=1}^n \omega_i(X \natural_t A_i)$ and $Y = \sum_{i=1}^n \omega_i(Y \natural_s A_i)$. Therefore

$$\begin{aligned} d(X, Y) &= d(Y, X) = d\left(\sum_{i=1}^n \omega_i(Y \natural_s A_i), \sum_{i=1}^n \omega_i(X \natural_t A_i)\right) \\ &\leq \max_{1 \leq i \leq n} \{d(Y \natural_s A_i, X \natural_t A_i)\} \\ &\leq \max_{1 \leq i \leq n} \{d(Y \natural_s A_i, X \natural_s A_i) + d(X \natural_s A_i, X \natural_t A_i)\} \\ &\leq \max_{1 \leq i \leq n} \{(s-1)d(Y, X) + (s-t)d(X, A_i)\} \\ &\leq (s-1)d(X, Y) + (s-t) \left[\frac{t}{2-t} \Delta(\mathbb{A}) + \frac{1}{2-t} \log K(m/M, (M/m)^t, t) \right] \end{aligned}$$

and hence we have

$$d(X, Y) \leq \frac{s-t}{2-s} \left[\frac{t}{2-t} \Delta(\mathbb{A}) + \frac{1}{2-t} \log K(m/M, (M/m)^t, t) \right].$$

$\square$

Theorem 11. Let $\mathbb{A} = (A_1, \dots, A_n)$ and $\mathbb{B} = (B_1, \dots, B_n)$ such that $0 < m_1 \leq A_i \leq M_1$ and $0 < m_2 \leq B_i \leq M_2$ for $i = 1, \dots, n$ for some scalars $0 < m_1 \leq M_1$ and $0 < m_2 \leq M_2$. Then for each $1 < t < 2$

$$d(P_t(\omega; \mathbb{A}), P_t(\omega; \mathbb{B})) \leq \frac{t}{2-t} \max_{1 \leq i \leq n} \{d(A_i, B_i)\} + \frac{1}{2-t} \log K_1(t),$$

where

$$K_1(t) = \max\{K(m_2/m_1^{1-t}M_1^t, m_2^{1-t}M_2^t/m_1, t), K(m_2/M_1, M_2/m_1, t)\}.$$

Proof. Put $X = P_t(\omega; \mathbb{A})$ and $Y = P_t(\omega; \mathbb{B})$. Then it follows that

$$\begin{aligned} d(X, Y) &= d\left(\sum_{i=1}^n \omega_i(X \natural_t A_i), \sum_{i=1}^n \omega_i(Y \natural_t B_i)\right) \\ &\leq \max_{1 \leq i \leq n} \{d(X \natural_t A_i, Y \natural_t B_i)\} \\ &\leq \max_{1 \leq i \leq n} \{(t-1)d(X, Y) + td(A_i, B_i) + \log K_1(t)\} \\ &= (t-1)d(X, Y) + t \max_{1 \leq i \leq n} \{d(A_i, B_i)\} + \log K_1(t) \end{aligned}$$

and hence we have

$$d(X, Y) \leq \frac{t}{2-t} \max_{1 \leq i \leq n} \{d(A_i, B_i)\} + \frac{1}{2-t} \log K_1(t).$$

□

Conclusion and problems: We were able to extend the range of the power means $P_t(\omega; \mathbb{A})$ to the $1 < |t| < 2$. Unfortunately, we do not know whether the power means are defined for $t \geq 2$ or not. For example, we put $t = 2$ and $\mathbb{A} = (A, B, C)$. Then the power mean $P_2(\omega; \mathbb{A})$ is the unique positive invertible solution of

$$X = \omega_1 A X^{-1} A + \omega_2 B X^{-1} B + \omega_3 C X^{-1} C.$$

What is X ?

Moreover, we do not know whether the power means are monotone increasing or not for $1 < t < 2$:

$$P_t(\omega; \mathbb{A}) \leq P_s(\omega; \mathbb{A}) \quad \text{for } 1 \leq t \leq s < 2$$

holds or not.

REFERENCES

- [1] R. Bhatia, *On the exponential metric increasing property*, Linear Algebra Appl., **375** (2003), 211–220.
- [2] G. Corach, H. Porta and L. Recht, *Convexity of the geodesic distance on spaces of positive operators*, Illinois J. Math. **38**(1994), 87–94.
- [3] T. Furuta, J. Mičić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 1, Element, Zagreb, 2005.
- [4] J. Lawson and Y. Lim, *Karcher means and Karcher equations of positive definite operators*, Trans. Amer. Math. Soc., Series B, **1** (2014), 1–22.
- [5] Y. Lim and M. Pálfi, *Matrix power means and the Karcher mean*, J. Funct. Anal., **262** (2012), 1498–1514.
- [6] Y. Seo, *Operator power means due to Lawson-Lim-Pálfi for $1 < t < 2$* , Linear Algebra Appl., **459** (2014), 342–356.
- [7] A. C. Thompson, *On certain contraction mappings in a partially ordered vector space*, Proc. Amer. Math. Soc., **14**(1963), 438–443.

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