

TRANSLATION INVARIANT MODELS IN NONRELATIVISTIC QUANTUM ELECTRODYNAMICS

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1 The Pauli-Fierz Hamiltonian

In this paper we discuss translation invariant nonrelativistic quantum electrodynamics by *functional integrations*. We assume that an electron is in low energy, its density of charge is smoothly localized. In particular, the ultraviolet divergence does not exist. Let us see some classical model. Let $E(t, x)$ and $B(t, x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, be an electric field and a magnetic field respectively, and $q(t)$ the position of an electron at time $t \in \mathbb{R}$. The Maxwell equation with form factor φ is given by

$$\begin{aligned}\dot{B} &= -\nabla \times E, \\ \nabla \cdot B &= 0, \\ \dot{E} &= \nabla \times B - e\varphi(\cdot - q(t))\dot{q}(t), \\ \nabla \cdot E &= e\varphi(\cdot - q(t)).\end{aligned}$$

Let $(J, \rho) = (e\varphi(x - q(t))\dot{q}(t), e\varphi(x - q(t)))$. Then the Lagrangian density is given by

$$\mathcal{L}(t, x) = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}(E^2 - B^2) + J \cdot A - \rho\phi, \quad (1.1)$$

where A and ϕ are a vector potential and a scalar potential related to E and B such as $E = -\dot{A} - \nabla\phi$ and $B = \nabla \times A$. Let $L = \int_{\mathbb{R}^3} \mathcal{L}(t, x)dx$. Then the conjugate momenta are given by

$$p(t) := \frac{\partial L}{\partial \dot{q}} = m\dot{q}(t) + e \int A(t, x)\varphi(x - q(t))dx, \quad \Pi(t, x) := \frac{\delta L}{\delta \dot{A}} = \dot{A}(t, x).$$

Then the Hamiltonian is given through the Legendre transformation as

$$\begin{aligned} H_{cl} &= p \cdot \dot{q} + \int \dot{A} \Pi dx - L \\ &= \frac{1}{2m} \left(p - e \int A(t, x) \varphi(x - q(t)) dx \right)^2 + \frac{1}{2} \int \left\{ \dot{A}(t, x)^2 + (\nabla \times A(t, x))^2 \right\} dx + V_{cl}(q), \end{aligned}$$

where V is a smeared external potential given by

$$V_{cl}(q) := \frac{1}{2} e^2 \int \frac{\varphi(q - y) \varphi(q - y')}{4\pi |y - y'|} dy dy'.$$

We quantized H_{cl} to define the Pauli-Fierz Hamiltonian.

Let us assume that the dimension of the state space is d and the photon is polarized to $d - 1$ directions. Physically reasonable choice is $d = 3$. Let $\mathcal{F}_b$ be the Boson Fock space over $h_b := \oplus^{d-1} L^2(\mathbb{R}^d)$, i.e., $\mathcal{F}_b := \bigoplus_{n=0}^{\infty} [\otimes_s^n h_b]$, where $\otimes_s^n h_b$ denotes the n -fold symmetric tensor product of h_b with $\otimes_s^0 h_b := \mathbb{C}$. $\Omega = \{1, 0, 0, \dots\} \in \mathcal{F}_b$ is called the Fock vacuum. The annihilation operator and the creation operator on $\mathcal{F}_b$ are denoted by $a(f)$ and $a^*(f)$, $f \in W$, respectively, and are defined by

$$(a^*(f)\Psi)^{(n)} := \sqrt{n} S_n(f \otimes \Psi^{(n-1)}) \quad (1.2)$$

and $a(f) := (a^*(\bar{f}))^*$, where S_n denotes the symmetrizer. Let $\mathcal{F}_{b, \text{fin}}$ be the finite particle subspace of $\mathcal{F}_b$. The annihilation operator and the creation operator leave $\mathcal{F}_{b, \text{fin}}$ invariant and satisfy the canonical commutation relations on it:

$$[a(f), a^*(g)] = (\bar{f}, g)1, \quad [a(f), a(g)] = 0, \quad [a^*(f), a^*(g)] = 0.$$

For $f = (f_1, \dots, f_{d-1}) \in \oplus^{d-1} L^2(\mathbb{R}^d)$, we informally write $a^\sharp(f)$, where $a^\sharp$ stands for a or a^* , as $a^\sharp(f) = \sum_{j=1}^{d-1} \int a^\sharp(k, j) f_j(k) dk$. The quantized radiation field $A_\mu(x)$ with a form factor φ is defined by

$$A_\mu(x) = \frac{1}{\sqrt{2}} \sum_{j=1}^{d-1} \int e_\mu(k, j) \left(\frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} a^*(k, j) e^{-ik \cdot x} + \frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}} a(k, j) e^{ik \cdot x} \right) dk.$$

Here $e(k, 1), \dots, e(k, d-1)$ denote generalized polarization vectors satisfying $k \cdot e(k, j) = 0$ and $e(k, i) \cdot e(k, j) = \delta_{ij}1$, $i, j = 1, \dots, d-1$, and $\hat{\varphi}$ is the Fourier transform of form factor φ . Note that

$$\sum_{j=1}^{d-1} e_\alpha(k, j) e_\beta(k, j) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2} := \delta_{\alpha\beta}^\perp(k), \quad \alpha, \beta = 1, \dots, d.$$

Thus

$$(A_\mu(x)\Omega, A_\nu(x)\Omega)_{\mathcal{F}_b} = \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} \delta_{\mu\nu}^\perp(k) dk$$

Translation invariant Hamiltonian

holds. Throughout this paper we use Assumption (A) below.

(A) Form factor $\hat{\varphi}$ satisfies $\sqrt{\omega}\hat{\varphi}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^d)$ and $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k) = \hat{\varphi}(k)$.

$A_\mu(x)$ is essentially self-adjoint on $\mathcal{F}_{b,\text{fin}}$, and its unique self-adjoint extension is denoted by the same symbol. Next we define the second quantization. Let $\mathcal{C}(\mathcal{K} \rightarrow \mathcal{L})$ be the set of contraction operators from $\mathcal{K}$ to $\mathcal{L}$. The second quantization Γ is the functor:

$$\Gamma : \mathcal{C}(L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)) \rightarrow \mathcal{C}(\mathcal{F}_b \rightarrow \mathcal{F}_b)$$

given by

$$\Gamma(T) := \bigoplus_{n=0}^{\infty} \otimes^n (\oplus^{d-1} T).$$

For a self-adjoint operator h on $L^2(\mathbb{R}^d)$, $\{\Gamma(e^{ith})\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group on $\mathcal{F}_b$. Then there exists a unique self-adjoint operator $d\Gamma(h)$ on $\mathcal{F}_b$ such that $\Gamma(e^{ith}) = e^{itd\Gamma(h)}$. The number operator is defined by $N := d\Gamma(1)$. Let $\omega(k) = |k|$ be the multiplication operator on $L^2(\mathbb{R}^d)$. Define the free Hamiltonian H_{rad} on $\mathcal{F}_b$ by

$$H_{\text{rad}} := d\Gamma(\omega). \quad (1.3)$$

The Hilbert space $\mathcal{H}$ of state vectors for the total system under consideration is given by

$$\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}_b. \quad (1.4)$$

Under the identification $\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathcal{F}_b dx$, we define the self-adjoint operator A on $\mathcal{H}$ by $A_\mu := \int_{\mathbb{R}^d}^{\oplus} A_\mu(x) dx$. The total Hamiltonian H , the so-called Pauli-Fierz Hamiltonian, is described by

$$H := \frac{1}{2}(-i\nabla \otimes 1 - eA)^2 + V \otimes 1 + 1 \otimes H_{\text{rad}}, \quad (1.5)$$

where $e \in \mathbb{R}$ is a coupling constant. The proposition below is established in [H00b, H02].

Proposition 1.1 *Assume that V is relatively bounded with respect to $-\Delta$ with a relative bound strictly smaller than one. Then H is self-adjoint on $D(H_0)$ and essentially self-adjoint on any core of self-adjoint operator $-(1/2)\Delta \otimes 1 + 1 \otimes H_{\text{rad}}$, and bounded from below,*

Define the field momentum by $P_{f\mu} := d\Gamma(k_\mu)$ and the total momentum

$$P_\mu^T := \overline{-i\nabla_\mu \otimes 1 + 1 \otimes P_{f\mu}}, \quad (1.6)$$

where $\overline{X}$ denotes the closure of closable operator X . Now we set $V = 0$. Then it is seen that H is translation invariant;

$$e^{isP_\mu^T} H e^{-isP_\mu^T} = H, \quad s \in \mathbb{R}, \quad \mu = 1, \dots, d.$$

Then we can decompose H on $\sigma(P_\mu^T) = \mathbb{R}$. Define

$$H(P) := \frac{1}{2}(P - P_f - eA(0))^2 + H_{\text{rad}}, \quad P \in \mathbb{R}^d. \quad (1.7)$$

Note that $H(P)$ is a well defined symmetric operator on $D(H_{\text{rad}}) \cap D(P_f^2)$ by assumption (A). The next proposition is established in [H06, LMS06].

Proposition 1.2 $H(P)$ is self-adjoint on $D(H_{\text{rad}}) \cap (\cap_{\mu=1}^d D(P_{\text{f}\mu}^2))$ and it follows that

$$\int_{\mathbb{R}^d}^{\oplus} H(P) dP \cong H. \quad (1.8)$$

So $H(P)$ is our main object and $P \in \mathbb{R}^d$ is called the total momentum. We want to investigate spectral properties of $H(P)$ by making use of functional integrations.

2 Functional integral representations

Let $(b(t))_{t \geq 0} = (b_1(t), \dots, b_d(t))_{t \geq 0}$ be the d -dimensional Brownian motion starting at 0 on a probability space $(W, \mathcal{B}, db)$. Set $X_s := x + b(s)$, $x \in \mathbb{R}^d$, and $dX := dx \otimes db$.

2.1 Functional integral representations for e^{-tH}

Let $\mathcal{A}_0(f)$ be a Gaussian random process on a probability space (Q_0, Σ_0, μ_0) indexed by real $f = (f_1, \dots, f_d) \in \bigoplus^d L^2(\mathbb{R}^d)$ with mean zero and covariance given by

$$\int_{Q_0} \mathcal{A}_0(f) \mathcal{A}_0(g) d\mu_0 = q_0(f, g), \quad (2.1)$$

where

$$q_0(f, g) := \frac{1}{2} \sum_{\alpha, \beta=1}^d \int_{\mathbb{R}^d} \delta_{\alpha\beta}^{\perp}(k) \overline{\hat{f}_{\alpha}(k)} \hat{g}_{\beta}(k) dk.$$

The existence of probability space (Q_0, Σ_0, μ_0) and Gaussian random variable $\mathcal{A}_0(f)$ are known by the Minlos theorem. In a similar way, we can construct two other Gaussian random variables. Let $\mathcal{A}_1(f)$ indexed by real $f \in \bigoplus^d L^2(\mathbb{R}^{d+1})$ and $\mathcal{A}_2(f)$ by real $f \in \bigoplus^d L^2(\mathbb{R}^{d+2})$ be Gaussian random processes on probability spaces (Q_1, Σ_1, μ_1) and (Q_2, Σ_2, μ_2) , respectively, with mean zero and covariances given by

$$\int_{Q_1} \mathcal{A}_1(f) \mathcal{A}_1(g) d\mu_1 = q_1(f, g), \quad \int_{Q_2} \mathcal{A}_2(f) \mathcal{A}_2(g) d\mu_2 = q_2(f, g), \quad (2.2)$$

where

$$q_1(f, g) := \frac{1}{2} \sum_{\alpha, \beta=1}^d \int_{\mathbb{R}^{d+1}} \delta_{\alpha\beta}^{\perp}(k) \overline{\hat{f}_{\alpha}(k, k_0)} \hat{g}_{\beta}(k, k_0) dk dk_0,$$

$$q_2(f, g) := \frac{1}{2} \sum_{\alpha, \beta=1}^d \int_{\mathbb{R}^{d+1+1}} \delta_{\alpha\beta}^{\perp}(k) \overline{\hat{f}_{\alpha}(k, k_0, k_1)} \hat{g}_{\beta}(k, k_0, k_1) dk dk_0 dk_1.$$

From now on $q = 0, 1, 2$. We extend it for $f = f_{\text{R}} + i f_{\text{I}}$ with $f_{\text{R}} = (f + \bar{f})/2$ and $f_{\text{I}} = (f - \bar{f})/(2i)$ as $\mathcal{A}_q(f) = \mathcal{A}_q(f_{\text{R}}) + i \mathcal{A}_q(f_{\text{I}})$. The n -particle subspace $L_n^2(Q_q)$ of $L^2(Q_q)$ is defined by

$$L_n^2(Q_q) = \overline{\text{L.H.}\{\mathcal{A}_q(f_1) \cdots \mathcal{A}_q(f_n) : |f_j| \in L^2(\mathbb{R}^{d+q}), j = 1, \dots, n\}}.$$

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Here $:X:$ denotes the Wick product of X . The identity $L^2(Q_q) = \oplus_{n=0}^{\infty} L_n^2(Q_q)$ is known as the Wiener-Itô decomposition. We also define the second quantization on $L^2(Q_q)$. Let $\Gamma_{qq'} : \mathcal{C}(L^2(\mathbb{R}^{d+q}) \rightarrow L^2(\mathbb{R}^{d+q'})) \rightarrow \mathcal{C}(L^2(Q_q) \rightarrow L^2(Q_{q'}))$ be defined by

$$\Gamma_{qq'} T1 = 1, \quad \Gamma_q(T) : \mathcal{A}_q(f_1) \cdots \mathcal{A}_q(f_n) := \mathcal{A}_{q'}([T]_d f_1) \cdots \mathcal{A}_{q'}([T]_d f_n) :.$$

Set $\Gamma_{qq} = \Gamma_q$ for simplicity. In particular since $\{\Gamma_q(e^{ith})\}_{t \in \mathbb{R}}$ with a self-adjoint operator h on $L^2(\mathbb{R}^d)$ is a strongly continuous one-parameter unitary group, there exists a self-adjoint operator $d\Gamma_q(h)$ on $L^2(Q_q)$ such that $\Gamma_q(e^{ith}) = e^{itd\Gamma_q(h)}$, $t \in \mathbb{R}$. We set $N_q := d\Gamma_q(1)$. Let h be a multiplication operator in $L^2(\mathbb{R}^d)$. We define the families of isometries,

$$L^2(\mathbb{R}^d) \xrightarrow{j_s} L^2(\mathbb{R}^{d+1}) \xrightarrow{\xi_t = \xi_t(h)} L^2(\mathbb{R}^{d+2}), \quad s, t \in \mathbb{R}, \quad (2.3)$$

by

$$\begin{aligned} j_s f(k, k_0) &:= \frac{e^{-isk_0}}{\sqrt{\pi}} \left(\frac{\omega(k)}{\omega(k)^2 + |k_0|^2} \right)^{1/2} \hat{f}(k), \quad (k, k_0) \in \mathbb{R}^d \times \mathbb{R}, \\ \xi_t f(k, k_0, k_1) &:= \frac{e^{-itk_1}}{\sqrt{\pi}} \left(\frac{h(k)}{h(k)^2 + |k_1|^2} \right)^{1/2} \hat{f}(k, k_0), \quad (k, k_0, k_1) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}. \end{aligned} \quad (2.4)$$

Next, define the families of operators J_s and $\Xi_t = \Xi_t(h)$, $s, t \in \mathbb{R}$;

$$L^2(Q_0) \xrightarrow{J_s} L^2(Q_1) \xrightarrow{\Xi_t} L^2(Q_2)$$

by

$$J_s = \Gamma_{01}(j_s), \quad \Xi_t = \Gamma_{12}(\xi_t). \quad (2.5)$$

Define $\mathcal{A}_{q,\mu}(f) = \mathcal{A}_q(\oplus_{\ell=1}^d \delta_{\ell\mu} f)$. We see that $d\Gamma_0(-i\nabla) \cong P_f$ and $d\Gamma_0(\omega(-i\nabla)) \cong H_{\text{rad}}$. We can see that $\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} L^2(Q_0) dx$ i.e., $F \in \mathcal{H}$ can be regarded as an $L^2(Q_0)$ -valued L^2 -function on $\mathbb{R}^d$. Note that in the Fock representation the test function $\hat{f}$ of $\mathcal{A}_\mu(\hat{f})$ is taken in the momentum representation, but in the Schrödinger representation, f of $\mathcal{A}_{0,\mu}(f)$ in the position representation. We can see that

$$H \cong \frac{1}{2}(-i\nabla \otimes 1 - e\mathcal{A}_0^\varphi)^2 + V \otimes 1 + 1 \otimes H_{\text{rad}},$$

where $\tilde{\varphi} := (\hat{\varphi}/\sqrt{\omega})^\vee$. By the Feynman-Kac formula and the fact $J_0^* J_t = e^{-tH_{\text{rad}}}$ we can see that

$$(F, e^{-t(-(1/2)\Delta + V + H_{\text{rad}})} G)_{\mathcal{H}} = \int_{\mathbb{R}^d \times W} e^{-\int_0^t V(X_s) ds} (J_0 F(X_0), J_t G(X_t))_{L^2(Q_1)} dX.$$

Adding the minimal perturbation: $-i\nabla_\mu \otimes 1 \rightarrow -i\nabla_\mu \otimes 1 - e\mathcal{A}_0^\varphi$, we have the functional integral representation below [H97].

$$(F, e^{-tH} G)_{\mathcal{H}} = \int_{\mathbb{R}^d \times W} e^{-\int_0^t V(X_s) ds} (J_0 F(X_0), e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(x))} J_t G(X_t))_{L^2(Q_1)} dX, \quad (2.6)$$

where $\mathcal{K}_1^{[0,t]}(x) := \oplus_{\mu=1}^d \int_0^t j_s \tilde{\varphi}(\cdot - X_s) db_\mu(s) \in \oplus^d L^2(\mathbb{R}^{d+1})$.

2.2 Functional integral representations for $e^{-tH(P)}$

We now construct the functional integral representation of $(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b}$. We use the identification $\mathcal{F}_b \cong L^2(Q_0)$ without notices. For $\Psi \in L^2(Q_0)$, we set $\Psi_t := J_t e^{-iP_t \cdot b(t)} \Psi$, $t \geq 0$.

Theorem 2.1 *Let $\Psi, \Phi \in \mathcal{F}_b$. Then*

$$(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b} = \int_W (\Psi_0, e^{-ieA_1(\kappa_1^{[0,t]}(0))}\Phi_t)_{L^2(Q_1)} e^{iP \cdot b(t)} db, \quad (2.7)$$

where $\kappa_1^{[0,t]}(0) := \oplus_{\mu=1}^d \int_0^t j_s \tilde{\varphi}(\cdot - b(s)) db_\mu(s)$.

Proof: We show an outline of the proof. See [H06] for detail. Set $F_s = \rho_s \otimes \Psi \in L^2(\mathbb{R}^d) \otimes \mathcal{F}_{b,\text{fin}}$ and $G_r = \rho_r \otimes \Phi \in L^2(\mathbb{R}^d) \otimes \mathcal{F}_{b,\text{fin}}$, where ρ_s is the heat kernel:

$$\rho_s(x) = (2\pi s)^{-d/2} e^{-|x|^2/(2s)}, \quad s > 0. \quad (2.8)$$

By the fact that $H = U^{-1} \left(\int_{\mathbb{R}^d}^{\oplus} H(P) dP \right) U$ and $U e^{-i\xi \cdot P^T} U^{-1} = \int_{\mathbb{R}^d}^{\oplus} e^{-i\xi \cdot P} dP$, we have

$$(F_s, e^{-tH} e^{-i\xi \cdot P^T} G_r)_{\mathcal{H}} = \int_{\mathbb{R}^d} dP ((UF_s)(P), e^{-tH(P)} e^{-i\xi \cdot P} (UG_r)(P))_{\mathcal{F}_b}, \quad \xi \in \mathbb{R}^d.$$

Here $(UF_s)(P) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot P} e^{ix \cdot P_t} \rho_s(x) \Psi dx$. Note that

$$\lim_{s \rightarrow 0} (UF_s)(P) = \frac{1}{\sqrt{(2\pi)^d}} \Psi \quad (2.9)$$

strongly in $\mathcal{F}_b$ for each $P \in \mathbb{R}^d$. Hence we have by the Lebesgue dominated convergence theorem,

$$\lim_{s \rightarrow 0} (F_s, e^{-tH} e^{-i\xi \cdot P^T} G_r)_{\mathcal{F}_b} = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} dP (\Psi, e^{-tH(P)} e^{-i\xi \cdot P} (UG_r)(P))_{\mathcal{F}_b}. \quad (2.10)$$

On the other hand we see that by (2.6)

$$\lim_{s \rightarrow 0} (F_s, e^{-tH} e^{-i\xi \cdot P^T} G_r)_{\mathcal{H}} = \int_W \rho_r(b(t) - \xi) (J_0 \Psi, e^{-ieA_1(\kappa_1^{[0,t]}(0))} J_t e^{-i\xi \cdot P_t} \Phi)_{L^2(Q_1)} db. \quad (2.11)$$

Here we used that $\int_W db \rho_r(b_t + x - \xi) (J_0 \Psi, e^{-ieA_1(\kappa_1^{[0,t]}(x))} J_t e^{-i\xi \cdot P_t} \Phi)$ is continuous at $x = 0$ and $e^{-i\xi \cdot \tilde{P}_t} (\rho(X_t) \otimes \Phi) = \rho(X_t - \xi) \otimes e^{-i\xi \cdot P_t} \Phi$. Then we obtained by (2.10) and (2.11) that

$$\begin{aligned} & \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} e^{-i\xi \cdot P} (\Psi, e^{-tH(P)} (UG_r)(P))_{\mathcal{F}_b} dP \\ &= \int_W \rho_r(b(t) - \xi) (J_0 \Psi, e^{-ieA_1(\kappa_1^{[0,t]}(0))} J_t e^{-i\xi \cdot P_t} \Phi)_{L^2(Q_1)} db. \end{aligned} \quad (2.12)$$

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Since

$$\int_{\mathbb{R}^d} \|e^{-tH(P)}UG_r(P)\|_{\mathcal{F}_b}^2 dP \leq \int_{\mathbb{R}^d} \|UG_r(P)\|_{\mathcal{F}_b}^2 dP = \|G_r\|_{\mathcal{H}}^2 < \infty,$$

we have $(\Psi, e^{-tH(\cdot)}(UG_r)(\cdot))_{\mathcal{F}_b} \in L^2(\mathbb{R}^d)$ for $r \neq 0$. Then taking the inverse Fourier transform of both sides of (2.12) with respect to P , we have

$$\begin{aligned} & (\Psi, e^{-tH(P)}(UG_r)(P))_{\mathcal{F}_b} \\ &= \frac{1}{\sqrt{(2\pi)^d}} \int_W db \int_{\mathbb{R}^d} d\xi e^{iP \cdot \xi} \rho_r(b(t) - \xi) (J_0 \Psi, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-i\xi \cdot P_t} \Phi)_{L^2(Q_1)} \end{aligned} \quad (2.13)$$

for almost every $P \in \mathbb{R}^d$. Both sides (2.13) are continuous in P , then (2.13) is true for all $P \in \mathbb{R}^d$. Taking $r \rightarrow 0$ on both sides of (2.13), we have by the Lebesgue dominated convergence theorem and (2.9) that

$$(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b} = \int_W (J_0 \Psi, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_t \cdot b(t)} \Phi)_{L^2(Q_1)} e^{iP \cdot b(t)} db = (2.7).$$

Thus the theorem follows for $\Psi, \Phi \in \mathcal{F}_{b,\text{fin}}$. Let $\Psi, \Phi \in \mathcal{F}_b$, and $\Psi_n, \Phi_n \in \mathcal{F}_{b,\text{fin}}$ such that $\Psi_n \rightarrow \Psi$ and $\Phi_n \rightarrow \Phi$ strongly as $n \rightarrow \infty$. Since

$$|(J_0 \Psi_n, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_t \cdot b(t)} \Phi_n)_{L^2(Q_1)}| \leq \|\Psi_n\|_{\mathcal{F}_b} \|\Phi_n\|_{\mathcal{F}_b} \leq c$$

with some constant c independent of n , we have by the Lebesgue dominated convergence theorem

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_W (J_0 \Psi_n, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_t \cdot b(t)} \Phi_n)_{L^2(Q_1)} e^{iP \cdot b(t)} db \\ &= \int_W (J_0 \Psi, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_t \cdot b(t)} \Phi)_{L^2(Q_1)} e^{iP \cdot b(t)} db, \end{aligned}$$

and it is immediate that $\lim_{n \rightarrow \infty} (\Psi_n, e^{-tH(P)}\Phi_n)_{\mathcal{F}_b} = (\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b}$. Hence (2.7) is proven. qed

2.3 Applications

Let $L_{\text{fin}}^2(Q_q) := \bigcup_{N=0}^{\infty} [\oplus_{n=0}^N L_n^2(Q_q)]$ and T a self-adjoint operator on $L^2(\mathbb{R}^{d+q})$. Let us define the operator $\Pi_{q,\mu}(Tf)$ on $L_{\text{fin}}^2(Q_q)$ by

$$\Pi_{q,\mu}(Tf) := i[d\Gamma_q(T), \mathcal{A}_{q,\mu}(f)]$$

for $f \in D(T)$. In the case f is real-valued, $\Pi_{q,\mu}(Tf)$ is a symmetric operator. The self-adjoint extension of $\Pi_{q,\mu}(f)$ with real f is denoted by the same symbol.

Let $\mathcal{K}_+ := \{\Psi \in L^2(Q_0) | \Psi \geq 0\}$ and $\mathcal{K}_+^0 := \{\Psi \in \mathcal{K}_+ | \Psi > 0\}$. It is well known that $e^{iP_t \cdot v} \mathcal{K}_+ \subset \mathcal{K}_+$ for $v \in \mathbb{R}^d$. Fundamental fact is that for real $f \in L^2(\mathbb{R}^{d+1})$,

$$J_0^* e^{i\Pi_{1,\mu}(f)} J_t [\mathcal{K}_+ \setminus \{0\}] \subset \mathcal{K}_+^0, \quad (2.14)$$

i.e., $J_0^* e^{i\Pi_{1,\mu}(f)} J_t$ is positivity improving. See [H00a]. We define $\vartheta := \exp(i\frac{\pi}{2}N)$.

Theorem 2.2 $\vartheta e^{-tH(0)}\vartheta^{-1}$ is positivity improving.

Proof: Let $\Psi, \Phi \in \mathcal{K}_+ \setminus \{0\}$. It is seen that

$$(\Psi, \vartheta e^{-tH(0)}\vartheta^{-1}\Phi)_{\mathcal{F}_b} = \int_W (\Psi, J_0^* e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_t \cdot b(t)} \Phi)_{L^2(Q_0)} db. \quad (2.15)$$

Here we used the facts that $J_t e^{-iP_t \cdot b(t)} e^{-i(\pi/2)N} = e^{-i(\pi/2)\tilde{N}} J_t e^{-iP_t \cdot b(t)}$ and

$$e^{i(\pi/2)\tilde{N}} e^{-ieA_1(f)} e^{-i(\pi/2)\tilde{N}} = e^{-ie\Pi_1(f)},$$

where $\tilde{N} = d\Gamma_1(1)$. Since $J_0^* e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_t \cdot b(t)}$ is positivity improving for each $b \in W$, specifically the integrand in (2.15) is strictly positive for each $b \in W$. Hence the right-hand side of (2.15) is strictly positive, which implies that $\vartheta e^{-tH(0)}\vartheta^{-1}\mathcal{K}_+ \setminus \{0\} \subset \mathcal{K}_+^0$. Thus the theorem follows. qed

Immediate corollaries are as follows.

Corollary 2.3 The ground state $\varphi_g(0)$ of $H(0)$ is unique up to multiple constants, if it exists, and it can be taken as $\vartheta\varphi_g(0) > 0$ in the Schrödinger representation.

Corollary 2.4 It follows that

$$|(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b}| \leq (|\Psi|, e^{-t(\frac{1}{2}P_t^2 + H_{\text{rad}})}|\Phi|)_{L^2(Q_0)}, \quad (2.16)$$

$$|(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_b}| \leq (|\Psi|, \vartheta e^{-tH(0)}\vartheta^{-1}|\Phi|)_{L^2(Q_0)}. \quad (2.17)$$

Proof: When L is positivity preserving, we have $|L\Psi| \leq L|\Psi|$. Furthermore,

$$|(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b}| \leq \int_W (J_0|\Psi|, J_t e^{-iP_t \cdot b(t)}|\Phi|)_{L^2(Q_1)} db = (|\Psi|, e^{-t(\frac{1}{2}P_t^2 + H_{\text{rad}})}|\Phi|)_{L^2(Q_0)}$$

where we used that $b(t)$ is Gaussian with $\int |b_\mu(t)|^2 db = 1/2$. Thus (2.16) follows. We have

$$(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_b} = \int_W (\Psi_0, e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))}\Phi_t)_{L^2(Q_1)} e^{iP \cdot b(t)} db. \quad (2.18)$$

Then

$$|(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_b}| \leq (|\Psi|, \vartheta e^{-tH(0)}\vartheta^{-1}|\Phi|)_{L^2(Q_0)}.$$

Hence (2.17) follows. qed

Let $E(P, e^2) = \inf \sigma(H(P))$.

Corollary 2.5 (1) $0 = E(0, 0) \leq E(0, e^2) \leq E(P, e^2)$, (2) Assume that the ground state $\varphi_g(0)$ of $H(0)$ exists for $e \in [0, e_0)$ with some $e_0 > 0$. Then $E(0, e^2)$ is concave, continuous and monotonously increasing function on e^2 , (3) $E(0, e^2) \leq \inf \sigma(H)$.

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Proof: (2.17) implies $|(\Psi, \vartheta e^{-tH(P)} \vartheta^{-1} \Psi)_{\mathcal{F}_b}| \leq e^{-tE(0, e^2)} \|\Psi\|_{\mathcal{F}_b}^2$. Since ϑ is unitary, (1) follows. Let $\varphi_g(0)$ be the ground state of $H(0)$. Thus by Corollary 2.3, $(1, \varphi_g(0))_{L^2(Q_0)} \neq 0$. Hence

$$E(0, e^2) = \lim_{t \rightarrow \infty} -\frac{1}{t} \log(\Omega, e^{-tH(0)} \Omega)_{\mathcal{F}_b} = \lim_{t \rightarrow \infty} -\frac{1}{t} \log \int_W e^{-\frac{e^2}{2} q_0(\mathcal{K}_1^{[0, t]}(0), \mathcal{K}_1^{[0, t]}(0))} db.$$

Since $e^{-\frac{e^2}{2} q_0(\mathcal{K}_1^{[0, t]}(0), \mathcal{K}_1^{[0, t]}(0))}$ is log convex on e^2 , $E(0, e^2)$ is concave. Then $E(0, e^2)$ is continuous on $(0, e_0)$. Since $E(0, e^2)$ is also continuous at $e^2 = 0$ by the fact that $H(0)$ converges as $e^2 \rightarrow 0$ in the uniform resolvent sense, $E(0, e^2)$ is continuous on $[0, e_0)$. Then $E(0, e^2)$ can be expressed as $E(0, e^2) = \int_0^{e^2} \phi(t) dt$ with some positive function ϕ . Thus $E(0, e^2)$ is monotonously increasing on e^2 . Then (2) is obtained. We have

$$(F, (1 \otimes \vartheta) e^{-tH} (1 \otimes \vartheta^{-1}) G)_{\mathcal{H}} = \int_{\mathbb{R}^d} dP (F(P), \vartheta e^{-tH(P)} \vartheta^{-1} G(P))_{\mathcal{F}_b}.$$

Then by (2.17) it is seen that

$$|(F, (1 \otimes \vartheta) e^{-tH} (1 \otimes \vartheta^{-1}) F)_{\mathcal{H}}| \leq e^{-tE(0, e^2)} \int_{\mathbb{R}^d} dP \|F(P)\|_{\mathcal{F}_b}^2 = e^{-tE(0, e^2)} \|F\|_{\mathcal{H}}^2.$$

Thus (3) follows. qed

3 The n point Euclidean Green functions

The functional integral representations derived in the previous section can be extended to the n point Euclidean Green functions.

Theorem 3.1 *Let $K = d\Gamma(h)$ with a multiplication operator h in $L^2(\mathbb{R}^d)$. We assume that $\Phi_0, \Phi_m \in \mathcal{F}_b$ and $\Phi_j \in \mathcal{F}_b^\infty$ for $j = 1, \dots, m-1$ with $\Phi_j = \Phi_j(A(f_1^j), \dots, A(f_{n_j}^j))$. Then for $P_0, \dots, P_{m-1} \in \mathbb{R}^d$,*

$$\begin{aligned} & (\Phi_0, \prod_{j=1}^m e^{-(s_j - s_{j-1})K} e^{-(t_j - t_{j-1})H(P_{j-1})} \Phi_j)_{\mathcal{F}_b} \\ &= \int_W (\hat{\Phi}_0, e^{-ieA_2(\mathcal{K}_2(0))} \prod_{j=1}^m \hat{\Phi}_j)_{L^2(Q_2)} e^{+i \sum_{j=1}^m (b(t_j) - b(t_{j-1})) P_{j-1}} db, \end{aligned} \quad (3.1)$$

where $\mathcal{K}_2(0) := \oplus_{\mu=1}^d \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \xi_{s_j} j_s \tilde{\varphi}(\cdot - b(s)) db_\mu(s)$ and

$$\hat{\Phi}_j := \Xi_{s_j} J_{t_j} e^{-iP_{j-1} \cdot b(t_j)} \Phi_j = \Phi_j \left(\mathcal{A}_2 \left(\xi_{s_j} j_{t_j} f_1^j(\cdot - b(t_j)) \right), \dots, \mathcal{A}_2 \left(\xi_{s_j} j_{t_j} f_{n_j}^j(\cdot - b(t_j)) \right) \right).$$

Proof: See [H06] for detail.

We shall show some applications of Theorem 3.1, by which we can construct a sequence of measures on W converging to $(\varphi_g(P), T\varphi_g(P))_{\mathcal{F}_b}$ for some bounded operator T . In particular $T = e^{-\beta N}$ and $T = e^{-iA(f)}$ are taken as examples. It is known that $H(P)$ has a unique ground state $\varphi_g(P)$ and $(\varphi_g(P), \Omega)_{\mathcal{F}_b} \neq 0$ for sufficiently small e .

Corollary 3.2 *We suppose that $H(P)$ has the unique ground state $\varphi_g(P)$ and it satisfies $(\varphi_g(P), \Omega)_{\mathcal{F}_b} \neq 0$. Then for $\beta > 0$,*

$$(\varphi_g(P), e^{-\beta N} \varphi_g(P)) = \lim_{t \rightarrow \infty} \int_W e^{(e^2/2)(1-e^{-\beta})D(t)} e^{iP \cdot b(2t)} d\mu_{2t},$$

where $D(t) := q_1(\mathcal{K}_1^{[0,t]}(0), \mathcal{K}_1^{[t,2t]}(0))$ and μ_{2t} is a measure on W given by

$$d\mu_{2t} := \frac{1}{Z} e^{-(e^2/2)q_1(\mathcal{K}_1^{[0,2t]}(0), \mathcal{K}_1^{[0,2t]}(0))} db$$

with normalizing constant Z such that $\int_W e^{iP \cdot b(2t)} d\mu_{2t} = 1$.

Proof: We define the family of isometries $\xi_s = \xi_s(1)$, $s \in \mathbb{R}$, by (2.3). By Theorem 3.1 we have

$$\begin{aligned} (e^{-tH(P)}\Omega, e^{-\beta N} e^{-tH(P)}\Omega)_{\mathcal{F}_b} &= \int_W db e^{iP \cdot b(2t)} (1, e^{-ieA_2(\xi_0 \mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0))} 1)_{L^2(Q_2)} \\ &= \int_W db e^{iP \cdot b(2t)} e^{-(e^2/2)q_2(\xi_0 \mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0))}. \end{aligned}$$

Noticing that $q_2(\xi_s f, \xi_t g) = e^{-|s-t|} q_1(f, g)$, we have

$$q_2(\xi_0 \mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0)) = q_1(\mathcal{K}_1^{[0,2t]}(0), \mathcal{K}_1^{[0,2t]}(0)) - (1 - e^{-\beta}) q_1(\mathcal{K}_1^{[0,t]}(0), \mathcal{K}_1^{[t,2t]}(0)).$$

Then

$$\frac{(e^{-tH(P)}\Omega, e^{-\beta N} e^{-tH(P)}\Omega)}{(e^{-tH(P)}\Omega, e^{-tH(P)}\Omega)} = \int_W e^{(e^2/2)(1-e^{-\beta})D(t)} e^{iP \cdot b(2t)} d\mu_{2t}. \quad (3.2)$$

The corollary follows from (3.2) and

$$s - \lim_{t \rightarrow \infty} \frac{e^{-tH(P)}\Omega}{\|e^{-tH(P)}\Omega\|_{\mathcal{F}_b}} = \frac{(\varphi_g(P), \Omega)_{\mathcal{F}_b}}{|(\varphi_g(P), \Omega)_{\mathcal{F}_b}|} \cdot \varphi_g(P)$$

qed

Corollary 3.3 *Assume the same assumptions as in Corollary 3.2. Then*

$$(\varphi_g(P), e^{-iA(f)} \varphi_g(P))_{\mathcal{F}_b} = \lim_{t \rightarrow \infty} \int_W e^{-eq_1(\mathcal{K}_1^{[0,2t]}(0), f^t) - \frac{1}{2}q_0(f, f)} e^{iP \cdot b(2t)} d\mu_{2t}, \quad (3.3)$$

where $f^t := \oplus_{\mu=1}^d j_t f_\alpha(\cdot - b(t))$.

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Proof: We have by Theorem 3.1

$$\begin{aligned}
 (\varphi_g(P), e^{-iA(f)} \varphi_g(P))_{\mathcal{F}_b} &= \lim_{t \rightarrow \infty} \frac{(e^{-tH(P)} \Omega, e^{-iA(f)} e^{-tH(P)} \Omega)_{\mathcal{F}_b}}{(e^{-tH(P)} \Omega, e^{-tH(P)} \Omega)_{\mathcal{F}_b}} \\
 &= \lim_{t \rightarrow \infty} \frac{1}{Z} \int_W db e^{iP \cdot b(2t)} (1, e^{-i(eA_1(\mathcal{K}_1^{[0,2t]}(0)) + A_1(jt))} 1)_{L^2(Q_1)} \\
 &= \lim_{t \rightarrow \infty} \frac{1}{Z} \int_W db e^{iP \cdot b(2t)} e^{-\frac{1}{2} q_1(e\mathcal{K}_1^{[0,2t]}(0) + f^t)}.
 \end{aligned}$$

Note that $q_1(f^t, f^t) = q_0(f, f)$. Then the corollary follows. qed

Remark 3.4 *It is informally written as*

$$\begin{aligned}
 &q_1(\mathcal{K}_1^{[S,T]}(0), \mathcal{K}_1^{[S',T']}(0)) \\
 &= \frac{1}{2} \sum_{\alpha, \beta=1}^d \int_S^T db_\alpha(s) \int_{S'}^{T'} db_\beta(r) \int_{\mathbb{R}^d} \delta_{\alpha\beta}^\perp(k) \frac{|\hat{\varphi}(k)|^2}{\omega(k)} e^{-|s-r|\omega(k)} e^{-ik \cdot (b(s) - b(r))} dk.
 \end{aligned}$$

and

$$q_1(\mathcal{K}_1^{[0,2t]}(0), f^t) = \frac{1}{2} \sum_{\alpha, \beta=1}^d \int_0^{2t} db_\alpha(s) \int_{\mathbb{R}^d} \delta_{\alpha\beta}^\perp(k) \frac{\overline{\hat{\varphi}(k)}}{\sqrt{\omega(k)}} \hat{f}_\beta(k) e^{ik \cdot (b(s) - b(t))} e^{-|s-t|\omega(k)} dk.$$

4 The Pauli-Fierz Hamiltonian with spin 1/2

Let us include the spin of the electron. Let $d = 3$ and $\sigma_1, \sigma_2, \sigma_3$ be the 2×2 Pauli matrices given by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Pauli-Fierz Hamiltonian with spin 1/2 is defined by

$$H_\sigma(P) = \frac{1}{2} (P - P_f - eA(0))^2 + H_{\text{rad}} - \frac{e}{2} \sum_{\mu=1}^3 \sigma_\mu B_\mu(0),$$

where $B(0) = \text{rot} A(x)$. Although $H_\sigma(P)$ acts on $\mathbb{C}^2 \otimes \mathcal{F}_b$, it can be reduced to the self-adjoint operator on $L^2(\mathbb{Z}/2\mathbb{Z}; Q_0)$. The functional integral representation of $e^{-tH_\sigma(P)}$ can be also constructed by making use of 3 + 1 dimensional Lévy process $(b(t), N_t)$ with values in $\mathbb{R}^3 \times (\mathbb{N} \cup \{0\})$, where N_t denotes the Poisson process on a measure space (S, Σ, P_P) with $\mathbb{E}_{P_P}[N_t = N] = e^{-t} t^N / N!$. For $\sigma \in \mathbb{Z}/2\mathbb{Z}$ we define $\sigma_t = \sigma(-1)^{N_t}$. Let $\mathcal{B}_q(x) = \text{rot} \mathcal{A}_q(x)$. The net result is

Theorem 4.1 Let $\Phi, \Psi \in L^2(\mathbb{Z}/2\mathbb{Z}; Q_0)$. Then

$$(\Phi, e^{-tH_\sigma(P)}\Psi) = \lim_{\epsilon \rightarrow 0} e^t \sum_{\sigma \in \mathbb{Z}/2\mathbb{Z}} \int_{W \times S} db \otimes dP_P \left[e^{iP \cdot b(t)} \int_{Q_1} d\mu_1 \overline{J_0 \Phi(\sigma)} e^{X_t^\epsilon} J_t e^{-iP \cdot b(t)} \Psi(\sigma_t) \right], \quad (4.1)$$

where

$$\begin{aligned} X_t = & -ie \sum_{\mu=1}^3 \int_0^t \mathcal{A}_{1,\mu}(\lambda(\cdot - b(s))) db_s^\mu - \int_0^t \left(-\frac{e}{2}\right) \sigma_s \mathcal{B}_{1,3}(j_s \lambda(\cdot - b(s))) ds \\ & + \int_0^{t+} \log(-H_{\text{od}}(b(s), -\sigma_{s-}, s) - \epsilon \psi_\epsilon(H_{\text{od}}(b(s), -\sigma_{s-}, s))) dN_s \end{aligned}$$

and

$$H_{\text{od}}(x, -\sigma, s) = \frac{e}{2} (\mathcal{B}_{1,1}(j_s \lambda(\cdot - b(s))) - i\sigma \mathcal{B}_{1,2}(j_s \lambda(\cdot - b(s))))$$

with the indicator function $\psi_\epsilon(x) = \begin{cases} 1, & |x| < \epsilon/2, \\ 0, & |x| \geq \epsilon/2. \end{cases}$

Proof: See [HL07] for detail.

qed

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