

On Projective Planes of Order 15 Admitting a Collineation of Order 7

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1. Introduction

In his paper [3], Ho showed that a collineation group of a projective plane of order 15 is solvable whose order divides $2^6, 2^3 \cdot 3^3, 2 \cdot 5, 3 \cdot 5, 2^3 \cdot 3 \cdot 7$ or $2^6 \cdot 7$.

In the paper we prove the following result:

THEOREM 1. *There is no projective plane of order 15 admitting a collineation group of order 21.*

This theorem and Ho's theorem (Theorem B in [3]) yield

COROLLARY. *Let G be a collineation group of a projective plane of order 15. If $7 \parallel |G|$, then $|G| \mid 2^6 \cdot 7$.*

The terminology we use is standard and can be found in [1] and [4]. Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a projective plane and G a subset of $\text{Aut } \Pi$. Then set $\mathcal{P}(G) = \{X \mid X \in \mathcal{P}, X^\sigma = X \text{ for all } \sigma \in G\}$, $\mathcal{L}(G) = \{x \mid x \in \mathcal{L}, x^\sigma = x \text{ for all } \sigma \in G\}$ and $\text{Fix}(G) = (\mathcal{P}(G), \mathcal{L}(G))$. When G is a collineation group of Π , for a point orbit Ω and a line orbit Δ of G , set $(\Omega\Delta) = |\Omega \cap (l)|$, where l is a line in Δ . Here we remark that the number $(\Omega\Delta)$ depends only on Ω and Δ , not on l .

2. Projective Planes of Order 15

The following proposition will be frequently used in the paper.

PROPOSITION 1(see [1] or Proposition 4.1 of [3]). *Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a pro-*

be a projective plane of order n and G a collineation group of Π . Let $\mathcal{L}_1, \dots, \mathcal{L}_w$ be the line orbits of G . Then for any two point orbits $\mathcal{P}_1$ and $\mathcal{P}_2$ of G ,

$$\sum_{1 \leq j \leq w} |\mathcal{L}_j|(\mathcal{P}_1 \mathcal{L}_j)(\mathcal{P}_2 \mathcal{L}_j) = |\mathcal{P}_1||\mathcal{P}_2| + n|\mathcal{P}_1 \cap \mathcal{P}_2|.$$

The following condition will be assumed in the rest of the paper.

HYPOTHESIS 1. Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a projective plane of order 15 and G a collineation group of order 21 of Π .

PROPOSITION 2([2], [3]). (i) $G = \langle \varphi, \tau | \varphi^7 = \tau^3 = 1, \tau^{-1}\varphi\tau = \varphi^2 \rangle$.

(ii) τ is either a generalized elation or planar.

(iii) If τ is a generalized elation, then $|\mathcal{P}(\tau)| = 1$ or 7.

(iv) If τ is planar, then $\text{Fix}(\tau)$ is a subplane of order 3.

(v) φ is either triangular or a homology.

(vi) If φ is triangular, then τ is either planar or $|\mathcal{P}(\tau)| = 1$.

For a subset H of $\langle \varphi \rangle$, set $\widehat{H} = \sum_{\mu \in H} \mu(\in Z[\langle \varphi \rangle])$, $H^{-1} = \{\mu^{-1} | \mu \in H\}$.

3. The Case That φ is Triangular

In this section we assume that φ is triangular. Then one of the following 3 cases occurs by Proposition 2 (iv,vi):

Case 1. $\mathcal{P}(\varphi) \cap \mathcal{P}(\tau) = \emptyset, |\mathcal{P}(\tau)| = 1$.

Case 2. $\mathcal{P}(\varphi) \cap \mathcal{P}(\tau) = \emptyset, |\mathcal{P}(\tau)| = 13$.

Case 3. $\mathcal{P}(\tau) \supseteq \mathcal{P}(\varphi), |\mathcal{P}(\tau)| = 13$.

4. The Case That φ is a Homology

In this section we assume that φ is a homology. Let P_0 be the center of φ and l_0 the axis of φ . Since τ fixes the point P_0 and normalizes $\langle \varphi \rangle$, there exists a line l_1 through P_0 such that $|\mathcal{P}(\tau) \cap (l_1)| \geq 2$. Then from Proposition 2 (ii, iii) the following 2 cases occur:

Case 4. $(l_1) \supseteq \mathcal{P}(\tau), |\mathcal{P}(\tau)| = 7$.

Case 5. $|(l_1) \cap \mathcal{P}(\tau)| = 4, |\mathcal{P}(\tau)| = 13$.

5. Case 5

In this section we consider Case 5. Clearly φ is a homology and $\text{Fix}(\tau)$ is a plane of order 3. Let $\mathcal{P}(\varphi) = \{P_0, P_1, P_2, P_3, P_4, P_{13}, P_{14}, \dots, P_{24}\}$ and $\mathcal{L}(\varphi) = \{l_0, l_1, l_2, l_3, l_4, l_{13}, l_{14}, \dots, l_{24}\}$, where P_0 is a center of φ and l_0 is an axis of φ . Let $\mathcal{P}_0, \dots, \mathcal{P}_{48}$ be 49 $\langle \varphi \rangle$ -orbits on $\mathcal{P}$ and $\mathcal{L}_0, \dots, \mathcal{L}_{48}$ be 49 $\langle \varphi \rangle$ -orbits on $\mathcal{L}$. Then we may assume the following:

$$\mathcal{P}_i = \{P_i\}, \mathcal{L}_i = \{l_i\} (0 \leq i \leq 4, 13 \leq i \leq 24),$$

$$|\mathcal{P}_i| = 7, |\mathcal{L}_i| = 7 (5 \leq i \leq 12, 25 \leq i \leq 48),$$

$$(l_0) = \{P_1, P_2, P_3, P_4, P_{13}, P_{14}, \dots, P_{24}\}, \quad (P_0) = \{l_1, l_2, l_3, l_4, l_{13}, l_{14}, \dots, l_{24}\},$$

$$(l_1) = \mathcal{P}_5 \cup \mathcal{P}_6 \cup \{P_0, P_1\}, \quad (P_1) = \mathcal{L}_5 \cup \mathcal{L}_6 \cup \{l_0, l_1\},$$

$$(l_2) = \mathcal{P}_7 \cup \mathcal{P}_{10} \cup \{P_0, P_2\}, \quad (P_2) = \mathcal{L}_7 \cup \mathcal{L}_{10} \cup \{l_0, l_2\},$$

$$(l_3) = \mathcal{P}_8 \cup \mathcal{P}_{11} \cup \{P_0, P_3\}, \quad (P_3) = \mathcal{L}_8 \cup \mathcal{L}_{11} \cup \{l_0, l_3\},$$

$$(l_4) = \mathcal{P}_9 \cup \mathcal{P}_{12} \cup \{P_0, P_4\}, \quad (P_4) = \mathcal{L}_9 \cup \mathcal{L}_{12} \cup \{l_0, l_4\},$$

$$(l_{13}) = \mathcal{P}_{25} \cup \mathcal{P}_{37} \cup \{P_0, P_{13}\}, \quad (P_{13}) = \mathcal{L}_{25} \cup \mathcal{L}_{37} \cup \{l_0, l_{13}\},$$

$$(l_{14}) = \mathcal{P}_{26} \cup \mathcal{P}_{38} \cup \{P_0, P_{14}\}, \quad (P_{14}) = \mathcal{L}_{26} \cup \mathcal{L}_{38} \cup \{l_0, l_{14}\},$$

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$$(l_{24}) = \mathcal{P}_{36} \cup \mathcal{P}_{48} \cup \{P_0, P_{24}\}, \quad (P_{24}) = \mathcal{L}_{36} \cup \mathcal{L}_{48} \cup \{l_0, l_{24}\}.$$

Set $m_{ij} = (\mathcal{P}_j \mathcal{L}_i)$ for $i, j, 0 \leq i, j \leq 48$ and $M = (m_{ij})_{0 \leq i, j \leq 48}$. Choose a line $l_i \in \mathcal{L}_i$ and a point $P_i \in \mathcal{P}_i$ for $i, 5 \leq i \leq 12, 25 \leq i \leq 48$. Set $D_{ij} = \{\mu | \mu \in \langle \varphi \rangle, P_j^\mu \in (l_i)\}$. Clearly $|D_{ij}| = m_{ij}$ for $i, j, 5 \leq i, j \leq 12, 25 \leq i, j \leq 48$.

LEMMA 5.1. $\sum_{j \in \{5, 6, \dots, 12\} \cup \{25, 26, \dots, 48\}} \widehat{D_{ij}} \widehat{D_{i'j}}^{-1} =$

$$\begin{cases} 15 & \text{if } i = i' \in \{5, 6, \dots, 12\} \cup \{25, 26, \dots, 48\}, \\ 0 & \text{if } \{i, i'\} \in \{\{5, 6\}, \{7, 10\}, \{8, 11\}, \{9, 12\}, \{25, 37\}, \{26, 38\}, \dots, \{36, 48\}\}, \\ \langle \varphi \rangle & \text{otherwise.} \end{cases}$$

Proof. We only show the case $i = i'$ (one can obtain the others by a similar argument). Set $A = \{5, 6, \dots, 12\} \cup \{25, 26, \dots, 48\}$, $l = l_i$ and $\Phi =$

$\sum_{j \in A} \sum_{P_j^\mu, P_j^\xi (\neq) \in (l)} \mu \xi^{-1}$. For example, let $i = 5$. Then since $(l) \cap (l^{\alpha^{-1}}) = \{P_1\}$ for all $\alpha \in \langle \varphi \rangle - \{1\}$, $\Phi = 0$. Therefore $\sum_{j \in A} \widehat{D_{ij}} \widehat{D_{ij}}^{-1} = \sum_{j \in A} \sum_{P_j^\mu \in (l), \mu \in \langle \varphi \rangle} \mu \mu^{-1} = |(\mathcal{P}_7 \cup \mathcal{P}_{10} \cap (l))| + |(\mathcal{P}_8 \cup \mathcal{P}_{11} \cap (l))| + |(\mathcal{P}_9 \cup \mathcal{P}_{12} \cap (l))| + |(\mathcal{P}_{25} \cup \mathcal{P}_{37} \cap (l))| + |(\mathcal{P}_{26} \cup \mathcal{P}_{38} \cap (l))| + \cdots + |(\mathcal{P}_{36} \cup \mathcal{P}_{48} \cap (l))| = 15$.

LEMMA 5.2. (i) $\sum_{j \in \{5,6,\dots,12\} \cup \{25,26,\dots,48\}} m_{ij} = 15$ for $i \in \{5,6,\dots,12\} \cup \{25,26,\dots,48\}$.

(ii) $\sum_{j \in \{5,6,\dots,12\} \cup \{25,26,\dots,48\}} m_{ij} m_{i'j} =$

$$\begin{cases} 15 & \text{if } i = i' \in \{5,6,\dots,12\} \cup \{25,26,\dots,48\}, \\ 0 & \text{if } \{i, i'\} \in \{\{5,6\}, \{7,10\}, \{8,11\}, \{9,12\}, \{25,37\}, \{26,38\}, \dots, \{36,48\}\}, \\ 7 & \text{otherwise.} \end{cases}$$

Proof. (i) follows immediately from the definition of m_{ij} 's. We get by considering the action of the trivial character of $\langle \varphi \rangle$ on the equations of Lemma 5.1.

We consider the action of τ . We may assume that $\mathcal{P}(\tau) = \{P_0, P_1, \dots, P_{12}\}$ and $\mathcal{L}(\tau) = \{l_0, l_1, \dots, l_{12}\}$. Moreover we may assume that τ induces the actions on both $\{P_i | 0 \leq i \leq 48\}$ and $\{l_i | 0 \leq i \leq 48\}$, and they are $(P_0)(P_1) \cdots (P_{12}) (P_{13}, P_{14}, P_{15})(P_{16}, P_{17}, P_{18}) \cdots (P_{22}, P_{23}, P_{24})(P_{25}, P_{26}, P_{27})(P_{28}, P_{29}, P_{30}) \cdots (P_{46}, P_{47}, P_{48})$ and $(l_0)(l_1) \cdots (l_{12})(l_{13}, l_{14}, l_{15})(l_{16}, l_{17}, l_{18}) \cdots (l_{22}, l_{23}, l_{24})(l_{25}, l_{26}, l_{27}) \cdots (l_{28}, l_{29}, l_{30}) \cdots (l_{46}, l_{47}, l_{48})$ respectively.

Let $\Omega_0, \dots, \Omega_{24}$ be G -orbits on $\mathcal{P}$ and $\Delta_0, \dots, \Delta_{24}$ be G -orbits on $\mathcal{L}$. Then we may assume the following:

$$\begin{aligned} \Omega_0 &= \mathcal{P}_0, & \Delta_0 &= \mathcal{L}_0, \\ \Omega_1 &= \mathcal{P}_1, & \Delta_1 &= \mathcal{L}_1, \\ & \dots & & \\ \Omega_{12} &= \mathcal{P}_{12}, & \Delta_{12} &= \mathcal{L}_{12}, \\ \Omega_{13} &= \mathcal{P}_{13} \cup \mathcal{P}_{14} \cup \mathcal{P}_{15}, & \Delta_{13} &= \mathcal{L}_{13} \cup \mathcal{L}_{14} \cup \mathcal{L}_{15}, \\ \Omega_{14} &= \mathcal{P}_{16} \cup \mathcal{P}_{17} \cup \mathcal{P}_{18}, & \Delta_{14} &= \mathcal{L}_{16} \cup \mathcal{L}_{17} \cup \mathcal{L}_{18}, \\ & \dots & & \\ \Omega_{24} &= \mathcal{P}_{46} \cup \mathcal{P}_{47} \cup \mathcal{P}_{48}, & \Delta_{24} &= \mathcal{L}_{46} \cup \mathcal{L}_{47} \cup \mathcal{L}_{48}. \end{aligned}$$

Set $l_{ij} = (\Omega_j \Delta_i)$ for $0 \leq i, j \leq 24$ and $L = (l_{ij})_{0 \leq i, j \leq 24}$. Without loss of generality we may assume that $(l_5 \ 7 \ l_5 \ 8 \ l_5 \ 9) = (111)$, $(l_7 \ 5 \ l_8 \ 5 \ l_9 \ 5) = (111)$ and $(l_{17} \ 5 \ l_{18} \ 5 \ l_{19} \ 5 \ l_{20} \ 5) = (1111)$. Then we have the following.

LEMMA 5.3. *There are exactly the following two possibilities, L_1, L_2 for L up to equivalence and transposition.*

[illegible]

$$L_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 & 0 & 3 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 3 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 3 & 0 & 3 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 7 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 7 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

It follows that each $L_i (i = 1, 2)$ of Lemma 5.3 can be extended 162 M 's which satisfy the conditions of Lemma 5.2, using a computer. Therefore we get the following lemma.

LEMMA 5.4. *There are exactly 324 possibilities $M_1, \dots, M_{324}$ for M up to equivalence and transposition. Actually,*

[illegible]

for $i, 163 \leq i \leq 324$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$, $E =$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, P = (0 \ 0 \ 0), Q = (1 \ 1 \ 1), R = {}^tP \text{ and } S = {}^tQ.$$

Next we examine whether there are projective planes of order 15 corresponding to each M_i or not. For example

$$M_1 = \begin{pmatrix} & * & & & & ** & & & \\ & & & & & & & & \\ & & C & C & C & C & B & Z & Z & Z \\ & & Y & C & Z & B & B & B & C & Y \\ & & Y & B & B & Y & B & Y & C & B \\ & & Y & Z & C & B & B & C & Z & C \\ *** & & B & Z & Z & Z & C & C & C & C \\ & & B & B & C & Y & Y & C & Z & B \\ & & B & Y & C & B & Y & B & B & Y \\ & & B & C & Z & C & Y & Z & C & B \end{pmatrix},$$

$$\text{where } B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\text{and } Z = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \text{ To construct projective planes corresponding to } M_1,$$

we must determine D_{ij} 's satisfying the conditions of Lemma 5.1. We identify an element φ^i of $\langle \varphi \rangle$ with an element i of the additive group F^+ of the ring $F = \{0, 1, \dots, 6\} \pmod{7}$. Set $\Omega = \{5, 6, 25, 26, \dots, 33\}$ and $\Delta = \{5, 6, \dots, 12\} \cup \{25, 26, \dots, 48\}$. Then by Lemma 5.1, $\sum_{j \in \Delta} \widehat{D_{ij}} \widehat{D_{i'j}}^{-1} = \widehat{\langle \varphi \rangle}$ for distinct $i, i' \in \Omega$. We want to investigate whether D_{ij} 's ($i \in \Omega, j \in \Delta$) satisfying these conditions exist or not. Set $\Delta' = \{5, 6, \dots, 12\}$, $\Delta'' = \{25, 26, \dots, 36\}$ and $\Delta''' = \{37, 38, \dots, 48\}$. Changing of $P_5, P_7, P_8, \dots, P_{12}, P_{25}, P_{28}, P_{31}, P_{34}, P_{37}, P_{40}$,

Here we omit components D_{ij} 's if they are empty sets. For example, $\sum_{j \in \Delta} \widehat{D_{25} j} \widehat{D_{30} j}^{-1} = \widehat{\langle \varphi \rangle}$ means $\{0, 1 - 4e_3, d_4 - 4e_4, d_6 - 4e_7, d_9 - 4e_{11}, d_{12} - 4e_{12}, d_{14} - 4e_{14}\} = F$. But it follows that there is no $\{d_i\}, \{e_i\}$ and $\{f_i\}$ satisfying these conditions, using programs of sortings. By a similar argument for each other M_i , we have the following. (It follows that it is sufficient to examine only 108 M_i 's by considering the forms of M_i 's.)

LEMMA 5.4. *There is no $\{D_{ij} | i \in \{5, 6, 25, 26, \dots, 33\}, j \in \{0, 1, \dots, 48\}\}$ satisfying the conditions of Lemma 5.1.*

LEMMA 5.5. *There is no projective plane of order 15 satisfying Case 5.*

6. The Other Cases

For each case except Case 5, we can also define similar matrices M and L to those of case 5.

For Case 1, there are exactly 221 possibilities $L_1, \dots, L_{221}$ for L up to equivalence, but we can not extend any L_i to M . For Case 2, Case 3 or Case 4, there is no L . Therefore by Lemma 5.5 we have Theorem 1.

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