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On the Stable Reduction of $X_0(5^4)$ and $X_0(7^4)$

By

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Abstract

R. Coleman and K. McMurdy calculated the stable reduction of $X_0(p^3)$ on the basis of the rigid geometry in [CM]. In [T], we determine the stable model of $X_0(p^4)$ for primes $p \geq 13$ on the basis of their idea. In this paper, we compute the stable model of $X_0(p^4)$ for the remaining cases $p = 5, 7$. The stable model of $X_0(11^4)$ is expected to be calculated in the same way as the cases $p = 5, 7$.

§ 1. Introduction

By a model for a scheme X over a complete local field K, we mean a scheme $\mathcal{X}$ over the ring of integers $\mathcal{O}_K$ of K such that $X \simeq \mathcal{X} \otimes_{\mathcal{O}_K} K$. When a curve C over K does not have a model with good reduction over $\mathcal{O}_K$, it may have the "next best thing," i.e., *a stable model*. The stable model is unique up to isomorphism if it exists, and it does over the ring of integers in some finite extension of K, as long as the genus of the curve is at least 2, which is proved by Deligne and Mumford in [DM]. Moreover, if $\mathcal{C}$ is a stable model for C over $\mathcal{O}_K$, and $K \subset L \subset \mathbb{C}_p$, then $\mathcal{C} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ is a stable model for $C \otimes_K L$ over $\mathcal{O}_L$. The special fiber of any stable model for C is called the *stable reduction*.

In the following, we focus on the modular curve $X_0(p^n)$. Let n be an integer and p a prime number. It is known that if $n \geq 3$ and $p \geq 5$, or if $n \geq 1$ and $p \geq 11$ except for $(n, p) = (1, 13)$, the modular curve $X_0(p^n)$ does not have a model with good reduction over the ring of integers of any complete subfield of $\mathbb{C}_p$. The stable models of $X_0(p)$ and $X_0(p^2)$ were previously known, due to works of Igusa and Deligne-Rapoport [DR, Section 7.6], and B. Edixhoven [E, Theorem 2.1.2] respectively. In [CM], Coleman-McMurdy determine the stable reduction of $X_0(p^3)$ for primes $p \geq 13$. Furthermore, they

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also compute the stable model of $X_0(p^3)$ for $p = 5, 7, 11$ in [CM2]. In [T], we determine the stable model of $X_0(p^4)$ for $p \geq 13$. The stable model of $X_0(3^4)$ is calculated in [Mc]. See [CM, Introduction] for other prior results regarding the stable models of modular curves at prime power levels. Using the type theory of Bushnell-Kutzko, J. Weinstein conjectures on the stable model of the modular curve $X_0(p^n)$ in [W].

In order to compute the stable reduction of $X_0(p^3)$ and $X_0(p^4)$, it is necessary to approximate the forgetful map $\pi_f : X_0(p) \longrightarrow X_0(1)$, over some supersingular locus. To apply de Shalit's approximation formula for π_f in [CM, Theorem 3.5], [dSh] or (3.1), we need an existence of an elliptic curve over $\mathbb{F}_p$, whose j-invariant is not equal to $0, 1728$. By a result of E. Howe in [CM, Theorem A.1], one always has such an elliptic curve as long as $p \geq 13$. In cases $p = 5, 7, 11$, there is *no* such supersingular elliptic curve. Therefore, Coleman-McMurdy use a direct approximation formula for π_f to compute the stable reduction of $X_0(p^3)$ for $p = 5, 7, 11$ in [CM2, subsections 7.2-7.4]. In the following, we briefly explain the reason why the approximation formulas for π_f mentioned above is needed to determine the stable reduction of $X_0(p^3)$ in [CM] and in [CM2, subsections 7.2-7.4]. Coleman-McMurdy define a subspace $\mathbf{Z}^A_{1,1} \subset X_0(p^3)$, whose reduction becomes an irreducible component in the stable reduction of $X_0(p^3)$. The main part in loc. cit. is in computing the reduction of the space $\mathbf{Z}^A_{1,1}$. To compute the reduction of this space, they consider an embedding of it into a product of $X_0(p)$ and determine its image as in (3.2) and (3.3). Under these identifications, by the approximation formulas for π_f mentioned above, we know explicit defining equations of the spaces $\mathbf{Z}^A_{1,1}$. Hence, we can compute the reduction $\overline{\mathbf{Z}}^A_{1,1}$.

In my talk at the conference, I reported on the stable model of $X_0(p^4)$ for $p \geq 13$. In this paper, we review the shape of the stable model of $X_0(p^4)$ with $p \geq 13$, which is given in [T], in section 2, and compute explicitly the stable models of $X_0(5^4)$ and $X_0(7^4)$ in section 3. To compute the stable models of $X_0(p^4)$ $(p = 5, 7)$, we use the explicit approximation formulas for $\pi_f : X_0(p) \longrightarrow X_0(1)$ over a supersingular locus, which are given in [CM2, subsections 7.2 and 7.3]. The main part in an explicit computation of the stable reduction of $X_0(p^4)$ is in calculating the reduction of a subspace $\mathbf{Y}^A_{2,2} \subset X_0(p^4)$. The stable reduction of $X_0(11^4)$ is expected to be computed in the same way as the cases $p = 5, 7$. However, to compute the stable reduction of $X_0(11^4)$, we need a more precise approximation formula for π_f than the one given in [CM2, subsection 7.4]. Computations in this paper should serve not only to extend the result of [T], but also to make the construction more concrete.

Notation . We fix some p-adic notation. We let $\mathbb{C}_p$ be the completion of a fixed algebraic closure of $\mathbb{Q}_p$, with integer ring $\mathbb{R}_p$ and with $m_{\mathbb{R}_p}$ the maximal ideal of $\mathbb{R}_p$. For any finite field $\mathbb{F}$ contained in $\bar{\mathbb{F}} := \mathbb{R}_p/m_{\mathbb{R}_p}$, an algebraic closure of $\mathbb{F}_p$, let $W(\mathbb{F}) \subset \mathbb{R}_p$ denote the ring of Witt vectors of $\mathbb{F}$. Let v denote the unique valuation on $\mathbb{C}_p$ with

$v(p) = 1$, $|\cdot|$ the absolute value given by $|x| = p^{-v(x)}$ and $\mathcal{R} = |\mathbb{C}_p^*| = p^{\mathbb{Q}}$. Throughout the paper, we let K be a complete subfield of $\mathbb{C}_p$ with ring of integers R_K and residue field $\mathbb{F}_K$. For $r \in \mathcal{R}$, we let $B_K[r]$ and $B_K(r)$ denote the closed and open disks over K of radius r around 0, i.e. the rigid spaces over K whose $\mathbb{C}_p$-valued points are $\{x \in \mathbb{C}_p : |x| \leq r\}$ and $\{x \in \mathbb{C}_p : |x| < r\}$ respectively. If $r, s \in \mathcal{R}$ and $r \leq s$, let $A_K[r, s]$ and $A_K(r, s)$ be the rigid spaces over K whose $\mathbb{C}_p$-valued points are $\{x \in \mathbb{C}_p : r \leq |x| \leq s\}$ and $\{x \in \mathbb{C}_p : r < |x| < s\}$, which we call closed annuli and open annuli. By the width of such an annulus, we mean $\log_p(s/r)$. A closed annuli of width 0 will be called a circle, which we will also denote the circle, $A_K[s, s]$, by $C_K[s]$.

§ 2. Overview of the stable model of $X_0(p^4)$ when $p \geq 13$

In this section, we review a construction of the stable model of $X_0(p^4)$ ($p \geq 13$) given in [T].

Over $\mathbb{C}_p$, we may think of points on the modular curve $X_0(p^n)$ as corresponding to isomorphism classes of pairs (E, C) where $E/\mathbb{C}_p$ is an elliptic curve and C is a cyclic subgroup of order p^n.

The approach of [T] is rigid analytic as in [CM]. Our strategy to find the stable model is the same as the one in loc. cit. Namely, we construct a stable model of $X_0(p^4)$ by actually constructing a stable covering by wide opens. The concept of the stable covering is invented by Coleman to compute the stable reduction of a curve over a local field. Roughly speaking, the wide open subspaces in a stable covering intersect each other in disjoint annuli and have underlying affinoids with good reduction. Each irreducible component in the stable reduction is the reduction of one of these underlying affinoids and the annuli of intersection reduce to the ordinary double points where components intersect. See [CM, subsections 2.2 and 2.3] or [CW, Section 1] for the notions of wide open space and stable covering. The groundwork in the rigid analytic setting has been done in [CM, Section 2]. See also [C1].

The ordinary (resp. supersingular) region or locus of $X_0(p^n)$ means a set of isomorphism classes of pairs $(E, C) \in X_0(p^n)$ with the reduction $\overline{E}$ an ordinary (resp. supersingular) elliptic curve. The geometry of the ordinary region of $X_0(p^n)$ is well-understood. A covering of the ordinary locus of the modular curve $X_0(p^n)$ can be obtained by extending the ordinary affinoids $\mathbf{X}_{a,b}^{\pm}$ with $a + b = n, a \geq 0, b \geq 0$ defined in [C2] and recalled in [CM2, subsection 2.1] to wide open neighborhoods $W_{a,b}^{\pm}$. See also [KM, Section 13] for the treatment of the ordinary locus. The ordinary regions of $X_0(p^2)$ and $X_0(p^3)$ are covered by four wide opens $W_{2,0}, W_{1,1}^{\pm}, W_{0,2}$ and six wide opens $W_{3,0}, W_{2,1}^{\pm}, W_{1,2}^{\pm}, W_{0,3}$ respectively as in [CM, subsection 3.2, Theorem 5.3 and Theorem 9.2]. Similarly as the stable coverings of $X_0(p^2)$ and $X_0(p^3)$, the ordinary region of $X_0(p^4)$ is covered by eight wide opens, which are denoted by $W_{4,0}, W_{3,1}^{\pm}, W_{2,2}^{\pm}, W_{1,3}^{\pm}$ and

$W_{0,4}$. These spaces contain affinoid subdomains $\mathbf{X}^{\pm}_{a,b}$ $(a+b=4, a\geq 0, b\geq 0)$, whose reduction are known to be the Igusa curves $\mathrm{Ig}(p^{\min(a,b)})$ by [C2] or [CM, Proposition 3.6]. Let $\overline{W}^{\pm}_{a,b}$ denote the reduction of the space $W^{\pm}_{a,b}$.

The supersingular locus essentially breaks up into the union of finitely many deformation spaces of height 2 formal groups with level structure. We produce a covering of the supersingular locus on the basis of Coleman-McMurdy's ideas in [CM] and [Mc]. Finally, we compute the "genus" of the covering to show that it is equal to the genus of $X_0(p^4)$, and then conclude that the overall covering is stable.

For a fixed supersingular elliptic curve $A/\mathbb{F}_{p^2}$, let $W_A(p^n)$ be the subspace of $X_0(p^n)$ consisting of pairs (E,C) where $\overline{E}\simeq A$. We set $i(A)=|\mathrm{Aut}(A)|/2$. To analyze the supersingular locus $W_A(p^n)$, we use the theory of canonical subgroups due to Katz-Lubin-Buzzard. See [Ka] and [B] for the canonical subgroup. Let $E/\mathbb{C}_p$ be an elliptic curve such that $\overline{E}\simeq A$. The size of the canonical subgroup of E, denoted by $K(E)$, is measured by the valuation of the Hasse invariant of A. We denote the valuation by $h(E)$. We have the following result

$$|K(E)|>p^n \Longleftrightarrow h(E)<p^{1-n}/(p+1).$$

The space $W_A(p)$ is known to be isomorphic to an annulus $A(p^{-i(A)},1)$. We fix an isomorphism $W_A(p)\simeq A(p^{-i(A)},1)$ satisfying $v(x_A(E,C))=i(A)h(E)$ if C is a canonical subgroup, and $v(x_A(E,C))=i(A)(1-h(E/C))$ otherwise. For a rational number α such that $1<\alpha<p^{-i(A)}$, let $\mathbf{C}^A_\alpha\subset W_A(p)$ be the subspace corresponding to the circle $C[p^{-\alpha}]$ under the identification $W_A(p)\simeq A(p^{-i(A)},1)$. The space $W_A(p^2)$ is known to be a basic wide open space by [CM, Section 5]. Unlike $W_A(p^2)$, however, $W_A(p^3)$ and $W_A(p^4)$ must themselves be covered by smaller wide opens, because their reduction contains multiple irreducible components as mentioned in [CM, Section 1.1].

We define several rigid analytic subspaces of $W_A(p^n)$ on the basis of the idea of [Mc, subsection 5.1], whose reduction plays a key role in the construction of the stable model of the modular curves.

We focus on the circles $\mathbf{TS}_A:=C[p^{-i(A)\frac{p}{p+1}}]\subset W_A(p)$ and $\mathbf{SD}_A:=C[p^{-i(A)/2}]\subset W_A(p)$ under the above identification $W_A(p)\simeq A(p^{-i(A)},1)$. Let $\pi_f,\pi_v: X_0(p^n)\longrightarrow X_0(p^{n-1})$ be level-lowering maps given by $\pi_f(E,C)=(E,pC)$ and $\pi_v(E,C)=(E/p^{n-1}C, C/p^{n-1}C)$. Let $a,b\in\mathbb{Z}_{\geq 0}$. We put $\pi_{a,b}:=\pi_v^a\circ\pi_f^b$.

Assume that a,b are positive integers. We define as follows

$$\mathbf{Y}^A_{a,b}:=\pi^{-1}_{a,b-1}(\mathbf{TS}_A)\subset W_A(p^n)$$

with $a+b=n\geq 2$ and

$$\mathbf{Z}^A_{a,b}:=\pi^{-1}_{a,b}(\mathbf{SD}_A)\subset W_A(p^n)$$

with $a+b=n-1\geq 2$. As mentioned above, the reduction of these spaces plays a fundamental role in the stable models of modular curves. Let $\overline{\mathbf{Y}}^A_{a,b}$ and $\overline{\mathbf{Z}}^A_{a,b}$ denote

the reduction of the spaces $\mathbf{Y}^A_{a,b}$ and $\mathbf{Z}^A_{a,b}$ respectively. The main parts of the works [E] and [CM] are in calculating the reduction of $\mathbf{Y}^A_{1,1} \subset W_A(p^2)$ and $\mathbf{Z}^A_{1,1} \subset W_A(p^3)$ respectively. Similarly, the main part in [T] is in calculating the reduction $\overline{\mathbf{Y}}^A_{2,2}$. In [CM, Lemma 5.1 and Proposition 7.1], to calculate the reduction $\overline{\mathbf{Y}}^A_{1,1}$ and $\overline{\mathbf{Z}}^A_{1,1}$, Coleman-McMurdy consider an embedding of them into a product of the subspaces of $W_A(p)$ and apply de Shalit's approximation theorem for π_f.

In the following, we will explain the shape of the stable model of $X_0(p^4)$ for $p \geq 13$. We fix a supersingular elliptic curve $A/\mathbb{F}_p$ with $j(A) \neq 0, 1728$ and analyze the locus $W_A(p^4)$. The existence of such elliptic curve is guaranteed by a result of Howe in [CM, Theorem A. 1] as mentioned in the introduction. First of all, the reduction of $W_A(p^4)$ contains two isomorphic lifts $\overline{\mathbf{Y}}^A_{1,3}$ and $\overline{\mathbf{Y}}^A_{3,1}$ of a supersingular component $\overline{\mathbf{Y}}^A_{1,1}$ of $X_0(p^2)$, with each meeting exactly three of the ordinary components. For example, the reduction $\overline{\mathbf{Y}}^A_{3,1}$ meets the reduction of $W_{4,0}, W^{\pm}_{3,1}$. The component $\overline{\mathbf{Y}}^A_{1,1}$ of $X_0(p^2)$ is the "horizontal component" found by Edixhoven in [E, Theorem 2.1.2]. The curve $\overline{\mathbf{Y}}^A_{1,1}$ is defined by the equation

$$xy(x-y)^{p-1} = 1$$

and its genus is equal to $(p-1)/2$. Coleman-McMurdy give a rigid analytic interpretation to the horizontal component of Edixhoven in [CM, Proposition 5.2]. Furthermore, the reduction of $W_A(p^4)$ contains two isomorphic lifts $\overline{\mathbf{Z}}^A_{1,2}$ and $\overline{\mathbf{Z}}^A_{2,1}$ of a supersingular component $\overline{\mathbf{Z}}^A_{1,1}$ in the stable reduction of $X_0(p^3)$. The component $\overline{\mathbf{Z}}^A_{1,1}$ in the stable reduction of $X_0(p^3)$ is found by Coleman-McMurdy in [CM, Proposition 8.2], which they call the "bridging component". The curve $\overline{\mathbf{Z}}^A_{1,1}$ is defined by

$$Z^p + X^{p+1} + X^{-(p+1)} = 0$$

and its genus is equal to 0. This curve has $2(p+1)$ singular points at $X = \zeta$ with $\zeta^{2(p+1)} = 1$. Moreover, in the stable reduction of $X_0(p^3)$, the component $\overline{\mathbf{Z}}^A_{1,1}$ meets (in distinct points) a certain number of isomorphic copies of a curve of genus $(p-1)/2$ defined by $a^p - a = s^2$. This phenomenon is first observed by Coleman-McMurdy in the stable reduction of $X_0(p^3)$. Similarly as above, in the stable reduction of $X_0(p^4)$, the components $\overline{\mathbf{Z}}^A_{1,2}$ and $\overline{\mathbf{Z}}^A_{2,1}$ meet (in distinct points) a certain number of isomorphic copies of a curve of genus $(p-1)/2$ defined by $a^p - a = s^2$. In the stable reduction of $X_0(p^4)$, these two "old" components $\overline{\mathbf{Z}}^A_{1,2}$ and $\overline{\mathbf{Z}}^A_{2,1}$ are connected through a central component $\overline{\mathbf{Y}}^A_{2,2}$, which we call the "new bridging component" in the stable reduction of $X_0(p^4)$. This curve $\overline{\mathbf{Y}}^A_{2,2}$ is defined by the following equations

$$xy(x-y)^{p-1} = 1,\ Z^p + 1 + \frac{1}{x^{p+1}} + \frac{1}{y^{p+1}} = 0, \tag{2.1}$$

and its genus is equal to $(p-1)/2$. The component $\overline{\mathbf{Y}}^A_{2,2}$ meets exactly two ordinary

components $\overline{W}^{\pm}_{2,2}$. The curve (2.1) has $(p+1)$ singular points at $(x,y)=(-\zeta,\zeta)$ with $\zeta^{p+1}=-1$. To complete the picture, the new bridging component $\overline{\mathbf{Y}}^{A}_{2,2}$ then meets (in distinct points) a certain number of isomorphic copies of a curve of genus $p(p-1)/2$, defined by $a^p-a=t^{p+1}$. This curve is the *Deligne-Lusztig curve for* $SL_2(\mathbb{F}_p)$. This is a new phenomenon that is observed in the stable reduction of $X_0(p^4)$.

In the table below, we give the following pairs (c,d)

- c is the genus of the component $\overline{\mathbf{Y}}^{A}_{a,b}$ $(a+b=4)$ or $\overline{\mathbf{Z}}^{A}_{a,b}$ $(a+b=3)$.

- d is the number of copies of the curve $a^p-a=s^2$ or $a^p-a=t^{p+1}$ which intersect the component $\overline{\mathbf{Z}}^{A}_{a,b}$ $(a+b=3)$ or $\overline{\mathbf{Y}}^{A}_{a,b}$ $(a+b=4)$.

	$\overline{\mathbf{Z}}^{A}_{1,2}, \overline{\mathbf{Z}}^{A}_{2,1}$	$\overline{\mathbf{Y}}^{A}_{1,3}, \overline{\mathbf{Y}}^{A}_{3,1}$	$\overline{\mathbf{Y}}^{A}_{2,2}$
$j(A)=0$	$(0, \frac{2(p+1)}{3})$	$(\frac{p-5}{6}, 0)$	$(\frac{p-5}{6}, \frac{p+1}{3})$
$j(A)=1728$	$(0, p+1)$	$(\frac{p-3}{4}, 0)$	$(\frac{p-3}{4}, \frac{p+1}{2})$
otherwise	$(0, 2(p+1))$	$(\frac{p-1}{2}, 0)$	$(\frac{p-1}{2}, p+1)$

Table 1 : Genera of Supersingular Components of $X_0(p^4)$

Partial dual graphs of the stable reduction of $X_0(p^n)$ $(2 \le n \le 4,\ p \ge 13)$, including one complete supersingular region, are given below.

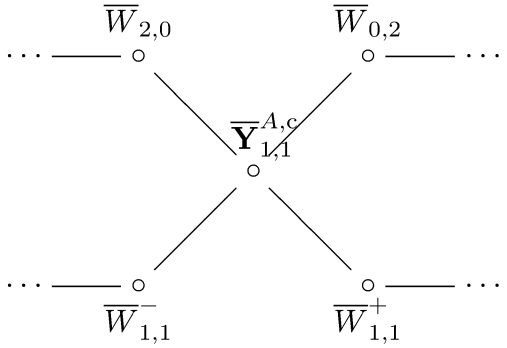

Figure 1 : A partial dual graph of $X_0(p^2)$.

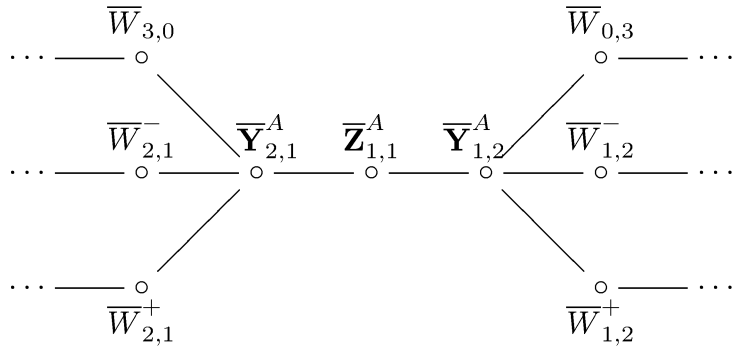

Figure 2 : A partial dual graph of $X_0(p^3)$

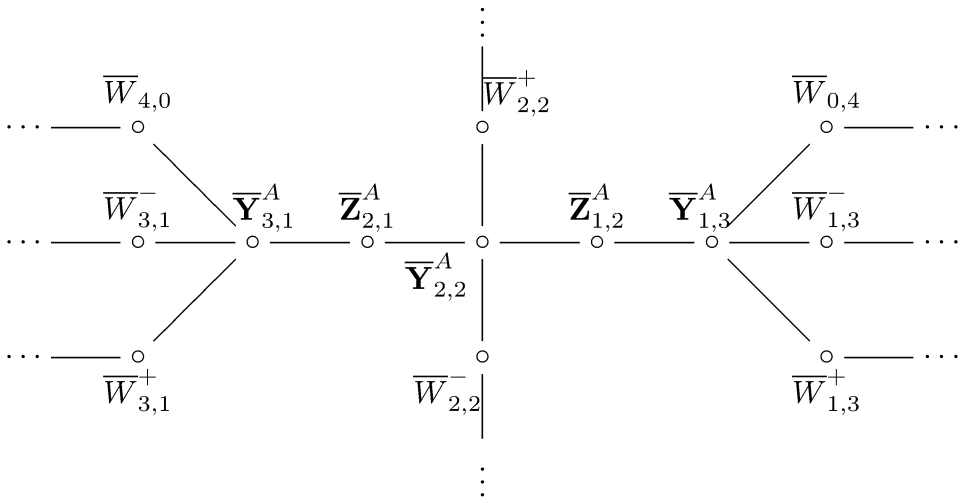

Figure 3 : A partial dual graph of $X_0(p^4)$

§ 3. Stable model of $X_0(p^4)$ for $p = 5, 7$

In this section, we recall briefly how to compute the irreducible components in the stable reduction of $X_0(p^3)$ and $X_0(p^4)$ when $p \geq 13$. We then use explicit equations for $X_0(p)$ when $p = 5, 7$, to derive formulas analogous to the case $p \geq 13$, and subsequently construct the analogous stable reduction components.

§ 3.1. Explicit analysis of a good supersingular region

In 3.1, suppose that $p \geq 13$, and hence by the result of Howe there is a supersingular elliptic curve $A/\mathbb{F}_p$ with $j(A) \neq 0, 1728$. All computations of irreducible components in the stable reduction of $X_0(p^4)$ in [T] can be summarized as follows. First of all, let $w_1 : X_0(p) \longrightarrow X_0(p)$ be the Atkin-Lehner involution given by $w_1(E, C) = (E/C, E[p]/C)$. Then, we have $\pi_v = \pi_f \circ w_1$. We have parameters, T and S, on $W_A(p)$ and $W_A(1)$, which identify these regions with the annulus, $0 < v(T) < 1$, and the disk $v(S) > 0$.

Namely, we fix identifications $W_A(p) \simeq A(p^{-1},1) \ni T$ and $W_A(1) \simeq B(1) \ni S$. Under these identifications, the maps π_f and w_1 satisfy

$$w_1(T) = \frac{\kappa}{T},\ S = \pi_f(T) \equiv T + \left(\frac{\kappa}{T}\right)^p \pmod{p}. \tag{3.1}$$

for some $\kappa \in W(\mathbb{F}_{p^2})$ with $v(\kappa) = 1$. These results follow from the de Shalit theorem in [CM, Theorem 3.5]. Finally, the two circles inside $W_A(p)$, namely $\mathbf{TS}_A$ and $\mathbf{SD}_A$ are defined by $v(t) = \frac{p}{p+1}$ and $v(t) = 1/2$ respectively. Using the above information, we now recall briefly how to explicitly compute the reduction of the affinoids $\mathbf{Z}^A_{1,1} \subset W_A(p^3)$ and $\mathbf{Y}^A_{2,2} \subset W_A(p^4)$. We consider the following embeddings

$$(\pi_{0,2}, \pi_{1,1}, \pi_{2,0}) : W_A(p^3) \hookrightarrow W_A(p)^{\times 3} \simeq A(p^{-1},1)^{\times 3} \ni (X,U,V)$$

and

$$(\pi_{0,3}, \pi_{1,2}, \pi_{2,1}, \pi_{3,0}) : W_A(p^4) \hookrightarrow W_A(p)^{\times 4} \simeq A(p^{-1},1)^{\times 4} \ni (X,U,V,Y)$$

where all isomorphisms are induced by $W_A(p) \simeq A(p^{-1},1) \ni T$ fixed above. These embeddings induce the following descriptions of $\mathbf{Z}^A_{1,1}$ and $\mathbf{Y}^A_{2,2}$ in [T]

$$\mathbf{Z}^A_{1,1} \simeq \{(X,U,Y) \in \mathbf{C}^A_{1/2p} \times \mathbf{C}^A_{1/2} \times \mathbf{C}^A_{1-(1/2p)} |\ \pi_v(X) = \pi_f(U),\ \pi_v(U) = \pi_f(Y)\}.$$

and

$$\mathbf{Y}^A_{2,2} \simeq \{(X,U,V,Y) \in \mathbf{C}^A_{1/p(p+1)} \times \mathbf{C}^A_{1/(p+1)} \times \mathbf{C}^A_{p/(p+1)} \times \mathbf{C}^A_{1-(1/p(p+1))} |$$

$$\pi_v(X) = \pi_f(U),\ \pi_v(U) = \pi_f(V),\ \pi_v(V) = \pi_f(Y),\ w_1(X) \neq Y\}. \tag{3.2}$$

We apply the above approximations (3.1) of π_f and w_1 to deduce the defining equations of $\overline{\mathbf{Z}}^A_{1,1}$ and $\overline{\mathbf{Y}}^A_{2,2}$ in [T]. See the previous section for the defining equations of them.

When $p = 5, 7$, the ideas described above are not able to be applied, because there is no such A. However, without the assumption $j(A) \neq 0, 1728$, the spaces $\mathbf{Y}^A_{2,2}$ and $\mathbf{Z}^A_{1,1}$ have the following descriptions, analogous to the above identifications,

(3.3)

$$\mathbf{Z}^A_{1,1} \simeq \{(X,U,Y) \in \mathbf{C}^A_{i(A)/2p} \times \mathbf{C}^A_{i(A)/2} \times \mathbf{C}^A_{i(A)\{1-(1/2p)\}} |\ \pi_v(X) = \pi_f(U),\ \pi_v(U) = \pi_f(Y)\}$$

and

$$\mathbf{Y}^A_{2,2} \simeq \{(X,U,V,Y) \in \mathbf{C}^A_{i(A)/p(p+1)} \times \mathbf{C}^A_{i(A)/(p+1)} \times \mathbf{C}^A_{pi(A)/(p+1)} \times \mathbf{C}^A_{i(A)\{1-(1/p(p+1))\}} |$$

$$\pi_v(X) = \pi_f(U),\ \pi_v(U) = \pi_f(V),\ \pi_v(V) = \pi_f(Y),\ w_1(X) \neq Y\}. \tag{3.4}$$

In following subsections, to compute the stable reduction of $X_0(p^4)$ for $p = 5, 7$, we use direct approximation formulas for w_1 and π_f given in [CM2, subsections 7.2 and 7.3] with the identifications (3.3) and (3.4).

§ 3.2. $X_0(5^3)$

We recall a parametrization on the genus 0 curve, $X_0(5)$, considered in [CM2, subsection 7.2] and [Mc2, Section 4]. We put $T := \eta_1^6/\eta_5^6$. See [Mc2] for η-function. The only supersingular j-invariant is $j = 0$. We denote by A the unique supersingular elliptic curve with j-invariant 0. Then, the unique supersingular annulus $W_A(p)$ is described by $0 < v(T) < 3$, because of $i(A) = 3$. Namely, we fix an identification $W_A(p) \simeq A(p^{-3}, 1) \ni T$. Furthermore, from [Mc2, Table 3], the formulas for the forgetful map $\pi_f : W_A(p) \to W_A(1) \simeq B(1) \ni j$ and the Atkin-Lehner involution $w_1 : W_A(p) \to W_A(p)$ are given by

$$\pi_f^*(j) = \frac{(T^2 + 2 \cdot 5^3 T + 5^5)^3}{T^5}, \ w_1^*(T) = \frac{5^3}{T}.$$

Therefore, we obtain the following by $\pi_v = w_1 \circ \pi_f$

$$\pi_v^*(j) = T^5 \left(1 + \frac{2 \cdot 5}{T} + \frac{5}{T^2}\right)^3.$$

The circles $\mathbf{SD}_A$ and $\mathbf{TS}_A$ are described by $v(T) = 3/2$ and $v(T) = 5/2$ respectively.

We briefly recall the computation of the reduction of the space $\mathbf{Z}_{1,1}^A := \pi_{1,1}^{-1}(\mathbf{SD}_A)$ from [CM2, subsection 7.2]. In the following, we show that the reduction $\overline{\mathbf{Z}}_{1,1}^A$ is defined by

$$Z^p + x^2 + x^{-2} = 0$$

in Proposition 3.3. This affine curve with genus 0 has singularities at $x \in \mu_4 = \mathbb{F}_5^\times$. After calculating the reduction $\overline{\mathbf{Z}}_{1,1}^A$, we find 4 irreducible components defined by $a^5 - a = s^2$, which attach to the component $\overline{\mathbf{Z}}_{1,1}^A$ at each singular point $x \in \mathbb{F}_5^\times$. Finally, by the genus computation, we conclude that, in the stable reduction of $X_0(5^3)$, no non-trivial component appears except for these components mentioned above. Hence, we obtain the stable reduction of $X_0(5^3)$.

To compute the reduction of $\mathbf{Z}_{1,1}^A$, we use the identification (3.3)

$$\mathbf{Z}_{1,1}^A \simeq \{(X, U, Y) \in \mathbf{C}_{3/10}^A \times \mathbf{C}_{3/2}^A \times \mathbf{C}_{27/10}^A |\ \pi_v(X) = \pi_f(U),\ \pi_v(U) = \pi_f(Y)\}.$$

We choose an element β such that $\beta^{10} = 5^3$. We have $v(\beta) = 3/10$.

For a rational number $r \geq 0$, if we have $v(f - g) > r$, we write $f \equiv g \pmod{r+}$.

We change variables as follows $X = \beta/x, U = \beta^5 u, Y = (5^3/\beta)y$. By $\pi_v(X) = \pi_f(U)$, we acquire the following congruence

$$u \equiv \frac{(1 + 3\frac{5}{\beta^2}x^2)}{x^5} \pmod{(1/2)+}. \tag{3.5}$$

In the same way as above, we obtain the following congruence by $\pi_v(U) = \pi_f(Y)$

$$u^{-1} \equiv \frac{(1 + 3\frac{5}{\beta^2}y^2)}{y^5} \pmod{(1/2)+}. \tag{3.6}$$

By (3.5) and (3.6), the following congruence holds

$$(xy)^5 \equiv 1 + 3\frac{5}{\beta^2}(x^2 + y^2) \pmod{(1/2)+}. \tag{3.7}$$

We choose an element γ such that $\gamma^5 = 3 \cdot 5/\beta^2$. We have $v(\gamma) = 2/25$. We introduce a new parameter Z as follows

$$xy = 1 + \gamma Z. \tag{3.8}$$

Substituting (3.8) to (3.7) and dividing it by $3{\cdot}5/\beta^2$, we acquire the following congruence

$$Z^5 \equiv x^2 + \left(\frac{1 + \gamma Z}{x}\right)^2 \pmod{(1/10)+}. \tag{3.9}$$

We put $F(Z, x) := x^2 + \left(\frac{1+\gamma Z}{x}\right)^2$.

Proposition 3.1 ([CM2, subsection 7.2 (4)]). *Let the notation be as above. Then, the reduction of the space $\mathbf{Z}^A_{1,1}$ is defined by the following equation*

$$Z^5 = x^2 + x^{-2}.$$

This affine curve has genus 0.

Proof. By considering (3.9) modulo 0+, the required assertion follows. □

Remark 3.2. The reduction $\overline{\mathbf{Z}}^A_{1,1}$ has singularities at $x \in \mathbb{F}_5^\times$.

In the following, we prove that there exist four irreducible components defined by $a^5 - a = t^2$ in the stable reduction of $X_0(5^3)$. These components are called "new components" in [CM]. These components attach to the curve $\overline{\mathbf{Z}}^A_{1,1}$ at each singular point $x = \bar{\zeta}$ with $\bar{\zeta} \in \mathbb{F}_5^\times$.

Let $\zeta \in \mu_4(\mathbb{Z}_5)$. We choose an element γ_0 such that $\gamma_0^5 = (\zeta^2 + \zeta^{-2})(1 + \gamma\gamma_0)$. We set $x_0 := \zeta(1 + \gamma\gamma_0)^{1/2}$. By the definitions of γ_0 and x_0, we acquire $\gamma_0^5 = F(\gamma_0, x_0)$.

Then, we can easily check that

- $\partial_x F(\gamma_0, x_0) = 0$.

- $v(\partial_Z F(\gamma_0, x_0)) = v(\gamma) = 2/25$.

- $\partial_x^2 F(\gamma_0, x_0)$ is a unit.

We choose elements α_1 and β_1 such that $\alpha_1^4 = \partial_Z F(\gamma_0, x_0)$ and $\alpha_1^5 = (1/2)\partial_x^2 F(\gamma_0, x_0)\beta_1^2$ respectively. Then, we have $v(\alpha_1) = 1/50, v(\beta_1) = 1/20$.

We change variables as follows

$$x = x_0 + \beta_1 t,\ Z = \gamma_0 + \alpha_1 a. \tag{3.10}$$

Substituting (3.10) to (3.9), we obtain the following congruence by the choices of α_1 and β_1

$$\alpha_1^5(a^5 - a - t^2) \equiv 0 \pmod{(1/10)+}.$$

By dividing this by α_1^5, we acquire $a^5 - a = t^2 \pmod{0+}$. Hence, we have proved the following proposition.

Proposition 3.3. *Let the notation be as above. Then, in the stable reduction of $X_0(5^3)$, there exist four irreducible components defined by $a^5 - a = t^2$.*

Let $g_0(p^n)$ denote the genus of the modular curve $X_0(p^n)$. We have $g_0(5^3) = 8$ by [Sh, Propositions 1.40 and 1.43]. Let $\mathcal{X}_0(p^n)$ denote the stable model of $X_0(p^n)$. In $\overline{\mathcal{X}_0(5^3)}$, we find the component $\overline{\mathbf{Z}}_{1,1}^A$ defined by $Z^5 = x^2 + x^{-2}$ with genus 0 and the four curves $\{\overline{\mathbf{X}}_\zeta\}_{\zeta\in\mu_4}$ defined by $a^5 - a = t^2$ with genus 2, which attach to the component $\overline{\mathbf{Z}}_{1,1}^A$. On the other hand, the graph of $\overline{\mathcal{X}_0(5^3)}$ is a tree by [CM, Theorem 9.4]. Hence, the sum of genera of all irreducible components and the Betti number of the dual graph is equal to 8. Therefore, no non-trivial component except for these components appears in $\overline{\mathcal{X}_0(5^3)}$. Hence, we conclude that $\overline{\mathcal{X}_0(5^3)}$ consists of $\overline{\mathbf{Z}}_{1,1}^A$ and $\{\overline{\mathbf{X}}_\zeta\}_{\zeta\in\mu_4}$.

§ 3.3. $X_0(5^4)$

In this subsection, we will compute the reduction of $\mathbf{Y}_{2,2}^A := \pi_{2,1}^{-1}(\mathbf{TS}_A)$. Recall the following identification given in (3.4)

$$\mathbf{Y}_{2,2}^A \simeq \{(X, U, V, Y) \in \mathbf{C}_{1/10}^A \times \mathbf{C}_{1/2}^A \times \mathbf{C}_{5/2}^A \times \mathbf{C}_{29/10}^A |$$

$$\pi_v(X) = \pi_f(U),\ \pi_v(U) = \pi_f(V),\ \pi_v(V) = \pi_f(Y),\ w_1(X) \neq Y\}.$$

In the following, under the above description, by using the approximation formula for π_f given in the previous subsection, we compute the reduction of the space $\mathbf{Y}_{2,2}^A$ as in Proposition 3.4. The component $\overline{\mathbf{Y}}_{2,2}^A$ has genus 0, and has singularities at two points $(r, s) = (0, \zeta)$ with $\zeta^2 = -1$. Then, we find two components defined by $a^5 - a = t^6$ with genus 10 in Proposition 3.6, which attach to $\overline{\mathbf{Y}}_{2,2}^A$ at each singular point. By the genus computation, we conclude that we have computed all irreducible components in the stable reduction of $X_0(5^4)$.

We choose an element β such that $\beta^{10} = 5$. We have $v(\beta) = 1/10$. Note that this β is not equal to β in the previous subsection. We change variables as follows $X = \beta/x$, $U = \beta^5/u$, $V = 5^2\beta^2 v$, $Y = (5^3/\beta)y$. By $\pi_v(U) = \pi_f(V)$, we acquire the following equality

$$\frac{(u^2+1+2\beta^5 u)^3}{u^5} = \frac{(v^2+1+2\beta^5 v)^3}{v^5}. \tag{3.11}$$

By $\pi_v(X) = \pi_f(U)$, we obtain the following congruence

$$u = x^5\left(1+2\frac{5}{\beta}x+\frac{5}{\beta^2}x^2\right)^{-3} = x^5\left(1-\frac{5}{\beta}x-3\frac{5}{\beta^2}x^2\right) \pmod{(4/3)+}. \tag{3.12}$$

By (3.12), the following congruence holds on the left hand side of the equality (3.11)

$$\frac{(u^2+1+2\beta^5 u)^3}{u^5} \equiv \frac{(1+x^{10}+2\beta^5 x^5)^3}{x^{25}} - 3\frac{5}{\beta^2}\cdot(1+x^2+2\beta x)^{10}\times\left(\frac{x^2+2\beta x}{x^{15}}-\frac{\beta^5}{x^{18}}\right) \tag{3.13}$$

modulo $(4/3)+$. By exchanging (u, x) for (v, y), we acquire the same congruence (3.13) for (v, y). Hence, we acquire the following congruence by (3.11) and (3.13)

$$\frac{(1+x^{10}+2\beta^5 x^5)^3}{x^{25}} - \frac{(1+y^{10}+2\beta^5 y^5)^3}{y^{25}} - 3\frac{5}{\beta^2}\cdot(1+x^2+2\beta x)^{10}\times\left(\frac{x^2+2\beta x}{x^{15}}-\frac{\beta^5}{x^{18}}\right)$$

$$+3\frac{5}{\beta^2}\cdot(1+y^2+2\beta y)^{10}\times\left(\frac{y^2+2\beta y}{y^{15}}-\frac{\beta^5}{y^{18}}\right) \equiv 0 \pmod{(4/3)+}. \tag{3.14}$$

We define $g(x, y)$ by the following equality

$$\frac{(1+x^{10}+2\beta^5 x^5)^3}{x^{25}} - \frac{(1+y^{10}+2\beta^5 y^5)^3}{y^{25}} = \left(\frac{(x^2+1+2\beta x)^3}{x^5} - \frac{(y^2+1+2\beta y)^3}{y^5}\right)^5 - 5g(x,y).$$

Let γ_1 be an element which satisfies $\gamma_1^5 = 5/\beta^2$. We have $v(\gamma_1) = 4/25$. Furthermore, we set

$$\frac{(x^2+1+2\beta x)^3}{x^5} - \frac{(y^2+1+2\beta y)^3}{y^5} = \gamma_1 Z. \tag{3.15}$$

By substituting (3.15) to (3.14) and dividing it by $(5/\beta^2)$, the following congruence holds

$$Z^5 \equiv 3(1+x^2+2\beta x)^{10}\times\left(\frac{x^2+2\beta x}{x^{15}}-\frac{\beta^5}{x^{18}}\right)$$

$$-3(1+y^2+2\beta y)^{10}\times\left(\frac{y^2+2\beta y}{y^{15}}-\frac{\beta^5}{y^{18}}\right)+\beta^2 g(x,y) \pmod{(8/15)+}. \tag{3.16}$$

We set

$$x_1 := x\left(1+\frac{2\beta}{x}\right)^{1/2},\ y_1 := y\left(1+\frac{2\beta}{y}\right)^{1/2}.$$

Then, we have

$$x_1^2 = x^2 + 2\beta x,\ y_1^2 = y^2 + 2\beta y. \tag{3.17}$$

Then, the following congruence holds by (3.15)

$$\frac{(x_1^2+1)^3}{x_1^5} - \frac{(y_1^2+1)^3}{y_1^5} = \gamma_1 Z \ (\mathrm{mod}\ (1/5)+). \tag{3.18}$$

We put as in [CM2, subsection 7.2]

$$x_1 := \frac{s}{(r-1)^3},\ y_1 = \frac{s}{(r+1)^3}. \tag{3.19}$$

Proposition 3.4. *Let the notation be as above. The reduction of the space* $\mathbf{Y}^A_{2,2}$ *is defined by the following equations*

$$\overline{\mathbf{Y}}^A_{2,2} : s^2 = r^2 - 1,\ Z^5 = \frac{r^{10}(r^4+1)^{10}}{s^{15}(r^2-1)^{10}}(1+2r^6+r^{10}).$$

Proof. By (3.15), we acqure the following

$$\frac{(x_1^2+1)^3}{x_1^5} \equiv \frac{(y_1^2+1)^3}{y_1^5}\ (\mathrm{mod}\ 0+) \tag{3.20}$$

with $x \neq y$ by $w_1(X) \neq Y$. Under the transformation (3.19), the curve (3.20) is isomorphic to an affine curve $s^2 = r^2 - 1$ as proved in loc. cit. The required assertion follows by considering the congruence (3.16) modulo 0+ and rewriting it under the variables (r, s). □

Remark 3.5. In the proof above, the curve (3.20) with $x \neq y$ is isomorphic to the curve $s^2 = r^2 - 1$. We set

$$f = f(x_1, y_1) := \frac{x_1^2(y_1^2+1)}{y_1^2(x_1^2+1)}.$$

The inverse of the transformation (3.19) is described as follows

$$r = \frac{f+1}{f-1},\ s = \frac{3x_1}{(f-1)^3}.$$

Note $f^3 = x_1/y_1$.

In the following, we will prove that there exist two irreducible components defined by $a^5 - a = t^6$ which attach to the reduction $\overline{\mathbf{Y}}^A_{2,2}$ at $(r, s) = (0, \zeta)$, $\zeta^2 = -1$. We focus on a locus $v(r) = 1/30$. By (3.18), we define g by the equality $s^2 = r^2 - 1 + \gamma_1 g$. Then, g satisfies the following congruence

$$\gamma_1 g \equiv \frac{(r^2-1)}{r^2(r^4+1)^2}\gamma_1 Z \pmod{(1/5)+}. \tag{3.21}$$

Hence, we acquire the following congruence

$$s^2 \equiv (r^2-1)\{1 + \gamma_1 \frac{Z_1}{r^2(r^4+1)^2}\} \pmod{(1/5)+}. \tag{3.22}$$

In the following, we compute the term $\beta^2 g(x, y)$ in the right hand side of the congruence (3.16) modulo $(8/15)+$ as in (3.29). We obtain the following equality by the definition of x_1

$$x^2 + 1 + 2\beta x = x_1^2 + 1 = \frac{r(r^4+1)}{(r-1)^5} + \frac{\gamma_1 g}{(r-1)^6} + 5\Delta_0 \tag{3.23}$$

with some function Δ_0. Hence, we have $v(x^2 + 1 + 2\beta x) = v(r)$. By (3.17), we define Δ by the following equality

$$x^{10} + 1 + 2\beta^5 x^5 = x_1^{10} + 1 + 5\Delta. \tag{3.24}$$

By (3.24), the following congruence holds

$$x^{10} + 1 + 2\beta^5 x^5 \equiv \frac{r^5(r^4+1)^5}{(r-1)^{25}} + \frac{(\gamma_1 g)^5}{(r-1)^{30}} + 5\Delta_1 \pmod{(4/3)+}$$

with some function Δ_1. Therefore, we acquire

$$\frac{(x^{10} + 1 + 2\beta^5 x^5)^3}{x^{25}} \equiv \frac{r^{15}(r^4+1)^{15}}{s^{25}} + 3\frac{(\gamma_1 g)^5 r^{10}(r^4+1)^{10}}{(r-1)^5 s^{25}} + 5r^{10}\Delta_2 \pmod{(4/3)+}$$

with some function Δ_2. We set

$$H := 3\frac{(\gamma_1 g)^5 r^{10}(r^4+1)^{10}}{(r^2-1)^5 s^{25}} \times \{(r+1)^5 - (r-1)^5\}.$$

Hence, we obtain

$$\frac{(x^{10} + 1 + 2\beta^5 x^5)^3}{x^{25}} - \frac{(y^{10} + 1 + 2\beta^5 y^5)^3}{y^{25}} \equiv H + 5r^{10}\Delta_2' \tag{3.25}$$

modulo $(4/3)+$ with some function Δ_2'. Since we have $v(x^2+1+2\beta x) = v(r)$, we acquire the following congruence

(3.26)

$$\left(\frac{(x^2 + 1 + 2\beta x)^3}{x^5} - \frac{(y^2 + 1 + 2\beta y)^3}{y^5}\right)^5 \equiv \frac{(x^2 + 1 + 2\beta x)^{15}}{x^{25}} - \frac{(y^2 + 1 + 2\beta y)^{15}}{y^{25}} \pmod{(4/3)+}.$$

We easily verify the following congruence using the equality (3.23)

$$\frac{(x^2+1+2\beta x)^{15}}{x^{25}} \equiv \frac{r^{15}(r^4+1)^{15}}{s^{25}} + 3\frac{(\gamma_1 g)^5}{(r-1)^5} \times \frac{r^{10}(r^4+1)^{10}}{s^{25}} \pmod{(4/3)+}. \tag{3.27}$$

By exchanging x for y, we obtain the same congruence (3.27) for (y, r, s). Hence, by (3.27) and (3.26), we acquire the following congruence by the definition of $g(x, y)$

$$\left(\frac{(x^2+1+2\beta x)^3}{x^5} - \frac{(y^2+1+2\beta y)^3}{y^5}\right)^5 \equiv H \pmod{(4/3)+}. \tag{3.28}$$

By (3.25) and (3.28), we obtain the following

$$\beta^2 g(x, y) \equiv -\beta^2 r^{10}\Delta_2' \equiv \beta^2 r^{10} d \pmod{(8/15)+} \tag{3.29}$$

where $-d$ is the constant term of $\Delta_2'(r, s)$.

Since we have $v(\gamma_1^5) = 4/5 > 8/15$ and $v(\beta^5 r^{10}) = 5/6 > 8/15$, we rewrite the congruence (3.16) by (3.29)

$$Z^5 \equiv \beta^2 r^{10} d + 3\frac{(1+x_1^2)^{10}}{x_1^{13}} - 3\frac{(1+y_1^2)^{10}}{y_1^{13}} \pmod{(8/15)+}. \tag{3.30}$$

Then, on the term in the right hand side of the congruence (3.30), we acquire the following congruence by (3.19) and $s^2 = r^2 - 1 + \gamma_1 g$

$$\frac{(1+x_1^2)^{10}}{x_1^{13}} \equiv \frac{r^{10}(r^4+1)^{10}}{s^{15}}\left(\frac{r+1}{(r-1)^{10}}\right)\left(1+\frac{\gamma_1 g}{r^2-1}\right) \pmod{(8/15)+}. \tag{3.31}$$

By exchanging x_1 for y_1, we acquire the same congruence (3.31) for (y_1, r, s) by (3.19). Hence, substituting (3.31) to the right hand side of (3.30), we acquire the following congruence

$$Z^5 \equiv \beta^2 r^{10} d + 3\frac{r^{10}(r^4+1)^{10}}{s^{15}(r^2-1)^{10}}\left(1+\frac{\gamma_1 g}{r^2-1}\right)\{(r+1)^{11} - (r-1)^{11}\} \tag{3.32}$$

modulo $(8/15)+$. Note the we have $v(Z) = v(r^2)$.

We set

$$Z_1 := \frac{1}{r^2(r^4+1)^2} Z.$$

Therefore, we obtain the following congruence by (3.21), (3.22) and (3.32) $\times \{r(r^4 + 1)\}^{-10}$

$$Z_1^5 \equiv 3\frac{1}{s^{25}}(1+\gamma_1 Z_1)\{(r+1)^{11} - (r-1)^{11}\} + \beta^2 d \pmod{(1/5)+}. \tag{3.33}$$

Since we have $(r+1)^{11} - (r-1)^{11} \equiv 2(1 + 2r^6 + r^{10}) \pmod{1+}$, the congruence (3.33) has the following form

$$Z_1^5 \equiv \frac{1}{s^{25}}(1+\gamma_1 Z_1)(1+2r^6) + \beta^2 d \pmod{(1/5)+}. \tag{3.34}$$

We put $s_0 := \pm 2$. We choose elements β_1 and α_1 such that $\beta_1^4 = \gamma_1/s_0^{15}$ and $\beta_1^5 = 2\alpha_1^6/s_0^{15}$ respectively. We have $v(\beta_1) = 1/25, v(\alpha_1) = 1/30$. We choose an element γ_0 such that $\gamma_0^5 = (1+\gamma_1\gamma_0)/s_0^{15} - d$.

We change variables as follows

$$Z_1 = \gamma_0 + \beta_1 a,\ r = \alpha_1 t,\ s = s_0 + \frac{\alpha_1^2}{2s_0}s_1. \tag{3.35}$$

Then, we have $s_1 = t^2 \pmod{0+}$. Substituting (3.35) to the congruence (3.34) and dividing it by β^2, we acquire the following

$$a^5 - a = t^6 \pmod{0+}.$$

Hence, we have proved the following proposition.

Proposition 3.6. *Let the notation be as above. Then, in the stable reduction of $X_0(5^4)$, there exist two irreducible components defined by*

$$a^5 - a = t^6.$$

The genus of this affine curve is equal to 10. *These two curves attach to the reduction $\overline{\mathbf{Y}}^A_{2,2}$ at two points $(r,s) = (0, \pm 2)$.*

Let $g_0(p^n)$ be the genus of the modular curve $X_0(p^n)$. Let $g_{p,e}$ be the genus of the Igusa curve $\mathrm{Ig}(p^e)$. We have $g_{5,1} = 0$ and $g_{5,2} = 6$ by [Ig, Section 0]. Furthermore, we have $g_0(5^4) = 48$ by [Sh, Propositions 1.40 and 1.43]. In $\overline{\mathcal{X}_0(5^4)}$, we find the two components $\overline{\mathbf{Z}}^A_{2,1}, \overline{\mathbf{Z}}^A_{1,2}$: $Z^5 = x^2 + x^{-2}$ with genus 0, the eight components defined by $a^5 - a = s^2$ with genus 2, $\overline{\mathbf{Y}}^A_{2,2}$ with genus 0, the 2 components $\overline{\mathbf{X}}_\pm$ with genus 10 and the two Igusa curves $\mathrm{Ig}(5^2)$ with genus 6. On the other hand, the graph of the stable model of $X_0(5^4)$ is a tree by [CM, Theorem 9.4]. Hence, the sum of the genera of all irreducible components and the Betti number of the dual graph is 48. Therefore, we conclude that the stable reduction $\overline{\mathcal{X}_0(5^4)}$ consists of the above components.

§ 3.4. $X_0(7^3)$

We recall a parametrization on the modular curve $X_0(7)$ from [CM2, subsection 7.3]. The genus 0 curve $X_0(7)$ has a unique supersingular annulus corresponding to $j = 1728$. Let A denote this unique supersingular elliptic curve with j-invariant 1728. If we take $T = \eta_1^4/\eta_7^4$ as a parameter as in [Mc2, Section 2], the supersingular annulus $W_A(p)$ is the region described by $0 < v(T) < 2$, because of $i(A) = 2$. Namely, we fix an identification $W_A(p) \simeq A(p^{-2}, 1) \ni T$. Then, the formulas for the forgetful map $\pi_f : W_A(p) \to W_A(1) \ni j - 1728$ and the Atkin-Lehner involution $w_1 : W_A(p) \to W_A(p)$ are given as follows

$$\pi_f^*(j - 1728) = \frac{(T^4 - 10\cdot 7^2T^3 - 9\cdot 7^4T^2 - 2\cdot 7^6T - 7^7)^2}{T^7},\ w_1^*(T) = \frac{7^2}{T}.$$

Hence, we obtain the following by $\pi_v = w_1 \circ \pi_f$

$$\pi_v^*(j - 1728) = T^7\left(1 + \frac{2\cdot 7}{T} + \frac{9\cdot 7}{T^2} + \frac{10\cdot 7}{T^3} - \frac{7}{T^4}\right)^2.$$

The circles $\mathbf{TS}_A$ and $\mathbf{SD}_A$ are given by $v(T) = 7/4$ and $v(T) = 1$ respectively.

We compute the reduction of the space $\mathbf{Z}^A_{1,1} = \pi_{1,1}^{-1}(\mathbf{SD}_A)$ in the same way as $X_0(5^3)$. We use the following identification (3.3)

$$\mathbf{Z}^A_{1,1} \simeq \{(X, U, Y) \in \mathbf{C}^A_{1/7} \times \mathbf{C}^A_1 \times \mathbf{C}^A_{13/7} |\ \pi_v(X) = \pi_f(U),\ \pi_v(U) = \pi_f(Y)\}.$$

Compare this identification with the one given in [CM, subsection 7.3]. In the following, we calculate the reduction of the space $\mathbf{Z}^A_{1,1}$ in Proposition 3.7. Furthermore, in Proposition 3.8, we find 8 components defined by $a^7 - a = s^2$ with genus 3, which attach to $\overline{\mathbf{Z}}^A_{1,1}$ at each singular point $x \in \mu_8$. Finally, by the genus computation, we conclude that we have obtained all irreducible components which appear in the stable reduction of $X_0(7^3)$.

We choose an element β such that $\beta^7 = 7$. We have $v(\beta) = 1/7$.

We change variables as follows $X = \beta/x$, $U = \beta^7 u$, $Y = (7^2/\beta)y$. Then, by $\pi_v(X) = \pi_f(U)$, we acquire the following congruence

$$u \equiv \frac{\left(1 - 2\frac{7}{\beta^4}x^4\right)}{x^7} \pmod{(1/2)+}. \tag{3.36}$$

In the same way as above, we obtain the following congruence by $\pi_v(U) = \pi_f(Y)$

$$u^{-1} \equiv \frac{\left(1 - 2\frac{7}{\beta^4}y^4\right)}{y^7} \pmod{(1/2)+}. \tag{3.37}$$

By (3.36) and (3.37), the following congruence holds

$$(xy)^7 \equiv 1 - 2\frac{7}{\beta^4}(x^4 + y^4) \pmod{(1/2)+}. \tag{3.38}$$

We choose an element γ such that $\gamma^7 = -2\cdot 7/\beta^4$. We have $v(\gamma) = 3/49$. We introduce a new parameter Z as follows

$$xy = 1 + \gamma Z. \tag{3.39}$$

Substituting (3.39) to (3.38) and dividing it by $-2\cdot 7/\beta^4$, we acquire the following congruence

$$Z^7 \equiv x^4 + \left(\frac{1+\gamma Z}{x}\right)^4 \pmod{(1/14)+}. \tag{3.40}$$

We put $F(Z, x) := x^4 + \left(\frac{1+\gamma Z}{x}\right)^4$. We have proved the following proposition.

Proposition 3.7. *Let the notation be as above. The reduction of the space* $\mathbf{Z}^A_{1,1}$ *is defined by the following equation*

$$Z^7 = x^4 + x^{-4}. \tag{3.41}$$

This affine curve has genus 0.

Proof. As in loc. cit., by considering (3.40) modulo 0+, the required assertion follows. □

Note that the curve (3.41) has singularities at $x \in \mu_8(\mathbb{F}_{49})$. In the following, we prove that there exist eight irreducible components defined by $a^7 - a = t^2$ in the stable reduction of $X_0(7^3)$. These components are called "new components" in [CM]. These components attach to the reduction (3.41) at each singular point $x = \bar{\zeta}$ with $\bar{\zeta} \in \mu_8(\mathbb{F}_{49})$.

Let $\zeta \in \mu_8(\mathbb{C}_p)$. We choose an element γ_0 such that $\gamma_0^7 = (\zeta^4 + \zeta^{-4})(1 + \gamma\gamma_0)^2$. We set $x_0 := \zeta(1 + \gamma\gamma_0)^{1/2}$. By the definitions of γ_0, x_0, we acquire $\gamma_0^7 = F(\gamma_0, x_0)$.

Then, we can easily check that

- $\partial_x F(\gamma_0, x_0) = 0$.
- $v(\partial_Z F(\gamma_0, x_0)) = v(\gamma) = 3/49$.
- $\partial_x^2 F(\gamma_0, x_0)$ is a unit.

We choose elements α_1 and β_1 such that $\alpha_1^6 = \partial_Z F(\gamma_0, x_0)$ and $\alpha_1^7 = (1/2)\partial_x^2 F(\gamma_0, x_0)\beta_1^2$ respectively. Then, we have $v(\alpha_1) = 1/98, v(\beta_1) = 1/28$.

We change variables as follows

$$x = x_0 + \beta_1 t,\ Z = \gamma_0 + \alpha_1 a.$$

Substituting them to (3.40), we acquire the following congruence by the choices of α_1 and β_1

$$\alpha_1^7(a^7 - a - t^2) \equiv 0 \ (\mathrm{mod}\ (1/14)+).$$

By dividing this by α_1^7, we acquire $a^7 - a = t^2 \ (\mathrm{mod}\ 0+)$. Hence, we have proved the following proposition.

Proposition 3.8. *Let the notation be as above. Then, there exist eight irreducible components defined by* $a^7 - a = t^2$ *in the stable reduction of* $X_0(7^3)$. *These eight components attach to the curve (3.41) at each singular point.*

By the same computations as the ones in [CM2, subsection 7.3], we can compute the reduction of the spaces $\mathbf{Y}^A_{2,1}$ and $\mathbf{Y}^A_{1,2}$ as follows

$$\overline{\mathbf{Y}}^A_{2,1},\ \overline{\mathbf{Y}}^A_{2,1}:\ z^7=\frac{s}{(r-1)^2},\ s^4=r^2-1.$$

This curve has genus 1.

Let $g_0(p^n)$ denote the genus of the modular curve $X_0(p^n)$. We have $g_0(7^3)=26$ by [Sh, Propositions 1.40 and 1.43]. We find the two irreducible curves $\overline{\mathbf{Y}}^A_{2,1}$ and $\overline{\mathbf{Y}}^A_{1,2}$ of genus 1, the component $\overline{\mathbf{Z}}^A_{1,1}$ with genus 0 and the eight curves $\{\overline{\mathbf{X}}_\zeta\}_{\zeta\in\mu_4}$ defined by $a^7-a=t^2$ with genus 3 in $\overline{\mathcal{X}_0(7^3)}$. On the other hand, the dual graph of the stable reduction of $X_0(7^3)$ is a tree by [CM, Theorem 9.4]. Hence, the sum of genera of all irreducible components and the Betti number of the dual graph is equal to $g_0(7^3)=26$. Therefore, we conclude that the stable reduction $\overline{\mathcal{X}_0(7^3)}$ consists of the above components.

§ 3.5. $X_0(7^4)$

In this subsection, we will compute the reduction of the space $\mathbf{Y}^A_{2,2}\subset X_0(7^4)$. We consider the following identification (3.4)

$$\mathbf{Y}^A_{2,2}\simeq\{(X,U,V,Y)\in\mathbf{C}^A_{1/28}\times\mathbf{C}^A_{1/4}\times\mathbf{C}^A_{7/4}\times\mathbf{C}^A_{55/28}|$$

$$\pi_v(X)=\pi_f(U),\ \pi_v(U)=\pi_f(V),\ \pi_v(V)=\pi_f(Y),\ w_1(X)\neq Y\}.$$

In Proposition 3.9, we compute the reduction $\overline{\mathbf{Y}}^A_{2,2}$. After that, in Proposition 3.11, we find four components defined by $a^7-a=t^8$ with genus 21, which attach to the component $\overline{\mathbf{Y}}^A_{2,2}$ at each singular point $(r,s)=(0,\zeta),\zeta^4=-1$. Finally, by the genus computation, we determine the stable reduction of $X_0(7^4)$.

We choose an element $\beta^{28}=-7$. We have $v(\beta)=1/28$. We change variables as follows $X=\beta/x,U=\beta^7/u,V=(7^2/\beta^7)v,Y=(7^2/\beta)y$. Let $h(T):=10\beta^7T^3+9\beta^{14}T^2+2\beta^{21}T\in\mathbb{Z}_7[\beta][T]$. By $\pi_v(U)=\pi_f(V)$, we acquire the following equality

$$\frac{(u^4+1-h(u))^2}{u^7}=\frac{(v^4+1-h(v))^2}{v^7}. \tag{3.42}$$

We put $h_1(T):=10\beta T^3+9\beta^2T^2+2\beta^3T$. By $\pi_v(X)=\pi_f(U)$, we obtain the following congruence

$$u=x^7\left(1+2\frac{7}{\beta^4}(x^4-h_1(x))\right)\ (\mathrm{mod}\ (9/8)+). \tag{3.43}$$

By (3.43), the following congruence holds

(3.44)

$$\frac{(u^4+1-h(u))^2}{u^7}\equiv\frac{(1+x^{28}+h(x^7))^2}{x^{49}}+2\frac{7}{\beta^4}\cdot(1+x^4-h_1(x))^7\times(x^4-h_1(x))\left(\frac{x^7+10\beta^7}{x^{28}}\right)$$

modulo $(9/8)+$. By exchanging (u,x) for (v,y), the same congruence (3.44) for (v,y) holds. By substituting (3.44) to (3.42), we acquire the following congruence

$$\frac{(1+x^{28}-h(x^7))^2}{x^{49}}-\frac{(1+y^{28}-h(y^7))^2}{y^{49}}+2\frac{7}{\beta^4}\cdot(1+x^4-h_1(x))^7\times(x^4-h_1(x))\left(\frac{x^7+10\beta^7}{x^{28}}\right)$$

$$-2\frac{7}{\beta^4}\cdot(1+y^4-h_1(y))^7\times(y^4-h_1(y))\left(\frac{y^7+10\beta^7}{y^{28}}\right)\equiv 0 \pmod{(9/8)+}. \tag{3.45}$$

We define $g(x,y)$ by the following equality

$$\frac{(1+x^{28}-h(x^7))^2}{x^{49}}-\frac{(1+y^{28}-h(y^7))^2}{y^{49}}=\left(\frac{(x^4+1-h_1(x))^2}{x^7}-\frac{(y^4+1-h_1(y))^2}{y^7}\right)^7-7g(x,y).$$

Let γ_1 be an element which satisfies $\gamma_1^7=7/\beta^4$. We have $v(\gamma_1)=6/49$. Furthermore, we set

$$\frac{(x^4+1-h_1(x))^2}{x^7}-\frac{(y^4+1-h(y))^2}{y^7}=\gamma_1 Z. \tag{3.46}$$

By substituting (3.46) to (3.45) and dividing it by $(7/\beta^4)$, the following congruence holds

$$Z^7\equiv\beta^4 g(x,y)-2\left(\frac{x^7+10\beta^7}{x^{28}}\right)(x^4-h_1(x))(1+x^4-h_1(x))^7$$

$$+2\left(\frac{y^7+10\beta^7}{y^{28}}\right)(y^4-h_1(y))(1+y^4-h_1(y))^7 \pmod{(15/56)+}. \tag{3.47}$$

We set

$$x_1:=x\left(1-\frac{h_1(x)}{x^4}\right)^{1/4},\ y_1:=y\left(1-\frac{h_1(y)}{y^4}\right)^{1/4}.$$

We put as in [CM2, subsection 7.3]

$$x_1:=\frac{s}{(r-1)^2},\ y_1=\frac{s}{(r+1)^2}. \tag{3.48}$$

Proposition 3.9. *Let the notation be as above. Then, the reduction of the space* $\mathbf{Y}^A_{2,2}$ *is defined by the following equations*

$$s^4=r^2-1,\ Z^7=3\frac{r^7(r^6+1)^7}{s^{21}(r^2-1)^{14}}(1+2r^8+r^{14}).$$

Proof. By considering the congruence (3.46) modulo $0+$, we acquire the following congruence

$$\frac{(x_1^4+1)^2}{x_1^7}\equiv\frac{(y_1^2+1)^2}{y_1^7} \pmod{0+} \tag{3.49}$$

with $x_1 \neq y_1$ by $w_1(X) \neq Y$. As proved in [CM2, subsection 7.3], by the map (3.48), the curve (3.49) is isomorphic to a curve $s^4 = r^2 - 1$. Hence, the required assertion follows by rewriting the congruence (3.47) under the variables (r, s). □

Remark 3.10. We write down the inverse map of (3.48). By setting

$$f := \frac{y_1^3(x_1^4+1)}{x_1^3(y_1^4+1)},$$

we have $f^2 = x_1/y_1$. Then, the inverse of the transformation (3.48) has the following form

$$r = \frac{f+1}{f-1}, s = \frac{4x_1}{(f-1)^2}.$$

In the following, we will prove that there exist four irreducible components defined by $a^7 - a = t^8$ which attach to the reduction $\overline{\mathbf{Y}}^A_{2,2}$ at $r = 0$.

We focus on a locus $v(r) = 1/56$. We set $s^4 = r^2 - 1 + \gamma_1 g$ by (3.46). By using (3.46), we can easily check the following congruence

$$\gamma_1 g \equiv \frac{r^2-1}{4r(r^6+1)^2}\gamma_1 Z \pmod{(1/7)+}. \tag{3.50}$$

Hence, we acquire the following by (3.50)

$$s^4 \equiv r^2 - 1 + \frac{r^2-1}{4r(r^6+1)^2}\gamma_1 Z \pmod{(1/7)+}. \tag{3.51}$$

In the following, we calculate the term $\beta^4 g(x, y)$ in the right hand side of the congruence (3.47) modulo $(15/56)+$ as in (3.59). We obtain the following equality by the definition of x_1

$$x^4 + 1 - h_1(x) = x_1^4 + 1 = \frac{r(r^6+1)}{(r-1)^7} + \frac{\gamma_1 g}{(r-1)^8} + 7\Delta_0. \tag{3.52}$$

Hence, we have $v(x^4 + 1 - h_1(x)) = v(r)$. By the definition of x_1, we obtain an equality

$$1 + x^{28} - h(x^7) = 1 + x_1^{28} + 7\Delta \tag{3.53}$$

with some function Δ. By (3.53), the following congruence holds

$$x^{28} + 1 - h(x^7) \equiv \frac{r^7(r^6+1)^7}{(r-1)^{49}} + \frac{(\gamma_1 g)^7}{(r-1)^{56}} + 7\Delta_0' \pmod{(9/8)+}$$

with some Δ_0'. Therefore, we acquire

$$\frac{(x^{28}+1-h(x^7))^2}{x^{49}} \equiv \frac{r^{14}(r^4+1)^{14}}{s^{49}} + 2\frac{(\gamma_1 g)^7 r^7 (r^6+1)^7}{(r-1)^7 s^{49}} + 7r^7\Delta_1 \pmod{(9/8)+} \tag{3.54}$$

with some function Δ_1. By exchanging x for y, we obtain the same congruence (3.54) for (y, r, s). We set as follows

$$G := 2\frac{(\gamma_1 g)^7 r^7 (r^6+1)^7}{(r^2-1)^7 s^{49}} \times \{(r+1)^7 - (r-1)^7\}$$

Hence, by (3.54), we obtain

$$\frac{(x^{28}+1-h(x^7))^2}{x^{49}} - \frac{(y^{28}+1-h(y^7))^2}{y^{49}} \equiv G + 7r^7\Delta_2' \tag{3.55}$$

modulo $(9/8)+$ with some function Δ_2'. Since we have $v(x^4+1-h_1(x)) = v(r)$, we acquire the following congruence

$$\left(\frac{(x^4+1-h_1(x))^2}{x^7} - \frac{(y^4+1-h_1(y))^2}{y^7}\right)^7 \equiv \frac{(x^4+1-h_1(x))^{14}}{x^{49}} - \frac{(y^4+1-h_1(y))^{14}}{y^{49}} \tag{3.56}$$

modulo $(9/8)+$. We easily verify the following congruence using the equality (3.52)

$$\frac{(x^4+1-h_1(x))^{14}}{x^{49}} \equiv \frac{r^{14}(r^6+1)^{14}}{s^{49}} + 2\frac{(\gamma_1 g)^7}{(r-1)^6} \times \frac{r^7(r^6+1)^7}{s^{49}} \pmod{(9/8)+}. \tag{3.57}$$

By exchanging x for y, we obtain the same congruence (3.57) for (y, r, s). Hence, by (3.57) and (3.56), we acquire the following congruence

$$\left(\frac{(x^4+1-h_1(x))^2}{x^7} - \frac{(y^4+1-h_1(y))^2}{y^7}\right)^7 \equiv G \pmod{(9/8)+}. \tag{3.58}$$

By (3.55) and (3.58), we obtain the following

$$\beta^4 g(x,y) \equiv -\beta^4 r^7 \Delta_2' \equiv \beta^4 r^7 d \pmod{(15/56)+} \tag{3.59}$$

where $-d$ is the constant term of $\Delta_2'(r, s)$.

Since we have $v(\beta^7 r^7) > 15/56$, we rewrite the congruence (3.47) by (3.59)

$$Z^7 \equiv \beta^4 r^7 d - 2\frac{(1+x_1^4)^7}{x_1^{17}} + 2\frac{(1+y_1^4)^7}{y_1^{17}}$$

$$\equiv \beta^4 r^7 d - 2\frac{r^7(r^6+1)^7}{s^{21}(r^2-1)^{14}}\left(1+\frac{\gamma_1 g}{r^2-1}\right)\{(r+1)^{15} - (r-1)^{15}\} \pmod{(15/56)+}. \tag{3.60}$$

We set

$$Z_1 := \frac{1}{4r(r^6+1)} Z.$$

Then, we obtain the following congruence by (3.60) $\times \{4r(r^6+1)\}^{-7}$ and $s^4 = (r^2-1)(1+\gamma_1 Z_1) \pmod{(1/7)+}$ by (3.51)

$$Z_1^7 \equiv -2\frac{1}{s^{21}}(1+\gamma_1 Z_1)\{(r+1)^{15} - (r-1)^{15}\} + \beta^4 d \pmod{(1/7)+}. \tag{3.61}$$

Since we have $(r+1)^{15}-(r-1)^{15} \equiv 2(1+2r^8+r^{14}) \pmod{1+}$, we acquire the following congruence by (3.61)

$$Z_1^7 \equiv 3\frac{1}{s^{21}}(1+\gamma_1 Z_1)(1+2r^8)+\beta^4 d \pmod{(1/7)+}. \tag{3.62}$$

We fix a root $z=s_0$ of the following equation $z^4=-1$. We choose elements β_1 and α_1 such that $\beta_1^6=3\gamma_1/s_0^{21}$ and $\beta_1^7=-\alpha_1^8/s_0^{21}$ respectively. We have $v(\beta_1)=1/49, v(\alpha_1)=1/56$. We choose an element γ_0 such that $\gamma_0^7=-(1+\gamma_1\gamma_0)/s_0^{21}-d$.

We change variables as follows

$$Z_1=\gamma_0+\beta_1 a,\ r=\alpha_1 t,\ s=s_0+\frac{\alpha_1^2}{4s_0}s_1. \tag{3.63}$$

Then, we have $s_1=t^2 \pmod{0+}$. Substituting (3.63) to the congruence (3.62) and dividing it by β^4, we acquire the following

$$a^7-a=t^8 \pmod{0+}.$$

Hence, we have proved the following proposition.

Proposition 3.11. *Let the notation be as above. Then, in the stable reduction of $X_0(7^4)$, there exist four irreducible components defined by $a^7-a=t^8$. The genus of the curve $a^7-a=t^8$ is equal to 21. Furthermore, these four components attach to the curve $\overline{\mathbf{Y}}^A_{2,2}$ at $(r,s)=(0,\zeta)$ with $\zeta^4=-1$.*

By the same computations as the ones in [CM2, subsection 7.3], we can compute the reduction of the spaces $\mathbf{Y}^A_{3,1}$ and $\mathbf{Y}^A_{1,3}$ as follows

$$\overline{\mathbf{Y}}^A_{3,1},\ \overline{\mathbf{Y}}^A_{1,3}:\quad z^{49}=\frac{s}{(r-1)^2},\ s^4=r^2-1.$$

This curve has genus 1. As in the previous subsection, we can also compute the reduction of the spaces $\mathbf{Z}^A_{2,1}$ and $\mathbf{Z}^A_{1,2}$ as follows

$$\overline{\mathbf{Z}}^A_{2,1},\ \overline{\mathbf{Z}}^A_{1,2}:\ Z^7=x^4+x^{-4}.$$

There exist eight curves defined by $a^7-a=s^2$ with genus 3, which attach to the component $\overline{\mathbf{Z}}^A_{2,1}$ at each singular point $x\in\mu_8$. The same things happen for $\overline{\mathbf{Z}}^A_{1,2}$.

Let $g_0(p^n)$ be the genus of the modular curve $X_0(p^n)$. Let $g_{p,e}$ be the genus of the Igusa curve $\mathrm{Ig}(p^e)$. We have $g_{7,1}=0$ and $g_{7,2}=33$ by [Ig, Section 0]. Furthermore, we have $g_0(7^4)=201$ by [Sh, Propositions 1.40 and 1.43]. In $\overline{\mathcal{X}_0(7^4)}$, we find the two Igusa curves $\mathrm{Ig}(7^2)$ with genus 33, the components $\overline{\mathbf{Y}}^A_{3,1}$ and $\overline{\mathbf{Y}}^A_{1,3}$ with genus 1, the two components $\overline{\mathbf{Z}}^A_{3,1}$, $\overline{\mathbf{Z}}^A_{1,3}$ with genus 0, the component $\overline{\mathbf{Y}}^A_{2,2}$ with genus 1, four irreducible

components defined by $a^7 - a = t^8$ with genus 21 and 16 irreducible components defined by $a^7 - a = s^2$ with genus 3. On the other hand, the graph of the stable model of $X_0(7^4)$ is a tree by [CM, Theorem 9.4]. Hence, the sum of the genera of all irreducible components and the Betti number of the dual graph is equal to $1 \times 3 + 16 \times 3 + 21 \times 4 + 2 \times 33 = 201 = g_0(7^4)$. Hence, the stable reduction of $\overline{\mathcal{X}_0(7^4)}$ consists of these components which we found.

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