数理解析研究所講究録
第 1872 巻 2014 年 151-157

ON MULTIPLICATIVE INDUCTION

FUMIHITO ODA
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND ENGINEERING
KINKI UNIVERSITY

ABSTRACT. Let G be a finite group and e be the proper trivial subgroup of G. We compute the value $\mathrm{Jnd}_H^G(\ell[H/e])$ for a subgroup H of G in the Burnside ring $\Omega(G)$ for an integer ℓ. Their values induce integer valued polynomials.

1. NOTATION

Let G be a finite group and s_G be the set of all subgroups of G. Denote by ${}^gH$ the conjugate subgroup gHg^{-1} for $H \leq G$ and $g \in G$. Let $[s_G]$ be a set of representatives of G-conjugacy classes of s_G. If X is a finite G-set, write $[X]$ for the isomorphism class of finite G-sets containing X. Denote by X^S the S-fixed points of the G-set X. If X is a finite set, write $|X|$ for the cardinality of X. Denote by e the identity element of G. The proper trivial subgroup $\{e\}$ of G is also denoted by e. For two subgroups S, $H \leq G$ denote by $[S\backslash G/H]$ a set of representatives of double cosets of G by S and H.

2. MULTIPLICATIVE INDUCTIONS FOR BURNSIDE RINGS

Let $\Omega(G)$ be the Burnside ring of G. Then $\Omega(G)$ is a free $\mathbb{Z}$-module with basis $\{[G/H] | H \in [s_G]\}$. The multiplication is defined by the Cartesian product. If $S \in s_G$, then there is a unique linear form $\varphi_S^G : \Omega(G) \to \mathbb{Z}$ such that $\varphi_S^G([X]) = |X^H|$ for any finite G-set X. It is a ring homomorphism. The *mark homomorphism* is a ring homomorphism $\varphi^G = \prod_{(S)\in[s_G]} \varphi_S^G : \Omega(G) \to \widetilde{\Omega}(G)$, where $\widetilde{\Omega}(G) = \prod_{(S)\in[s_G]} \mathbb{Z}$ and it is called the *ghost ring* of G.

Lemma 2.1. *The ring homomorphism φ^G is injective.*

We recall some properties for tensor induction of Burnside rings. We refer to [Yo90] for more details. Let $\mathbf{set}^G$ be the category of finite G-sets. If $H \leq G$, then there is a functor

$$\mathrm{Jnd}_H^G : \mathbf{set}^H \to \mathbf{set}^G$$

which has the values on objects

$$\mathrm{Jnd}_H^G : X \mapsto \mathrm{Map}_H(G, X),$$

where $\mathrm{Map}_H(G, X)$ is the set of H-maps $\alpha : G \to X$ such that $\alpha(h \cdot g) = h \cdot \alpha(g)$ for all $h \in H$, $g \in G$, with the action of G defined by $(k \cdot \alpha)(g) = \alpha(gk)$ for $k \in G$, for an H-set X.

Supported by JSPS Grant-in-Aid for Scientific Research (C) 21540031.

Lemma 2.2. *Let H be a subgroup of G and X be an H-set. If S is a subgroup of G, then*

$$\varphi_S^G(\mathrm{Jnd}_H^G(X)) = \prod_{g\in[S\backslash G/H]} \varphi_{H\cap{}^gS}^H(X).$$

Lemma 2.3. *Let H be a subgroup of G. If S is a subgroup of G and $q \in \mathbb{Z}$, then*

$$\varphi_S^G(\mathrm{Jnd}_e^G(q[e/e])) = q^{|G/S|}.$$

Proof. By Lemma 2.2, we have

$$\varphi_S^G(\mathrm{Jnd}_e^G(q[e/e])) = \prod_{g\in[S\backslash G/e]} \varphi_{e\cap{}^gS}^e(q[e/e]) = \prod_{g\in[S\backslash G]} q\varphi_e^e([e/e]) = \prod_{g\in[S\backslash G]} q.$$

□

It has been shown by Gluck ([Gl81]) and independently by Yoshida ([Yo83]) that a formula of primitive idempotent e_H^G of $\mathbb{Q}$-algebra $\mathbb{Q}\otimes_{\mathbb{Z}}\Omega(G)$ for $H \leq G$ can be expressed as

$$e_H^G = \frac{1}{|N_G(H)|}\sum_{K\subseteq H}|K|\mu(K,H)[G/K], \tag{2.1}$$

where $\mu(K,H)$ is the value of the Möbius function of s_G.

Denote by NH (resp. WH) $N_G(H)$ (resp. $H_G(H)/H$) for a subgroup H of G. Put $q^G = \mathrm{Jnd}_H^G(q[e/e])$ for $q \in \mathbb{Z}$.

Lemma 2.4. *If G is a finite group and q is an integer, then*

$$q^G = \sum_{(D)\in[s_G]} |WD|^{-1}\sum_{S\leq G}\mu(D,S)q^{|G/S|}[G/D]$$

Proof. By Lemma 2.3 and idempotent formula (2.1), we have that

$$\begin{aligned}
q^G &= \sum_{S\in[s_G]} \varphi_S^G(q^G)e_S^G \\
&= \sum_{S\in[s_G]} q^{|G/S|}|NS|^{-1}\sum_{D\leq S}|D|\mu(D,S)[G/D] \\
&= \sum_{S\leq G}(G:NS)^{-1}q^{|G/S|}|NS|^{-1}\sum_{D\leq G}|D|\mu(D,S)[G/D] \\
&= |G|^{-1}\sum_{D\leq G}|D|\left(\sum_{S\leq G}\mu(D,S)q^{|G/S|}\right)[G/D] \\
&= |G|^{-1}\sum_{D\in[s_G]}(G:ND)|D|\left(\sum_{S\leq G}\mu(D,S)q^{|G/S|}\right)[G/D] \\
&= \sum_{D\in[s_G]}|WD|^{-1}\left(\sum_{S\leq G}\mu(D,S)q^{|G/S|}\right)[G/D].
\end{aligned}$$

□

In particular, coefficients of $[G/D]$ in q^G as above are integers.

Proposition 2.5. *If G is a finite group and q is an integer, then*

$$|WD|^{-1}\sum_{S\leq G}\mu(D,S)q^{|G/S|}$$

is an integer for a subgroup D of G.

Substituting x for q we obtain integer-valued polynomials $f_D^G(x)$ as follows.

Theorem 2.6. *Let G be a finite group and put*

$$f_D^G(x)=\frac{1}{|WD|}\sum_{S\leq G}\mu(D,S)x^{|G/S|}$$

for subgroup D of G. Then $f_D^G(x)$ is an integer-valued polynomial.

3. Tambara functors

In this section, we recall some notes on Tambara functors. For a G-map $f:X\longrightarrow Y$ we consider a set

$$\Pi_f(A) \;=\; \left\{(y,\sigma)\;\middle|\; \begin{array}{l} y\in Y,\ \sigma: f^{-1}(y)\longrightarrow A:\ \text{map}\ , \\ \alpha\circ\sigma=\mathrm{id}_{f^{-1}(y)}\end{array}\right\}$$

with G-action defined by

$$g(y,\sigma):=(gy,{}^g\sigma),\quad {}^g\sigma(x):=g\sigma(g^{-1}x)$$

and denote by $\Pi_f\alpha$ the projection $(y,\sigma)\mapsto y$. For a G-map $\alpha:A\to X$ the pullback functor

$$\begin{array}{rcccl} f^* & : & \mathbf{set}^G/Y & \longrightarrow & \mathbf{set}^G/X, \\ & & (B\longrightarrow Y) & \longmapsto & (X\times_Y B\xrightarrow{\mathrm{pr}}X)\end{array}$$

has a left adjoint functor

$$\begin{array}{rcccl} \Sigma_f & : & \mathbf{set}^G/X & \longrightarrow & \mathbf{set}^G/Y, \\ & & (A\xrightarrow{\alpha}X) & \longmapsto & (A\xrightarrow{\alpha}X\xrightarrow{f}Y)\end{array}$$

and a right adjoint functor

$$\begin{array}{rcccl} \Pi_f & : & \mathbf{set}^G/X & \longrightarrow & \mathbf{set}^G/Y, \\ & & (A\xrightarrow{\alpha}X) & \longmapsto & (\Pi_f(A)\xrightarrow{\Pi_f\alpha}Y).\end{array}$$

Two natural transformations

$$\mathit{\Sigma}_f\xleftarrow{\;\mathit{\Sigma}_f\varepsilon'\;}\mathit{\Sigma}_f f^*\mathit{\Pi}_f\xrightarrow{\;\varepsilon\mathit{\Sigma}_f\;}\mathit{\Pi}_f,$$

give a commutative diagram

$$\begin{tikzcd}
& A \arrow[dl, "\alpha"'] & X \times_Y \Pi_f A \arrow[l, "e"'] \arrow[dd, "f'"] \\
X \arrow[d, "f"'] \arrow[rr, phantom, "EXP"] & & \\
Y & & \Pi_f A \arrow[ll, "\pi_f \alpha"']
\end{tikzcd}$$

where $e : X \times_Y \Pi_f A \ni (x, (y, \sigma)) \longmapsto \sigma(x) \in A$ and f' is projection. In order to discuss the TNR-functors, this diagram is introduced by Tambara in [Ta93]. Brun called it Tambara functor in [Br05]. There are some works concerning about Tambara functors ([Na12a], [Na12b], [Na13], [OY11]).

Denote by **Set** the category of sets and maps and by $\mathbf{set}^G$ the category of finite G-sets and G-maps. For any G-sets X and Y we denote by $X + Y$ the disjoint union of them.

For any G-map $f : X \to Y$ we consider the triplet of functors

$$\boldsymbol{T} = (\boldsymbol{T}_!, \boldsymbol{T}^*, \boldsymbol{T}_\star) : \mathbf{set}^G \longrightarrow \mathbf{Set},$$

consisting of a contravariant functor $\boldsymbol{T}^* : \mathbf{set}^G \longrightarrow \mathbf{Set}$ and two covariant functors $\boldsymbol{T}_!$, $\boldsymbol{T}_\star : \mathbf{set}^G \longrightarrow \mathbf{Set}$ which coincide on the objects, and so we write

$$\boldsymbol{T}(X) := \boldsymbol{T}_!(X) = \boldsymbol{T}^*(X) = \boldsymbol{T}_\star(X),$$

$$f_! := \boldsymbol{T}_!(f), f_\star := \boldsymbol{T}_\star(f) : \boldsymbol{T}(X) \longrightarrow \boldsymbol{T}(Y), f^* : \boldsymbol{T}(Y) \longrightarrow \boldsymbol{T}(X).$$

for any G-sets X, Y and any G-map $f : X \to Y$. A triplet $\boldsymbol{T} = (\boldsymbol{T}_!, \boldsymbol{T}^*, \boldsymbol{T}_\star)$ is called a *semi-Tambara functor* if these functors satisfy the following axioms:

(T.1) (Additivity) If

$$X \xrightarrow{i} X + Y \xleftarrow{j} Y$$

is a coproduct diagram of finite G-sets, then

$$\boldsymbol{T}(X) \xleftarrow{i^*} \boldsymbol{T}(X + Y) \xrightarrow{j^*} \boldsymbol{T}(Y)$$

is a product diagram of sets; and $\boldsymbol{T}(\emptyset) = 0 (:= \{0\})$.

(T.2) (Pullback formula)

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "PB"] & Y \arrow[d, "c"] \\
Z \arrow[r, "d"'] & W
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\boldsymbol{T}(X) \arrow[r, "a_!"] \arrow[dr, phantom, "\circlearrowleft"] & \boldsymbol{T}(Y) \\
\boldsymbol{T}(Z) \arrow[r, "d_!"'] \arrow[u, "b^*"] & \boldsymbol{T}(W), \arrow[u, "c^*"']
\end{tikzcd}
\quad
\begin{tikzcd}
\boldsymbol{T}(X) \arrow[r, "a_\star"] \arrow[dr, phantom, "\circlearrowleft"] & \boldsymbol{T}(Y) \\
\boldsymbol{T}(Z) \arrow[r, "d_\star"'] \arrow[u, "b^*"] & \boldsymbol{T}(W). \arrow[u, "c^*"']
\end{tikzcd}$$

(T.3) (Distributive law)

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l, "a"'] \arrow[dr, phantom, "EXP"] & X \times_Y \Pi_f A \arrow[l, "e"'] \arrow[d, "f'"] \\ Y & & \Pi_f A \arrow[ll, "q"] \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \boldsymbol{T}(X) \arrow[d, "f_\star"'] & \boldsymbol{T}(A) \arrow[l, "a_!"'] \arrow[r, "e^*"] \arrow[d, phantom, "\circlearrowleft"] & \boldsymbol{T}(X \times_Y \Pi_f A) \arrow[d, "f'_\star"] \\ \boldsymbol{T}(Y) & & \boldsymbol{T}(\Pi_f A). \arrow[ll, "q_!"] \end{tikzcd}$$

The axioms (T.1) and (T.2) mean that both of pairs $(\boldsymbol{T}^*, \boldsymbol{T}_!)$ and $(\boldsymbol{T}^*, \boldsymbol{T}_\star)$ form semi-Mackey functors (see 3.3 of [OY04]). If all $\boldsymbol{T}(X)$ are commutative ring and $f_!$, f^*, $f_\star$ are homomorphisms of additive groups, rings, multiplicative monoids, respectively, then $\boldsymbol{T}$ is called a *Tambara functor.*

For any finite G-set X, let $\Omega_+(X)$ be the set of isomorphism classes $[A \to X]$ of finite G-sets over X. Then $\Omega_+(X)$ is a semiring by coproducts and products in the comma category $\mathbf{set}^G/X$. A G-map $f : X \to Y$ induces three maps:

$$\begin{aligned} f_! &: \Omega_+(X) \longrightarrow \Omega_+(Y); [A \xrightarrow{\alpha} X] \longmapsto [A \xrightarrow{\alpha} X \xrightarrow{f} Y], \\ f^* &: \Omega_+(Y) \longrightarrow \Omega_+(X); [B \longrightarrow Y] \longmapsto [X \times_Y B \xrightarrow{px} X], \\ f_\star &: \Omega_+(X) \longrightarrow \Omega_+(Y); [A \xrightarrow{\alpha} X] \longmapsto [\Pi_f(A) \xrightarrow{\Pi_f \alpha} Y]. \end{aligned}$$

Then the family $\Omega_+(X)$, $f_!$, f^*, $f_\star$ form a semi-Tambara functor Ω_+. By the Grothendieck ring construction, we have the Burnside ring functor Ω, which is a Tambara functor.

Lemma 3.1. *Let $f : G/H \to G/G$ be the canonical surjection for a subgroup $H \leq G$. If $\alpha : A \to G/H$ is a G-map to transitive G-set G/H, then there exists a G-isomorphism*

$$\Pi_f(A) \cong \mathrm{Map}_H(G, \alpha^{-1}(eH)).$$

Proof. Since G/G is a set of cardinality 1 and f is surjective, we may identify

$$\Pi_f(A) = \{\sigma : G/H \to A \mid \sigma : \text{ map}, \alpha \circ \sigma = \mathrm{id}_{G/H}\}.$$

Then we see that the map $\varphi : \Pi_f(A) \to \mathrm{Map}_H(G, \alpha^{-1}(eH))$,

$$\varphi : s \mapsto \varphi(s) : G \to \alpha^{-1}(eH) : g \mapsto gs(g^{-1}H)$$

gives the isomorphism. □

Let $f : G/H \to G/G$ be the canonical surjection and Ω be the Burnside Tambara functor. Then by Lemma 3.1, we see that the image $\Omega_\star(f)([A \overset{\alpha}{\to} G/H])$ for the map $\Omega_\star(f) : \Omega(G/H) \to \Omega(G/G)$ is

$$\Omega_\star(f)([A \overset{\alpha}{\to} G/H]) = [\mathrm{Map}_H(G, \alpha^{-1}(eH)) \to G/G].$$

By Lemma 2.4, we have the following result.

Proposition 3.2. *If $f : G/e \to G/G$ is the canonical surjection, q is an integer, and Ω is the Burnside Tambara functor. Then we have*

$$\Omega_\star(f)(q[G/e \overset{\mathrm{id}}{\to} G/e]) = \sum_{(D) \in [s_G]} |WD|^{-1} \sum_{S \leq G} \mu(D, S) q^{|G/S|}[G/D \to G/G].$$

4. Necklace polynomials

In this section, we show that the polynomial $f^G_D(x)$ is a generalization of necklace polynomials. It is well known that the number $M(\alpha, n)$ of primitive necklaces of length n that can be constructed using a set of beads with α-colors is computed by a formula

$$M(\alpha,n) = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)\alpha^d = \frac{1}{n}\sum_{d|n}\mu(d)\,\alpha^{\frac{n}{d}},$$

where μ is the classical Möbius function (see [MR83] for instance). It is called *necklace polynomial.* In this section, we show that there is a relationship between the equation of Theorem 2.6 and the necklace polynomials. Denote by C_n the cyclic group of order n. Denote by $\mathscr{S}_G$ the poset $(s_G, \leq)$ of the subgroups of G ordered by inclusion. Denote by $\mathscr{D}(n)$ the divisor poset of a positive integer n ordered by divisibility relation. If m is a divisor of n, then there exists an isomorphism of posets from the closed interval $[C_m, C_n]_{\mathscr{S}_G}$ to $\mathscr{D}\left(\frac{n}{m}\right)$. The following lemma is well known.

Lemma 4.1. *If C_d is an element of $[C_m, C_n]_{\mathscr{S}_G}$, then*

$$\mu_{\mathscr{S}_G}(C_m, C_d) = \mu_{\mathscr{D}(\frac{n}{m})}\left(1, \frac{d}{m}\right).$$

In particular, $\mu_{\mathscr{S}_G}(C_m, C_d) = \mu\left(\frac{d}{m}\right)$.

Theorem 4.2. *If G is a cyclic group of order n, then $f^G_{C_m}(x) = M\left(x, \frac{n}{m}\right)$ for any divisor m of n.*

Proof. By the definition of $f^G_{C_m}(x)$ and Lemma 4.1,

$$\begin{aligned} f^G_{C_m}(x) &= |WC_m|^{-1}\sum_{S\leq C_n}\mu(C_m, S)x^{|G/S|} \\ &= \left|\frac{n}{m}\right|^{-1}\sum_{C_d\leq C_n}\mu(C_m, C_d)x^{|C_n/C_d|} \\ &= \left|\frac{n}{m}\right|^{-1}\sum_{\frac{d}{m}|\frac{n}{m}}\mu\left(\frac{d}{m}\right)x^{\frac{n/m}{d/m}} \\ &= M\left(x, \frac{n}{m}\right). \end{aligned}$$

□

Theorem 4.2 and Theorem 2.6 show the following.

Corollary 4.3. *If G is a cyclic group of order n and ℓ is a positive integer, then*

$$\ell^G = \sum_{m|n} M\left(\ell, \frac{n}{m}\right)[G/C_m].$$

REFERENCES

[Br05] Brun, M.: *Witt vectors and Tambara functors.* Advances in Mathematics **193** (2005), 233–256.
[Gl81] Gluck, D.: *Idempotent formula for the Burnside ring with applications to the p-subgroup simplicial complex.* Illinois J. Math. **25** No.1 (1981), 63–67.
[MR83] Metropolis, N.; Rota, Gian-Carlo.: *Witt vectors and the algebra of necklaces.* Adv. Math. **50** (2) (1983), 95–125.
[Na12a] Nakaoka, H.: *On the fractions of semi-Mackey and Tambara functors.* J. Algebra **352** (2012), 79–103.
[Na12b] Nakaoka, H.: *Ideals of Tambara functors.* Adv. Math. **230** (2012), 2295–2331.
[Na13] Nakaoka, H.: *A generalization of the Dress construction for a Tambara functor, and its relation to polynomial Tambara functors.* Adv. Math. **235** (2013), 237–260.
[OY04] Oda, F.; Yoshida, T.: *Crossed Burnside rings II: The Dress construction of a Green functor.* J. Algebra **282** (2004), 58–82.
[OY11] Oda, F.; Yoshida, T.: *The crossed Burnside rings III: The Dress construction for a Tambara functor.* J. Algebra **327** (2011), 31–49.
[Ta93] Tambara, D.: *On multiplicative transfer.* Comm. Algebra **21** (1993), no. 4, 1393–1420.
[Yo83] Yoshida, T.: *Idempotents of the Burnside rings and Dress induction theorem.* J. Algebra **80** (1983), 90–105.
[Yo90] Yoshida, T.: *On the unit groups of Burnside rings.* J. Math. Soc. Japan **42** (1990), 31–64.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND ENGINEERING
KINKI UNIVERSITY
HIGASHI-OSAKA 577-8502
JAPAN
E-MAIL ADDRESS : ODAF@MATH.KINDAI.AC.JP

近畿大学・理工学部理学科数学コース　小田 文仁