

SOME SUBMANIFOLDS OF COMPLEX PROJECTIVE SPACES

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We discuss a generalization of ruled surfaces in $\mathbb{R}^3$ to submanifolds in complex projective spaces. A ruled surface is generated by 1-parameter family of lines in $\mathbb{R}^3$. Examples are: hyperboloid of one sheet, hyperbolic paraboloid, circular cylinder, circular conic and right helicoid. It is classically known that a ruled surface M in $\mathbb{R}^3$ is minimal if and only if M is a part of a plane $\mathbb{R}^2$ or a right helicoid.

We denote by $\mathbb{P}^n(\mathbb{C})$ an n -dimensional complex projective space with Fubini-Study metric of holomorphic sectional curvature 1 unless otherwise stated. It is known that a totally geodesic submanifold of complex projective space $\mathbb{P}^n(\mathbb{C})$ is one of the following:

- (a) Kähler submanifold $\mathbb{P}^k(\mathbb{C})$ ($k < n$),
- (b) Totally real submanifold $\mathbb{P}^k(\mathbb{R})$ ($k \leq n$).

Now we study the following submanifolds in $\mathbb{P}^n(\mathbb{C})$:

- (1) Kähler submanifold M^{k+r} on which there is a holomorphic foliation of complex codimension r and each leaf is a totally geodesic $\mathbb{P}^k(\mathbb{C})$ in $\mathbb{P}^n(\mathbb{C})$ (for simplicity, we said that M is holomorphically k -ruled).
- (2) Totally real (Lagrangian) submanifold M^n on which there is a (real) foliation of real codimension $n - k$ and each leaf is a totally geodesic $\mathbb{P}^k(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{C})$.

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1. The set of k dimensional totally geodesic complex projective subspaces in $\mathbb{P}^n(\mathbb{C})$ is identified with the complex Grassmann manifold $\mathbb{G}_{k+1, n-k}(\mathbb{C})$ of $k+1$ dimensional complex linear subspaces in $\mathbb{C}^{n+1}$:

$$\{\mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})\} \cong \{\mathbb{C}^{k+1} \subset \mathbb{C}^{n+1}\} = \mathbb{G}_{k+1, n-k}(\mathbb{C}).$$

Hence there is a natural correspondence between Kähler submanifolds M^{k+r} holomorphically foliated by totally geodesic $\mathbb{P}^k(\mathbb{C})$ in $\mathbb{P}^n(\mathbb{C})$ and Kähler submanifolds Σ^r in the complex Grassmann manifold $\mathbb{G}_{k+1, n-k}(\mathbb{C})$.

Here we note that a holomorphically k -ruled Kähler submanifold M^{k+r} in $\mathbb{P}^n(\mathbb{C})$ is totally geodesic if and only if the corresponding Kähler submanifold $\Sigma^r \hookrightarrow \mathbb{G}_{k+1, n-k}(\mathbb{C})$ is contained in some totally geodesic $\mathbb{G}_{k+1, r}(\mathbb{C})$ in $\mathbb{G}_{k+1, n-k}(\mathbb{C})$ (cf. [CN]).

Examples of holomorphically k -ruled submanifolds in $\mathbb{P}^n(\mathbb{C})$.

- (1) totally geodesic $\mathbb{P}^{k+r}(\mathbb{C})$,
- (2) Segre imbedding $\mathbb{P}^k(\mathbb{C}) \times \mathbb{P}^r(\mathbb{C}) \hookrightarrow \mathbb{P}^{k+r+k+r}(\mathbb{C})$.

In the case $k = r = 1$, we can see that

Theorem 1. *Let M^2 be a Kähler surface in $\mathbb{P}^n(\mathbb{C})$ holomorphically foliated by totally geodesic projective lines $\mathbb{P}^1(\mathbb{C})$ in $\mathbb{P}^n(\mathbb{C})$. If the scalar curvature of M^2 is constant and M^2 is not totally geodesic, then the Gauss curvature of the corresponding holomorphic curve Σ^1 in $\mathbb{G}_{2,n-1}(\mathbb{C})$ is constant.*

Remark. The converse of the above Theorem does not hold. Consider the following holomorphic imbedding:

$$\mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^k(\mathbb{C}) \times \mathbb{P}^\ell(\mathbb{C}) \longrightarrow \mathbb{G}_{2,k+\ell}(\mathbb{C}),$$

where the first imbedding is the product of Veronese imbeddings of degree k and ℓ , and the second one is the totally geodesic holomorphic imbedding of $\mathbb{P}^k(\mathbb{C}) \times \mathbb{P}^\ell(\mathbb{C})$ into $\mathbb{G}_{2,k+\ell}(\mathbb{C})$ (cf. [CN]). By direct calculations, we can see that the Gauss curvature of the induced metric on $\mathbb{P}^1(\mathbb{C})$ is $\frac{1}{k+\ell}$. But the scalar curvature of the corresponding holomorphically 1-ruled Kähler surface M^2 in $\mathbb{P}^{k+\ell+1}(\mathbb{C})$ is constant only when $k = \ell$. In this case, the Kähler surface M^2 is holomorphically congruent to

$$\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}_{1/k}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^k(\mathbb{C}) \rightarrow \mathbb{P}^{2k+1}(\mathbb{C}),$$

where the first one is the product of the identity map and the Veronese imbedding of degree k , and the second one is the Segre imbedding. These examples show that contrary to Kähler submanifolds in $\mathbb{P}^n(\mathbb{C})$ (cf. [E]), the rigidity for Kähler submanifolds in complex Grassmann manifolds (of rank ≥ 2) does not hold (cf. [CZ]).

2. The set $W_{k,n}$ of k dimensional totally geodesic real projective subspaces $\mathbb{P}^k(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{C})$ is:

$$W_{k,n} = \begin{cases} \frac{SU(n+1)}{SO(n+1)}, & \text{if } n = k, \\ \frac{SU(n+1)}{K_{k+1,n-k}}, & \text{if } n > k, \end{cases}$$

where

$$K_{k+1,n-k} = \left\{ \left(\begin{array}{cc} e^{\frac{i\theta}{k+1}} P & 0 \\ 0 & e^{-\frac{i\theta}{n-k}} Q \end{array} \right) \mid \theta \in \mathbb{R}, P \in SO(k+1), Q \in SU(n-k) \right\}.$$

Let $G = SU(n+1)$, $\mathfrak{g} = \mathfrak{su}(n+1)$, and

$$\mathfrak{k} = \left\{ i \left(\begin{array}{cc} \frac{\theta}{k+1} E_{k+1} & 0 \\ 0 & -\frac{\theta}{n-k} E_{n-k} \end{array} \right) + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \mid \theta \in \mathbb{R}, U \in \mathfrak{so}(k+1), V \in \mathfrak{su}(n-k) \right\},$$

where E denotes identity matrix. Then $\mathfrak{k}$ is the Lie algebra of the Lie group $K = K_{k+1, n-k}$. Put

$$\mathfrak{m} = \left\{ \begin{pmatrix} iW & Z \\ -Z^* & 0 \end{pmatrix} \mid W \in S_0(k+1, \mathbb{R}), Z \in M(k+1, n-k, \mathbb{C}) \right\},$$

where $S_0(k+1, \mathbb{R})$ denotes the set of $(k+1) \times (k+1)$ (real) symmetric matrices with trace = 0. Then we have $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ (direct sum) and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. Since K is connected, the homogeneous space G/K is reductive. Note that G/K is naturally reductive with respect to some metric.

As §1, we can see that there is a one-to-one correspondence between $k+r$ ($0 < r < 2n-k$) dimensional submanifolds foliated by totally geodesic, totally real $\mathbb{P}^k(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{C})$ and r dimensional submanifolds in G/K .

There is a natural fibration on G/K as follows: Let $\pi : G/K \rightarrow G/H$ be the projection defined by $gK \mapsto gH$, where $K = K_{k+1, n-k}$ and $H = S(U(k+1) \times U(n-k))$. Note that $K \subset H$ and $G/H = \mathbb{G}_{k+1, n-k}(\mathbb{C})$. Then it can be seen that π is a Riemannian submersion with which each fibre $H/K \cong SU(k+1)/SO(k+1)$ is totally geodesic in G/K .

Since $G/H = \mathbb{G}_{k+1, n-k}(\mathbb{C})$ is a Kähler manifold, there is the almost complex structure on the horizontal distribution of G/K compatible to π and the canonical complex structure of G/H .

We claim

Proposition 2. *Let M^n be a submanifold in $\mathbb{P}^n(\mathbb{C})$ foliated by totally geodesic $\mathbb{P}^k(\mathbb{R})$. Then M^n is totally real (Lagrangian) if and only if the corresponding submanifold Σ^{n-k} in G/K is "horizontal" and "totally real" with respect to the almost complex structure on the horizontal distribution of G/K as above.*

Hence if M^n is a totally real submanifold in $\mathbb{P}^n(\mathbb{C})$ foliated by totally geodesic $\mathbb{P}^k(\mathbb{R})$, then there is a corresponding totally real submanifold Σ^{n-k} in $G/H = \mathbb{G}_{k+1, n-k}(\mathbb{C})$. So we consider the following question: If Σ^{n-k} is a (totally real) submanifold in $G/H = \mathbb{G}_{k+1, n-k}(\mathbb{C})$, then does there exist a "horizontal lift" of Σ^{n-k} in G/K with respect to π ? Note that a totally real submanifold M^n foliated by totally geodesic $\mathbb{P}^k(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{C})$ is totally geodesic if and only if the corresponding submanifold Σ^{n-k} in G/K is contained in some $\pi^{-1}(\mathbb{G}_{k+1, n-k}(\mathbb{R}))$, where $\mathbb{G}_{k+1, n-k}(\mathbb{R})$ is a maximal totally geodesic, totally real submanifold in $G/H = \mathbb{G}_{k+1, n-k}(\mathbb{C})$.

Hereafter we assume that: Let G be a linear Lie group, let $H \supset K$ be closed subgroups of G such that G/K is a reductive homogeneous space, G/H is a Riemannian symmetric space and the projection $\pi : G/K \rightarrow G/H$ defined by $gK \mapsto gH$ is a Riemannian submersion with which each fibre H/K is totally geodesic in G/K . We denote the projections as $\pi_H : G \rightarrow G/H$ and $\pi_K : G \rightarrow G/K$. Let $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of the Lie groups G, H, K , respectively, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = \mathfrak{h} + \mathfrak{p}$ be the canonical decompositions of $\mathfrak{g}$.

Let $f : \Sigma \rightarrow G/H$ be an isometric immersion. We would like to find the "condition" for existence of a "horizontal lift" of f . By using a local cross

section $G/H \rightarrow G$, we always have a "framing" $\Phi : \Sigma \rightarrow G$ satisfying $f = \pi_H \circ \Phi$ **locally**. To construct a "horizontal lift" of f , we may find the map $\Psi : \Sigma \rightarrow H$ such that $F := \pi_K \circ (\Phi \cdot \Psi) : \Sigma \rightarrow G \rightarrow G/K$ is the desirable horizontal lift. Here $\Phi \cdot \Psi$ is defined by the product of $\Phi(x) \in G$ and $\Psi(x) \in H \subset G$ for $x \in \Sigma$ as elements of the Lie group G .

Let $\alpha := \Phi^{-1}d\Phi$ (resp. $\beta := \Psi^{-1}d\Psi$) be the $\mathfrak{g}$ -valued (resp. $\mathfrak{h}$ -valued) 1-form on Σ which is the pull-back of the Maurer-Cartan form of G (resp. H). The following fact is well known (cf. [G]): Let ω be the Maurer-Cartan form on G . Suppose that α is a $\mathfrak{g}$ -valued 1-form on a connected and simply connected manifold Σ . Then there exists a C^∞ map $\Phi : \Sigma \rightarrow G$ with $\Phi^*\omega = \alpha$ if and only if $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$. Moreover, the resulting map is unique up to left translation.

We see that

$$\begin{aligned} & F = \pi_K \circ (\Phi \cdot \Psi) : \Sigma \rightarrow G/K \text{ is horizontal,} \\ \iff & (\mathfrak{h} \cap \mathfrak{m}) - \text{part of } (\Phi \cdot \Psi)^{-1}d(\pi_K \circ (\Phi \cdot \Psi)) = 0, \\ \iff & \beta_{\mathfrak{m}} + (\text{ad}(\Psi^{-1})\alpha)_{\mathfrak{h} \cap \mathfrak{m}} = 0. \end{aligned}$$

Taking an exterior derivation and using the integrability conditions of α and β , we get that the condition for existence of a horizontal lift of the isometric immersion $f : \Sigma \rightarrow G/H$ is

$$[\alpha_{\mathfrak{p}} \wedge \alpha_{\mathfrak{p}}]_{\mathfrak{h} \cap \mathfrak{m}} = 0.$$

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