

ON A $\{1,2\}$ -FACTOR OF A GRAPH

by

Jin Akiyama*

Hiroshi Era**

Abstract.

A criterion for the existence in a graph of a spanning regular subgraph of degree 1 was found by Tutte [4], [2, Theorem 9.4]. We now give an analogous criterion for the existence in a graph of a spanning subgraph whose point degrees are 1 or 2.

1. DEFINITIONS AND NOTATION

A factor of a graph G is a spanning subgraph of G which is not totally disconnected. An n -factor is regular of degree n . We define a new factor of a graph called a $\{1,2\}$ -factor of a graph, which is strongly related to a spanning linear forest of a graph.

Let G be a graph and H be a subgraph of G . A subgraph H is called a $\{1,2\}$ -subgraph of G if $1 \leq \deg_H v \leq 2$ for every point v of H . Then H is called a $\{1,2\}$ -factor of G if H is a factor of G . In other words, a $\{1,2\}$ -factor of G is a special kind of a spanning linear forest of G having no isolated points. Every graph has, of course, a spanning linear forests, but not necessarily $\{1,2\}$ -factor. We illustrate two graphs having no $\{1,2\}$ -factors in Figure 1.

* Dept. of Mathematics, Nippon Ika University, Kawasaki, 211, Japan

** Dept. of Mathematics, Tokai University, Hiratsuka, Japan

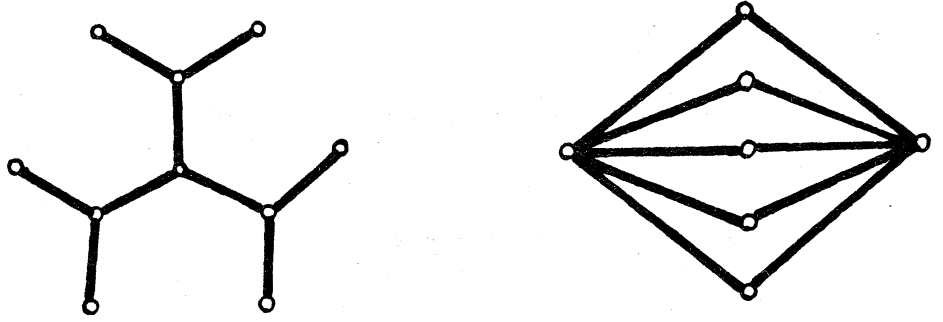


Figure 1. Two graphs having no $\{1,2\}$ -factors.

Throughout this sections we denote by $S_0(G)$ a set of isolated points of G . A $\{1,2\}$ -subgraph M of G is called maximal if the inequality $|V(M)| \geq |V(M')|$ holds for any $\{1,2\}$ -subgraph M' of G (M standing for maximal).

Let v, w be points of G and M be a maximal $\{1,2\}$ -subgraph of G . Then a vw -path $P = [v = v_0, v_1, \dots, v_\ell = w]$ in G is called a vw -alternating path with respect to M if the lines of P alternately lie in M , saying more precisely, $v_{2k}v_{2k+1} \notin M$ and $v_{2k+1}v_{2k+2} \in M$ for $k = 0, 1, \dots, \{\ell/2\}-1$. In Figure 2, we illustrate a maximal $\{1,2\}$ -subgraph M by the lines with slash and a vw -alternating path P with respect to M by the bold lines.

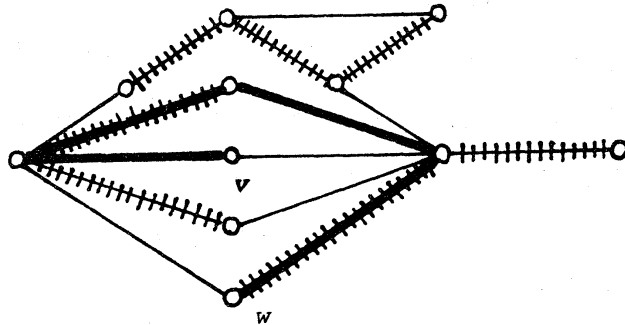


Figure 2 A maximal $\{1,2\}$ -subgraph M and a vw -alternating path with respect to M .

2. CHARACTERIZATION

The following theorem gives a characterization for graphs possessing a $\{1,2\}$ -factor. In general, this test for a $\{1,2\}$ -factor is quite inconvenient to apply.

THEOREM 1 A graph G has a $\{1,2\}$ -factor if and only if the following inequality holds:

$$|S_0(G - S)| \leq 2|S| \text{ for any point subset } S \text{ of } G.$$

NECESSITY OF THEOREM Suppose that G has a $\{1,2\}$ -factor F . Denote by $F_1, F_2, \dots, F_r$, the components of F . Let S be any point subset of G and

$$V_i = \{v | v \in S_0(G - S) \text{ and } v \in V(F_i)\}.$$

Then $\bigcup_{i=1}^r V_i = S_0(G - S)$, and $V_i \cap V_j = \emptyset, i \neq j$.

From the fact that every component F_i is either a path or a cycle, the following inequality follows at once:

$$2|S \cap V(F_i)| \geq |V_i|, i = 1, 2, \dots, r.$$

Thus we obtain the inequality:

$$2|S| = \sum_{i=1}^r 2|S \cap V(F_i)| \geq \sum_{i=1}^r |V_i| = |S_0(G - S)|.$$

We require three lemmas in order to prove the sufficiency of the theorem.

LEMMA 1 Let M be a maximal $\{1,2\}$ -subgraph of G , $u=u_0$ be a point of $G - M$ and $P = [u=u_0, u_1, \dots, u_{2\ell}]$ be an alternating path. Then

$$\deg_M(u_{2i-1}) = 2 \quad \text{and} \quad \deg_M(u_{2i}) = 1, i = 1, 2, \dots, \ell.$$

PROOF. Suppose that $\deg_M(u_{2i-1}) = 1$ for some i .

Denote by M' the subgraph of G obtained from M by deleting lines $u_{2j-1}u_{2j}$, $j = 1, \dots, i$ and adding lines $u_{2j}u_{2j+1}$, $j = 0, \dots, i-1$ instead. Figure 3 illustrates a path $p = [u=u_0, \dots, u_{12}]$ in M and M' when $i = 4$, respectively.

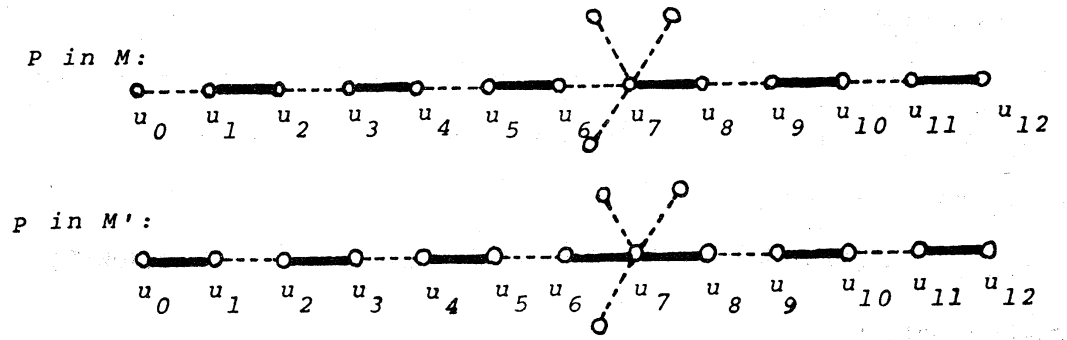


Figure 3 A step in the proof of Lemma 1

Then the following relations are easily verified:

$$\deg_{M'}(u_j) = \deg_M(u_j) \quad \text{for } j = 1, \dots, 2i-2.$$

and

$$\deg_{M'}(u_0) = 1, \quad \deg_{M'}(u_{2i-1}) = 2.$$

Furthermore,

$\deg_M(v) = \deg_{M'}(v)$ for every point v of M other than points u_i , $i = 0, 1, \dots, 2i-1$. Thus we see that M' is also a $\{1, 2\}$ -subgraph of G , contradicting the maximality of M since $|V(M')| > |V(M)|$.

In a quite similar way, we obtain that

$$\deg_M(u_{2i}) = 1 \quad \text{for } i = 1, 2, \dots$$

□

We denote by $A(u)$ (or $A_M(u)$) the set of all points v of G such that there exists a uv -alternating path with respect to a maximal $\{1, 2\}$ -subgraph M . Note that $A(u) \subseteq V(M)$.

LEMMA 2 Let u be a point of $G - M$ and $P = [u=u_0, u_1, \dots, u_k]$ be an alternating path. If a point w is adjacent to one of u_{2i} of G , then w is a point of $A(u)$ and its degree in M is 2:

that is,

$$w \in A(u) \quad \text{and} \quad \deg_M(w) = 2.$$

PROOF. Suppose that $w \notin A(u)$ or " $w \in A(u)$ and $\deg_M(w) = 1$ ",

then we could have another $\{1,2\}$ -subgraph M' of order greater than $|V(M)|$ in a quite similar method applied in the proof of the previous lemma. This contradicts the maximality of M . $\square$

LEMMA 3 Let u be a point of $G - M$ and v be a point in $A(u)$, then every component of M containing v is isomorphic to the path P_3 .

PROOF. Let v be a point in $A(u)$ and $P = [u = u_0, u_1, \dots, u_k = v]$ be an uv -alternating path. We divide the proof into two cases depending on the parity of k .

CASE 1. k : odd

It follows immediately from Lemma 1 that $\deg_M(v) = 2$, since $v = v_k$ and k is odd. We now suppose that the component of M containing v is not isomorphic to P_3 . Then M would contain either a path $P_4 = w_1, v, w_2, w_3$ or a triangle $C_3 = w_1, v, w_2, w_1$. Again applying the same method as in the proof of Lemma 1 we could construct a bigger $\{1,2\}$ -subgraph of G than M , contradicting the maximality of M .

CASE 2. k : even

Considering the fact that the line $u_{k-1}u_k \in M$ and that $k - 1$ is of course odd, the theorem follows at once from Case 1. $\square$

We are now ready to give the sufficiency of Theorem 1 by using the previous three lemmas.

SUFFICIENCY OF THEOREM

Suppose that G does not have a $\{1,2\}$ -factor. Let M be a maximal $\{1,2\}$ -subgraph, u be a point of $G - M$ and S be a set defined by:

$$S = \{v \mid v \in A(u), \deg_M(v) = 2\}.$$

Noting that $\deg_M(v') = 1$ for any point v' of $A(u) - S$, we see that the length

of every uv' -alternating path is even by Lemma 1. Thus every point w adjacent to v' belongs to S by Lemma 2, which implies that the removal of all the points in S from G results in v' isolated, that is, $\deg_{G-S} v' = 0$. Furthermore it follows at once from the definition of $A(u)$ that $\deg_{G-S}(u) = 0$. Hence the following relation holds:

$$S_0(G - S) \supset (A(u) - S) \cup \{u\}.$$

On the other hand, since every component of M containing $v \in A(u)$ is isomorphic to the path P_3 by Lemma 3, we obtain

$$2|S| = |A(u) - S|.$$

Therefore the following inequality holds:

$$|S_0(G - S)| \geq 2|S| + 1,$$

completing the proof. $\square$

COROLLARY 1. Every regular graph has a $\{1,2\}$ -factor.

PROOF. Let G be r -regular and S be any point subset of G . Consider the following two numbers D_1 and D_2 :

$$D_1 = \sum_{v \in S_0} \deg_G v = r|S|,$$

$$D_2 = \sum_{v \in S_0(G-S)} \deg_G v = r|S_0(G-S)|.$$

Then the inequality $D_1 \geq D_2$ holds, since every point of $S_0(G - S)$ is adjacent to only points of S in G . Thus we obtain

$$2|S| > |S| \geq |S_0(G - S)| \text{ for any point subset } S \text{ of } G.$$

The proof is completed by Theorem 1. $\square$

This result follows at once from Tutte's Theorem [5].

3. {1,2}-FACTORIZATION

If G is the sum of $\{1,2\}$ -factors, their union is called an $\{1,2\}$ -factorization and G itself is $\{1,2\}$ -factorable. Using this terminology, it has proved in [0] that every cubic graph is $\{1,2\}$ -factorable.

A criterion for the decomposability of a graph into 2-factors was obtained by Petersen [3].

THEOREM A A graph is 2-factorable if and only if it is regular of even degree.

By applying this, we obtain the following result:

THEOREM 2. Every regular graph is $\{1,2\}$ -factorable.

PROOF. Let G be r -regular. If r is even, then G is 2-factorable by Theorem A and thus trivially $\{1,2\}$ -factorable.

We thus assume r odd. By Corollary 1, we see that G has a $\{1,2\}$ -factor F . By G' we denote the graph obtained by deleting all lines of F from G . Note that the degree of every point in G' is either $r - 1$ or $r - 2$. By adding a set I of new points and applying the method in the proof of Theorem 1.2 in [1], G' can be embedded into some $(r - 1)$ -regular graph G'' as an induced subgraph. Applying Theorem A again, G'' can be decomposed into $(r - 1)/2$ 2-factors, F_i , $i = 1, \dots, (r-1)/2$. Removing a set I of points from G'' , every F_i results in a $\{1,2\}$ -factor of G , since the deficiency $e(v) = r - \deg v$ of each point v of G is at most one. Thus, these $\{1,2\}$ -factors F_i , $i = 1, \dots, (r-1)/2$, together with F constitutes a $\{1,2\}$ -factorization of G . $\square$

Theorem 2 has the following corollary, which was independently proved by Tutte [5].

COROLLARY 2. Every r -regular graph G has a $\{k, k+1\}$ -factor for every k , $1 \leq k \leq r$.

PROOF. The corollary is trivial when r is even, since G is 2-factorable by Theorem A. Thus we assume r odd. Then G can be decomposed into $\{r/2\}$ $\{1,2\}$ -factors $F_1, F_2, \dots, F_{\{r/2\}}$ by Theorem 2. Note here that for any point v of G there exists exactly one $\{1,2\}$ -factor F_j such that the degree of v in F_j is one. Deleting all lines of $F_1, \dots, F_i$ from G , we obtain an $\{r - 2i, r - 2i + 1\}$ -factor of G , for any $i, 1 \leq i \leq \{r/2\}$. The proof is thus completed. $\square$

Acknowledgement.

The authors would like to thank Professor Frank Harary for some helpful suggestions on the presentation of this paper.

References.

- [0] J.Akiyama, G.Exoo and F.Harary, Covering and packing in graphs III: Cyclic and acyclic invariants. Math. Slovaca. (to appear)
- [1] M.Behzad, G.Chartrand and L.Lesniak-Foster, Graphs and digraphs, Holt, Rinehart and Winston, N.Y. (1979).
- [2] F.Harary, Graph Theory, Addison-Wesley, Reading (1969).
- [3] J.Petersen, Die Theorie der regulären Graphen, Acta Math. 15 (1891)193-200.
- [4] W.Tutte, The factorizations of linear graphs, J. London Math. Soc. 22 (1947)107-111.
- [5] W.Tutte, The subgraph problem, Advances in Graph Theory (B.Bollobas, ed.), North-Holland, Amsterdam (1978)289-295.