

Surgery along a projective plane in a 4-manifold and D_4 -singularity

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1 Introduction

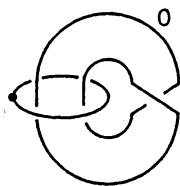
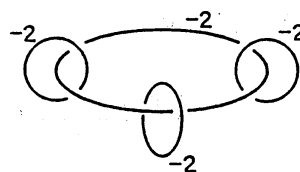
The *Price surgery* has been defined in [P, KSTY, Y3] as a cut and paste of a 4-manifold N_2 in the 4-sphere S^4 and also in general 4-manifolds, where N_2 is defined as a total space of a non-orientable D^2 -bundle over a projective plane with normal Euler number 2 (see [M1, M2, L1, Y1]). It may be expected to make a *fake pair* of 4-manifolds, which means a pair that are homotopy equivalent but non-diffeomorphic to each other, but such a trial seems not to be succeeded yet except the non-orientable example [A1, A2] (see also [KSTY]: Gluck surgery ([Gl]) is realized by Price surgery).

The 4-manifold N_2 is represented by the framed link (see [Ki, GoS]) in Figure 1(1). The boundary ∂N_2 is homeomorphic to the quaternion space Q , which is the quotient space of the unit sphere S^3 of the quaternion field $\mathbf{H} \cong \mathbf{R}^4$ by the quaternion group of order 8. This space Q is also homeomorphic to the linking 3-manifold of D_4 -singularity: $S^5 \cap \{f^{-1}(0)\}$, where

$$\begin{aligned} f : \quad \mathbf{C}^3 &\rightarrow \mathbf{C} \\ (x, y, z) &\mapsto x^2 + y^3 + z^3, \end{aligned}$$

and we regard S^5 as the unit sphere (the boundary of the unit disk D^6) in $\mathbf{C}^3 \cong \mathbf{R}^6$. Throughout the paper, by the notation D_4 we denote the compact 4-manifold obtained from $D^6 \cap \{f^{-1}(0)\}$ by resolve the singularity minimally, which is represented by the framed link in Figure 1(2).

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Title of the author's talk on 31 May was slightly different from that of this report.

Figure 1(1) : N_2 Figure 1(2) : D_4

The boundaries of the two 4-manifolds N_2 and D_4 are homeomorphic to each other and also to Q , thus we can define

Operation : “Cut D_4 off and paste N_2 on” a 4-manifold,

but the resulting 4-manifold is not well-defined because of the ambiguity of the gluing map (self-homeomorphisms on Q). Thus, for a given D_4 in an original 4-manifold M , we study the set $\Omega_M(D_4)$ (consisting of at most three elements, see Section 2) of diffeomorphic class of resulting 4-manifolds.

This operation changes some topological invariants of the ambient 4-manifold: it decreases the Euler characteristic number χ by 4, the negative second Betti number β_2^- by 4 and do not change the positive second Betti number β_2^+ , thus increases the signature σ of the 4-manifold by 4.

In this paper, we will report two lemmas related to the operation. One is Lemma 3.1 in § 3, which says that a certain operation consisting of four blowing up's and the operation above is reduced to Price surgery. The other is Lemma 4.1 in § 4 on the resulting manifolds of the operation on the simple elliptic surfaces. Before stating the results, in the next section, we will recall some facts on Price surgery in general 4-manifolds. In §5, we will show some key lemmas by “relative Kirby calculus” (see Section 5.5 in [GoS]), but we do not give the complete proof.

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2 Price surgery

We recall notations and facts on Price surgery from [KSTY, Y3].

- (1) We denote by N_2 the total space of a non-orientable D^2 -bundle over a projective plane with normal Euler number 2, which is a compact oriented 4-manifold with a boundary, and which is described by the Kirby diagram in Figure 1(1). Note that N_2 has a handlebody decomposition with one 0-handle, one 1-handle and one 2-handle.
- (2) The boundary ∂N_2 is diffeomorphic to the quaternion space Q , which admits a Seifert fibered structure whose Seifert invariants in the sense of [O, §5.2] are given by $\{-1; (\sigma_1, 0); (2, 1), (2, 1), (2, 1)\}$. We call the three singular fibers c_{-1}, c_0, c_1 .
- (3) In [P], Price has investigated the self-diffeomorphisms of the quaternion space Q and has shown that the mapping class group $\mathcal{M}(Q)$ (the group of isotopy classes of orientation preserving self-diffeomorphisms) is isomorphic to $\mathcal{S}_3$, the symmetric group on three letters $\{-1, 0, 1\}$. For each element σ in $\mathcal{S}_3$, there is a self-diffeomorphism f_σ of Q which preserves the Seifert fibered structure and satisfies $f_\sigma(c_i) = c_{\sigma(i)}$. Each map f_σ represents the class of $\mathcal{M}(Q)$.
- (4) Price has also shown that there is a self-diffeomorphism g (g_1 in [P, p.116]) of $Q = \partial N_2$ whose order is two in $\mathcal{M}(Q)$ and that can extend over N_2 as a self-diffeomorphism. (In fact, g is a bundle isomorphism “ $-$ ” : $N_2 \rightarrow N_2$ which maps each vector $\vec{v}$ to $-\vec{v}$.) Thus, for a given oriented 4-manifold E whose boundary is $-Q$, we have at most only three 4-manifolds up to diffeomorphism $E \cup_{i \circ \varphi} N_2$ obtained by gluing N_2 to E along the boundary. where we use the compositions of a fixed orientation reversing map i from ∂N_2 to ∂E and an orientation preserving self-diffeomorphism φ on Q as the gluing map. The three 4-manifolds correspond to the classes of φ in the right coset $\mathcal{M}(Q)/\{1, g\}$, which consists of three elements.

3 Equivalence of two operations

Let M be a closed oriented 4-manifold and K a smoothly embedded 2-sphere in M whose normal bundle is trivial. We define two operations **A** and **B** on M along K .

Operation A: Taking a pairwise connected sum of (M, K) with the (positive) standard projective plane (S^4, P_0) (see [PR], [L1], [Y1]), we have an embedded projective plane $(M, K \# P_0)$ in M whose normal Euler number 2. The tubular neighborhood $N(K \# P_0)$ is diffeomorphic to N_2 . Let $\Pi_M(K \# P_0)$ be the set of diffeomorphic class of 4-manifolds obtained by pasting N_2 to the exterior $M \setminus \text{int} N(K \# P_0)$ along the boundary. The original manifold M itself and the Gluck surgery $\Sigma_M(K)$ of M along K , by Theorem 4.1 in [KSTY], are contained in the set $\Pi_M(K \# P_0)$. By (4) in Section 2, $\Pi_M(K \# P_0)$ consists of at most three elements.

Operation B: This operation consists of five steps, see Figure 2: (1) Blow up at a point in K . (2) Blow up at the intersection point of the proper lift of K and the exceptional curve. (3) Blow up at a point on the newest exceptional curves. (4) Blow up at a point on the newest exceptional curves again. After this step, we have a D_4 in the ambient 4-manifold $M \# 4\overline{\mathbb{C}P^2}$. (5) Do the operation “Cut D_4 off and paste N_2 on” the 4-manifold. By $\Omega_M(D_4(K))$, we denote the set of the of diffeomorphic class of the resulting 4-manifolds.

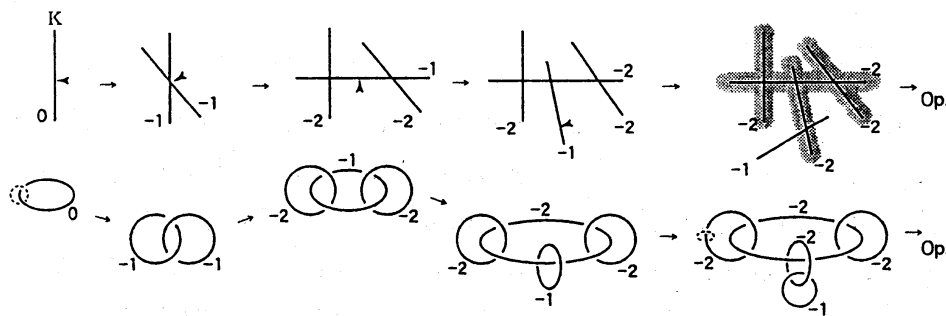


Figure 2

Lemma 3.1 *Two operations **A** and **B** along K on M are equivalent, i.e., it holds that $\Pi_M(K \# P_0) = \Omega_M(D_4(K))$ as sets.*

4 Operation on elliptic surfaces

Let $E(n)$ be the simply connected elliptic surface (with section) whose Euler characteristic is $12n$, ($E(1) \cong \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$). $E(n)$ is the fiber sum of n copies of $E(1)$. $E(2) \cong$ “the $K3$ surface”, $\dots$). In [BGo], using a method “relative Kirby diagram” (see Section 5.5 in [GoS]), a decomposition of $E(n)$ as a union of $n + 1$ pieces $N_n \cup W_n \cup W_{n-1} \cup \dots \cup W_1$ has been shown, where N_n is the *nuclei* of $E(n)$ ([Go]) and W_1 is the E_8 -plumbing. Each W_j ($j \geq 2$) is a cobordism represented by the relative Kirby diagram in Figure 3 (modified from Figure 27 in [BGo]), which clearly contains one E_8 -plumbing. An E_8 -plumbing contains an obvious D_4 . Thus we can do the operation “Cut D_4 off and paste N_2 on” $E(n)$ at most n times. To study the resulting 4-manifolds, we do the operation on W_j . For W_1 , see Lemma 5.2.

Lemma 4.1 *The resulting 4-manifold of the operation “Cut D_4 off and paste N_2 on” W_j ($j \geq 2$) does not depend on the gluing map of ∂N_2 and is diffeomorphic to $\mathcal{W}_j \# 4\overline{\mathbb{C}P^2}$, where $\mathcal{W}_j$ is the 4-manifold represented by the relative Kirby diagram in Figure 4.*

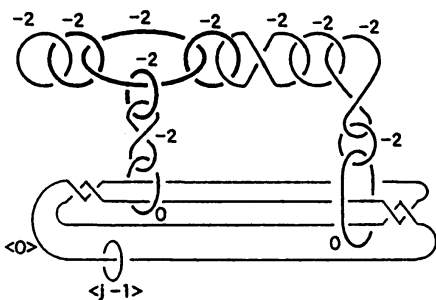


Figure 3 : W_j

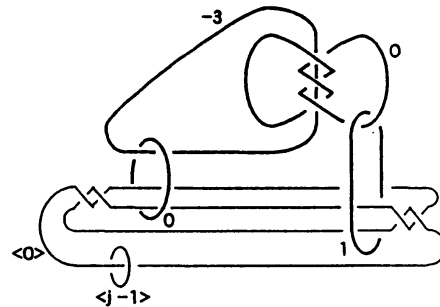


Figure 4 : $\mathcal{W}_j$

Note that W_j is, thus $\mathcal{W}_j$ is also a cobordism from the Seifert homology 3-sphere $-\Sigma(2, 3, 6(j-1)-1)$ to $\Sigma(2, 3, 6j-1)$ for $j \geq 2$. We conjecture that all the resulting 4-manifolds, the (non-trivial) union of possible W_j 's and $\mathcal{W}_j$'s capped by N_n and their “logarithmic transformation” (as a 4-manifold, not as a complex surface) in N_n are all diffeomorphic to $\beta_2^+(\mathbb{C}P^2) \# \beta_2^-(\overline{\mathbb{C}P^2})$.

5 Key of the proof

We show some key lemmas for Lemma 3.1 and give a proof of Lemma 4.1. They are shown by (ordinary) Kirby calculus and relative Kirby calculus (see Section 5.5 in [GoS]).

Lemma 5.1 See the Kirby calculus from the diagram (A) to (B) of 3-manifolds in Figure 5. It corresponds to a homeomorphism φ from the boundary ∂D_4 of D_4 to ∂N_2 . Calculating the curves c_i 's with 0-framing in (A) during the process of the Kirby calculus, we get the curves c_i 's with framings $(\cdot)$ in (B). They are $\varphi(c_i)$'s in ∂N_2 . Thus (under some conditions) the local change from (A) to (B) in a Kirby diagram of a 4-manifold M corresponds to (one of) the operation "Cut D_4 off and paste N_2 on" M .

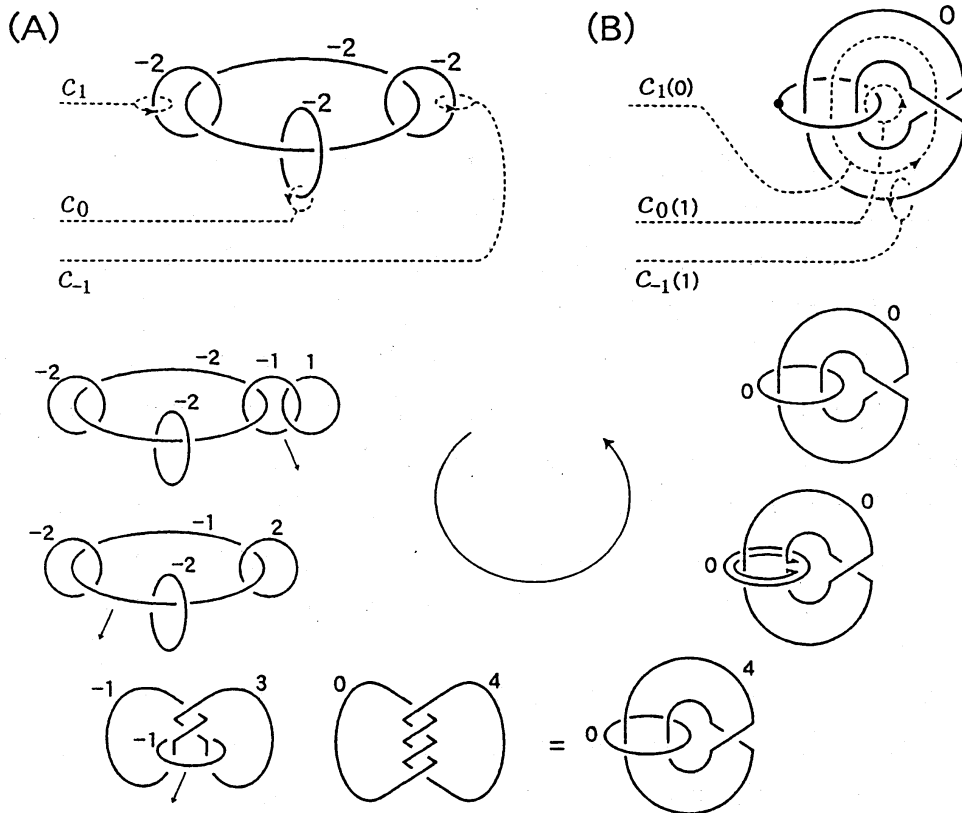


Figure 5

Of course, another Kirby calculus from the diagram (A) to (B) corresponds to another homeomorphism from ∂D_4 to ∂N_2 . To prove Lemma 4.1 completely, we need every (six or three) calculus from the diagram (A) to (B) for each element of the mapping class group $\mathcal{M}(Q)$ of order six, but in this paper, we omit the other calculus.

Now we use Lemma 5.1 to study the resulting 4-manifold of the operation "Cut D_4 off and paste N_2 on" the obvious D_4 in the E_8 -plumbing W_1 .

Lemma 5.2 The resulting 4-manifold is diffeomorphic to $\mathcal{W}_1 \# \overline{3\mathbb{CP}^2}$, where $\mathcal{W}_1$ is the 4-manifold represented by the final Kirby diagram (-1 -framed left-hand trefoil) in Figure 6.

Proof. See the Kirby calculus in Figure 6. $\square$

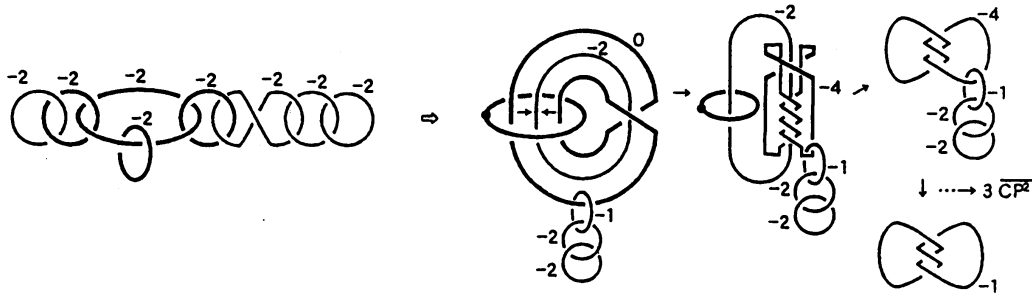


Figure 6

Lemma 4.1 is shown by application of such method.

Note that the action of $\mathcal{M}(Q) (\cong \mathcal{S}_3)$ on ∂D_4 is obvious. Thus we can calculate every resulting 4-manifold of the operation on D_4 in the E_8 -plumbing for each choice of the gluing map in $\mathcal{M}(Q)$. For a smoothly embedded 2-sphere K in S^4 , we can also study the resulting 4-manifold of the operation cut the D_4 and paste an exterior $-X(P_0 \# K)$ of a projective plane $P_0 \# K$ in S^4 instead of N_2 ($N_2 \cong -X(P_0)$, see [PR, P, L1, L2, Y1, Y2]) by the method “circle with a dot and with a symbol K ” in Kirby diagrams introduced in Appendix of [KSTY]. They are all diffeomorphic to $\mathcal{W}_1 \# 3\overline{\mathbb{C}P^2}$. Note that the Gluck surgery $\Sigma(K)$ along any K in S^4 satisfies that $\Sigma(K) \# \overline{\mathbb{C}P^2} \cong \overline{\mathbb{C}P^2}$.

Outline of the proof of Lemma 3.1: See the Kirby calculus in Figure 2 again. It describes the process of operation **B** near the 2-sphere K , but we have not done the final step yet. Doing the change in Lemma 5.1 to the final diagram, we finish the operation **B** and get the first diagram in Figure 7 (The dotted circle corresponds to a meridian to K in M . The thin circle corresponds to the boundary of a co-core of the 2-handle h . Once ignore them). The diagram describes a 4-manifold obtained by attaching a 2-handle h to N_2 . All we have to do is to verify that $(M \setminus \text{int} N(K)) \cup h^\perp \cong M \setminus \text{int} N(K \# P_0)$, where we use the notation $h^\perp$ for the piece h since we switch the core and the co-core. See Figure 4(1) and the proof of Theorem 4.1 in [KSTY] for the goal.

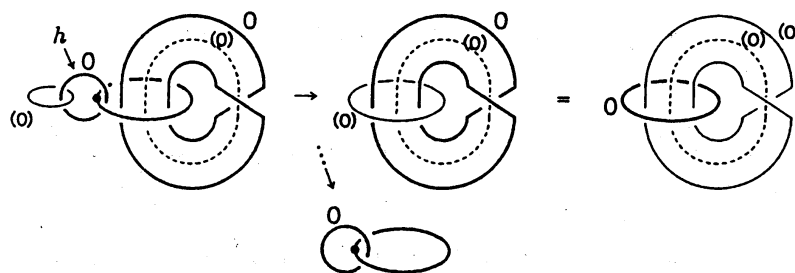


Figure 7

By the calculus in Figure 7, we have the attaching circle of $h^\perp$ in $\partial(M \setminus \text{int} N(K)) \cong S^1 \times S^2$ and the framing: it is the thin circle in the diagram. (If one care orientation of the diagram, it would be better take the mirror image.) We have the lemma. $\square$

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