

Existence of global solutions for a semilinear parabolic Cauchy problem

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1 Introduction

In this paper, we consider the following Cauchy problem

$$\begin{cases} w_t = \Delta w + |x|^l w^p, & x \in \mathbf{R}^n, t > 0, \\ w(x, 0) = f(x), & x \in \mathbf{R}^n, \end{cases} \quad (1.1)$$

where $p > 1$, $l > -2$ and $n \geq 3$ are parameters, and f is a nonnegative bounded continuous function in $\mathbf{R}^n$. This problem is more general version of

$$\begin{cases} w_t = \Delta w + w^p, & x \in \mathbf{R}^n, t > 0, \\ w(x, 0) = f(x), & x \in \mathbf{R}^n. \end{cases} \quad (1.2)$$

In 1966, Fijita [2] has proved that if $p < 1 + 2/n$, then the solution of (1.2) blows up in finite time for all $f \geq 0$ and $f \not\equiv 0$, and if $p > 1 + 2/n$, then (1.2) has a global classical solution when f satisfies $f(x) < \delta \exp(-|x|^2)$ where δ is sufficiently small positive number. Moreover, Lee and Ni [4] have shown that the global existence when the initial value f has polynomial decay near $x = \infty$. Their result is that if $p > 1 + 2/n$ and f satisfies $f(x) \sim (1 + |x|^2)^{-1/(p-1)}$, then (1.2) has a global classical solution and the solution $w(x, t)$ satisfies $\|w(\cdot, t)\|_{L^\infty(\mathbf{R}^n)} \sim t^{-1/(p-1)}$ as $t \rightarrow \infty$. Furthermore, for the problem (1.1), Wang [5] has derived the following result.

Theorem A. ([5]) Suppose $n \geq 3$, $l > -2$ and $p \geq (n + 2 + 2l)/(n - 2)$, then there exists a small $\mu > 0$ such that if $0 \leq f(x) \leq \mu(1 + |x|)^{-(2+l)/2(p-1)}$ in $\mathbf{R}^n$, then (1.1) has global solution $w(x, t)$ with $\|w(\cdot, t)\|_{L^\infty(\mathbf{R}^n)} \leq Mt^{-(2+l)/2(p-1)}$.

In order to prove Theorem A, we introduce the following semilinear elliptic equation with a gradient term

$$\Delta u + \frac{1}{2}x \cdot \nabla u + \lambda u + |x|^l u^p = 0, \quad x \in \mathbf{R}^n. \quad (1.3)$$

where, n, l, p and λ are parameters. It is easily seen that $w(x, t)$ given by

$$w(x, t) = t^{-(2+l)/2(p-1)} u(x/\sqrt{t}) \quad (1.4)$$

satisfies the equation of (1.1) if and only if $u(x) : \mathbf{R}^n \rightarrow \mathbf{R}$ in (1.4) satisfies

$$\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{2+l}{2(p-1)}u + |x|^l u^p = 0, \quad x \in \mathbf{R}^n. \quad (1.5)$$

Since we will consider the radial solutions ($u = u(r)$ with $r = |x|$) to (1.3), we need the following initial value problem

$$\begin{cases} u'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)u' + \lambda u + r^l u^p = 0, & r > 0, \\ u(0) = \alpha > 0, \quad u'(0) = 0. \end{cases} \quad (1.6)$$

Note that when $l > -2$, (1.6) has a unique solution $u(r) \in C^1([0, \infty)) \cap C^2((0, \infty))$, which is denoted by $u(r; \alpha)$. Then Wang has shown the following result.

Theorem B. ([5]) Suppose $n \geq 3$, $l > -2$ and $p \geq (n + 2 + 2l)/(n - 2)$. Then,

- (i) $\lim_{r \rightarrow \infty} r^{2\lambda} u(r; \alpha)$ exists.
- (ii) If $\lim_{r \rightarrow \infty} r^{2\lambda} u(r; \alpha) = 0$, then $\lim_{r \rightarrow \infty} r^m u(r; \alpha) = 0$ and $\lim_{r \rightarrow \infty} r^m u_r(r; \alpha) = 0$ for all positive m .
- (iii) Let $\lambda = (2+l)/2(p-1)$ Then there exists some positive number $\tilde{\alpha}$ such that $u(r; \tilde{\alpha}) > 0$ for all $r \geq 0$ and $u(r; \tilde{\alpha})$ satisfies $u(r; \tilde{\alpha}) \sim r^{-2\lambda}$ as $r \rightarrow \infty$.

Wang has shown Theorem A by using Theorem B. We introduce the sketch of the proof as follows:

Sketch of the proof of Theorem A. Let $\lambda = (2+l)/2(p-1)$. When the parameters n , l and p satisfy the assumptions, it follows from Theorem B that there exists some positive solution $u(r)$ with $u(r) \sim r^{-2\lambda}$ as $r \rightarrow \infty$ for the problem (1.6). Now, we set

$$\hat{w}(x, t) = (t+1)^{-\lambda} u(|x|/\sqrt{t+1}),$$

then $\hat{w}(x, t)$ is an upper solution of (1.1) if μ is sufficiently small positive number. Moreover, since the trivial solution can be a lower solution of (1.1), there exists a global solution of (1.1). ■

Namely, if we can show the existence of some positive solution $u(r)$ with $u(r; \alpha) \sim r^{-2\lambda}$ as $r \rightarrow \infty$ for the problem (1.6), then we can conclude the existence of a global solution to (1.1). Our first result is the following one.

Theorem 1.1 Suppose $n \geq 3$. Then there exist some positive numbers α_0 and $l^* = l^*(n) \in (0, 1)$ such that if $-2 < l < l^*$ and $1 + (2+l)/n < p < (n+2+2l)/(n-2)$, then

(i) For any $\alpha \in (0, \alpha_0)$, $u(r; \alpha) > 0$ for all $r \geq 0$ and $u(r; \alpha)$ satisfies $u(r; \alpha) \sim r^{-2\lambda}$ as $r \rightarrow \infty$.

(ii) $u(r; \alpha_0) > 0$ for all $r \geq 0$ and $u(r; \alpha_0)$ satisfies $\lim_{r \rightarrow \infty} r^{2\lambda} u(r; \alpha_0) = 0$.

(iii) For any $\alpha \in (\alpha_0, \infty)$, $u(\cdot, \alpha)$ has a zero in $(0, \infty)$.

(Especially, $l^*(1) = 2/3$ holds.)

Moreover, the following theorem follows from Theorem 1.1 and the sketch of the proof of Theorem A.

Theorem 1.2 Suppose $n \geq 3$. Then there exists some positive number $l^* = l^*(n) \in (0, 1)$ satisfies $-2 < l < l^*$ such that if $1 + (2+l)/n < p < (n+2+2l)/(n-2)$, then there exists a small $\mu > 0$ which satisfies

if $0 \leq f(x) \leq \mu(1 + |x|)^{-(2+l)/2(p-1)}$ in $\mathbf{R}^n$, then (1.1) has global solution $w(x, t)$.

In order to prove Theorem 1.1, we apply the classification theorem by Yanagida and Yotsutani [7]. Let $\varphi(r)$ be a solution of

$$\begin{cases} \varphi'' + \left(\frac{n-1}{r} + 2r\right) \varphi' + \lambda \varphi = 0, & r > 0, \\ \varphi(0) = 1, \quad \varphi'(0) = 0. \end{cases} \quad (1.7)$$

For a solution $u(r)$ of (1.6), if we put

$$u(r) = \varphi(r)v(r), \quad (1.8)$$

then we see that $v(r)$ satisfies

$$\begin{cases} (g(r)v')' + g(r)K(r)|v|^{p-1}v = 0, & r > 0, \\ v(0) = \alpha > 0, \quad v'(0) = 0, \end{cases} \quad (1.9)$$

where

$$g(r) := r^{n-1} \exp(r^2/4) \varphi(r)^2, \quad K(r) := r^l |\varphi(r)|^{p-1}. \quad (1.10)$$

We should note that $\varphi(r) > 0$ on $[0, \infty)$ if $\lambda > -n$ by (i) of Proposition 2.1 in Section 2. To see whether $u(r)$ has a zero or not, we have only to check this property for $v(r)$. For this purpose, we employ the classification theorem by Yanagida and Yotsutani [7], which is stated as follows. Let $g(r)$ and $K(r)$ satisfy

$$\begin{cases} g(r) \in C^2([0, \infty)); \\ g(r) > 0 \quad \text{on} \quad (0, \infty); \\ 1/g(r) \notin L^1(0, 1); \\ 1/g(r) \in L^1(1, \infty), \end{cases} \quad (g)$$

and

$$\begin{cases} K(r) \in C(0, \infty); \\ K(r) \geq 0 \quad \text{and} \quad K(r) \not\equiv 0 \quad \text{on} \quad (0, \infty); \\ h(r)K(r) \in L^1(0, 1); \\ g(r)(h(r)/g(r))^p K(r) \in L^1(1, \infty), \end{cases} \quad (K)$$

where

$$h(r) := g(r) \int_r^\infty g(s)^{-1} ds.$$

Moreover, define

$$G(r) := \frac{2}{p+1} g(r) h(r) K(r) - \int_0^r g(s) K(s) ds, \quad (1.11)$$

$$H(r) := \frac{2}{p+1} h(r)^2 \left(\frac{h(r)}{g(r)} \right)^p K(r) - \int_r^\infty h(s) \left(\frac{h(s)}{g(s)} \right)^p K(s) ds, \quad (1.12)$$

and

$$r_G := \inf\{r \in (0, \infty) : G(r) < 0\}, \quad r_H := \sup\{r \in (0, \infty) : H(r) < 0\}.$$

Theorem C. ([7]) *Assume that $g(r)$ and $K(r)$ satisfy the conditions (g) and (K). Let $v(r; \alpha)$ be a solution of*

$$\begin{cases} (g(r)v')' + g(r)K(r)(v^+)^p = 0, & r > 0, \\ v(0) = \alpha > 0, & v'(0) = 0, \end{cases} \quad (1.13)$$

where $v^+ := \max\{v, 0\}$, and suppose that $G(r) \not\equiv 0$ on $(0, \infty)$.

(i) *If*

$$0 < r_H \leq r_G < \infty, \quad (1.14)$$

then there exists a unique positive number α_0 such that the structure of solutions to (1.13) is as follows.

(a) *For every $\alpha \in (\alpha_0, \infty)$, $v(r; \alpha)$ has a zero in $(0, \infty)$.*

(b) *If $\alpha = \alpha_0$, then $v(r; \alpha) > 0$ on $[0, \infty)$ and*

$$0 < \lim_{r \rightarrow \infty} \left(\int_r^\infty g(s)^{-1} ds \right)^{-1} v(r; \alpha) < \infty. \quad (1.15)$$

(c) *For every $\alpha \in (0, \alpha_0)$, $v(r; \alpha) > 0$ on $[0, \infty)$ and*

$$\lim_{r \rightarrow \infty} \left(\int_r^\infty g(s)^{-1} ds \right)^{-1} v(r; \alpha) = \infty. \quad (1.16)$$

(ii) *If $r_G < \infty$ and $r_H = 0$ (i.e., $H(r) \geq 0$ on $[0, \infty)$), then $v(r; \alpha)$ is positive on $[0, \infty)$ and satisfies (1.16) for every $\alpha > 0$.*

(iii) *If $r_G = \infty$ (i.e., $G(r) \geq 0$ on $[0, \infty)$), then $v(r; \alpha)$ has a zero in $(0, \infty)$ for every $\alpha > 0$.*

2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we prepare the following proposition which is shown in [1] and [3].

Proposition 2.1 *Let φ be the solution of (1.7) and suppose $0 < \lambda < n/2$. Then,*

- (i) $\varphi(r) > 0$ and $\varphi'(r) < 0$ in $(0, \infty)$.
- (ii) $\lim_{r \rightarrow \infty} r^{2\lambda} \varphi(r)$ exists and positive.
- (iii) $\varphi(r) = 1 - \frac{\lambda}{2n} r^2 + o(r^2)$ as $r \rightarrow \infty$.
- (iv) $\exp(r^2/4) \varphi(r)$ is an increasing function of $r \in [0, \infty)$.
- (v) If $0 < \lambda \leq (n-2)/2$ and $(n-2)/2 < \lambda < n/2$, then

$$-2\lambda < \frac{r\varphi'(r)}{\varphi(r)} < 0 \quad (2.1)$$

and

$$-\frac{4\lambda}{n-2\lambda} < \frac{r\varphi'(r)}{\varphi(r)} < 0 \quad (2.2)$$

for all $r \in (0, \infty)$, respectively.

(vi) Set

$$m(\lambda) := \begin{cases} 2\lambda & \text{if } 0 < \lambda \leq \frac{n-2}{2}, \\ \frac{4\lambda}{n-2\lambda} & \text{if } \frac{n-2}{2} < \lambda < \frac{n}{2}. \end{cases}$$

Then for each $\lambda \in (0, n/2)$, $r^{m(\lambda)} \varphi(r)$ is an increasing function of $r \in [0, \infty)$.

By using Proposition 2.1, we can check the conditions imposed on the coefficients of equation of (1.9).

Lemma 2.1 *If $n \geq 3$ and $0 < \lambda < n/2$, then $g(r) := r^{n-1} \exp(r^2/4) \varphi(r)^2$ and $K(r) := r^l |\varphi(r)|^{p-1}$ satisfy (g) and (K), respectively.*

Therefore, $g(r)$ and $K(r)$ are admissible. Substituting their definition (1.10) into $G(r)$ and $H(r)$, we obtain

$$\begin{aligned}
 G(r) &:= \frac{2}{p+1} g(r) h(r) K(r) - \int_0^r g(s) K(s) ds \\
 &= \frac{2}{p+1} r^{2n-2+l} \exp\left(\frac{r^2}{2}\right) \varphi(r)^{p+3} \left\{ \int_r^\infty s^{1-n} \exp\left(-\frac{s^2}{4}\right) \varphi(s)^{-2} ds \right\} \\
 &\quad - \int_0^r s^{n-1+l} \exp\left(\frac{s^2}{4}\right) \varphi(s)^{p+1} ds, \\
 H(r) &:= \frac{2}{p+1} h(r)^2 \left(\frac{h(r)}{g(r)}\right)^p K(r) - \int_r^\infty h(s) \left(\frac{h(s)}{g(s)}\right)^p K(s) ds \\
 &= \frac{2}{p+1} r^{2n-2+l} \exp\left(\frac{r^2}{2}\right) \varphi(r)^{p+3} \left\{ \int_r^\infty s^{1-n} \exp\left(-\frac{s^2}{4}\right) \varphi(s)^{-2} ds \right\}^{p+2} \\
 &\quad - \int_r^\infty s^{n-1+l} \exp\left(\frac{s^2}{4}\right) \varphi(s)^{p+1} \left\{ \int_s^\infty t^{1-n} \exp\left(-\frac{t^2}{4}\right) \varphi(t)^{-2} dt \right\}^{p+1} ds.
 \end{aligned}$$

In order to prove Theorem 1.1, we must show condition (1.14). To do so, we will investigate the profiles of $G(r)$ and $H(r)$. First, we will study the increase and decrease. Differentiation yields

$$G'(r) = \left(\int_r^\infty g(s)^{-1} ds \right)^{-p-1} H'(r) = \frac{2}{p+1} g(r) K(r) \left(\Phi(r) - \frac{p+3}{2} \right), \quad (2.3)$$

where

$$\begin{aligned}
 \Phi(r) &:= \left(2g'(r) + \frac{g(r)K'(r)}{K(r)} \right) \int_r^\infty g(s)^{-1} ds \\
 &= r^{n-2} \exp\left(\frac{r^2}{4}\right) \varphi(r)^2 \left[r^2 + \{2(n-1) + l\} + (p+3) \left(\frac{r\varphi'(r)}{\varphi(r)} \right) \right] \\
 &\quad \times \int_r^\infty s^{1-n} \exp\left(-\frac{s^2}{4}\right) \varphi(s)^{-2} ds.
 \end{aligned} \quad (2.4)$$

In view of (2.3), $G(r)$ and $H(r)$ have the same extremal points, namely those $r > 0$ which satisfy $\Phi(r) = (p+3)/2$. So in order to know the sign of $G'(r)$ and $H'(r)$, we must study the relation between $\Phi(r)$ and $(p+3)/2$. We first consider the behaviour of $\Phi(r)$ near $r = 0$ and $r = \infty$.

Lemma 2.2 Suppose $n \geq 3$ and $0 < \lambda < n/2$. Then

$$\lim_{r \rightarrow 0} \Phi(r) = \frac{2n-2+l}{n-2} \quad \text{and} \quad \lim_{r \rightarrow \infty} \Phi(r) = 2.$$

Proof. Using l'Hospital's theorem, in view of the definition of $\Phi(r)$ we derive

$$\begin{aligned}\lim_{r \rightarrow \infty} \Phi(r) &= \lim_{r \rightarrow \infty} \frac{\frac{d}{dr} \int_r^\infty s^{1-n} \exp(-s^2/4) \varphi(s)^{-2} ds}{\frac{d}{dr} \left[r^{n-2} \exp(r^2/4) \varphi(r)^2 \left\{ r^2 + 2(n-1) + l + (p+3) \left(\frac{r\varphi'}{\varphi} \right) \right\} \right]^{-1}} \\ &= 2\end{aligned}$$

with noting (2.1) and (2.2). The case $r \rightarrow 0$ is done similarly. ■

Note that $\Phi(r)$ is continuous in $[0, \infty)$ and $(p+3)/2$ satisfies

$$(2 <) 2 + \frac{2+l}{2n} < \frac{p+3}{2} < \frac{2n-2+l}{n-2} \quad (2.5)$$

if and only if $1 + \frac{2+l}{n} < p < \frac{n+2+2l}{n-2}$. And we can get the following lemma which will be proved in the following Section 3.

Lemma 2.3 *Under the assumptions on n, l, p and λ in Theorem 1.1, there exists a unique number $r^* \in (0, \infty)$ satisfying $\Phi(r^*) = (p+3)/2$ and*

$$\begin{cases} \Phi(r) > \frac{p+3}{2} & \text{in } [0, r^*), \\ \Phi(r) \leq \frac{p+3}{2} & \text{in } (r^*, \infty). \end{cases} \quad (2.6)$$

Therefore, it follows from (2.3) and Lemma 2.3 that

Lemma 2.4 *Under the assumptions of Theorem 1.1, there exists a unique number $r^* \in (0, \infty)$ such that $G(r)$ and $H(r)$ are increasing in $[0, r^*)$ and decreasing in (r^*, ∞) .*

Moreover, in order to locate r_G and r_H , we need to determine the behaviour of $G(r)$ and $H(r)$ near $r = 0$ and $r = \infty$.

Lemma 2.5 *Under the assumptions of Theorem 1.1,*

- (i) $\lim_{r \rightarrow \infty} G(r) = -\infty$.
- (ii) $\lim_{r \rightarrow 0} G(r) = 0$.
- (iii) $\liminf_{r \rightarrow \infty} H(r) \geq 0$.
- (iv) $\limsup_{r \rightarrow 0} H(r) < 0$.

We can show this lemma by using Proposition 2.1. Now we can prove Theorem 1.1.

Proof of Theorem 1.1. First of all, we must note that

$$0 < \lambda < \frac{n}{2} \iff 1 + \frac{2+l}{n} < p < \infty$$

holds when $\lambda = (2+l)/2(p-1)$. As is already seen in Lemma 2.4, both $G(r)$ and $H(r)$ have exactly one local maximum at $r^* \in (0, \infty)$. Moreover, in view of Lemma 2.5 $H(r)$ is negative near $r = 0$ and positive for large r . Thus $H(r^*) > 0$ and $0 < r_H < r^*$. Besides, we obtain $G(r^*) > 0$ from $G(0) = 0$, and the negativity of $G(r)$ for large r yields $0 < r^* < r_G < \infty$; so we conclude that condition (1.14) holds. Thus from Theorem C there exists a unique positive number α_0 such that for every $\alpha \in (\alpha_0, \infty)$ $v(\cdot; \alpha)$, i.e., $u(\cdot; \alpha)$ has a zero in $(0, \infty)$. Moreover, for every $\alpha \in (0, \alpha_0]$ $v(\cdot; \alpha)$ is positive in $[0, \infty)$ and

$$\lim_{r \rightarrow \infty} \left\{ \int_r^\infty s^{1-n} \exp\left(-\frac{s^2}{4}\right) \varphi(s)^{-2} ds \right\}^{-1} v(r; \alpha) \begin{cases} \in (0, \infty) & \text{if } \alpha = \alpha_0, \\ = \infty & \text{if } \alpha \in (0, \alpha_0). \end{cases}$$

Integrating by parts, we obtain

$$\int_r^\infty s^{1-n} \exp\left(-\frac{s^2}{4}\right) \varphi(s)^{-2} ds = 2r^{-n} \exp\left(-\frac{r^2}{4}\right) \varphi(r)^{-2} (1 + o(1))$$

as $r \rightarrow \infty$. (Here we use the boundedness of $r\varphi'/\varphi$.) Taking the decay rate of $\varphi(r)$ and the definition of $v(r)$ into account (see Proposition 2.1 (ii)), this estimate immediately shows that (1.15) with $g(r) = r^{n-1} \exp(r^2/4)\varphi^2$ is equivalent to $\lim_{r \rightarrow \infty} r^{2\lambda} u(r) = 0$. At the same time, we obtain that (1.16) with $g(r) = r^{n-1} \exp(r^2/4)\varphi^2$ is equivalent to $\lim_{r \rightarrow \infty} r^{2\lambda} u(r) > 0$, i.e. $u(r; \alpha)$ satisfies $u(r; \alpha) \sim r^{-2\lambda}$ as $r \rightarrow \infty$ for every $\alpha \in (0, \alpha_0)$. ■

3 Proof of Lemma 2.3

We will show there is exactly one crossing point of $q = \Phi(r)$ and $q = (p+3)/2$ in (r, q) -plane. Our strategy is to investigate the sign of $\Phi'(r_*)$ for r_* satisfying $\Phi(r_*) = (p+3)/2$. Here the existence of r_* is guaranteed by Lemma 2.2, the continuity of $\Phi(r)$ and (2.5).

Define

$$\begin{cases} \Omega_1 := \{r_* \in (0, \infty); \Phi'(r_*) > 0\}, \\ \Omega_2 := \{r_* \in (0, \infty); \Phi'(r_*) = 0\}, \\ \Omega_3 := \{r_* \in (0, \infty); \Phi'(r_*) < 0\}. \end{cases}$$

Then we obtain the following result.

Lemma 3.1 *Suppose the assumptions on n, l, p and λ in Theorem 1.1. Then*

- (i) Ω_1 is empty.
- (ii) Ω_2 consists of at most one element.
- (iii) Ω_3 consists of at most one element.

Proof. Differentiating $\Phi(r)$, we get

$$\begin{aligned} \Phi'(r) = & \left[\frac{r^4}{2} + \{(2n-1) - \lambda(p+3)\}r^2 + 2(n-1)(n-2) \right. \\ & \left. + 2\{r^2 + 2(n-1) + l\} \left(\frac{r\varphi'}{\varphi} \right) + (p+3) \left(\frac{r\varphi'}{\varphi} \right)^2 \right] \\ & \times r^{n-3} \exp\left(\frac{r^2}{4}\right) \varphi(r)^2 \int_r^\infty s^{1-n} \exp\left(-\frac{s^2}{4}\right) \varphi(s)^{-2} ds \\ & - \frac{1}{r} \left\{ r^2 + 2(n-1) + l + (p+3) \left(\frac{r\varphi'}{\varphi} \right) \right\}. \end{aligned} \quad (3.1)$$

For any r_* satisfying $\Phi(r_*) = (p+3)/2$ the equality

$$\begin{aligned} & r_*^{n-3} \exp\left(\frac{r_*^2}{4}\right) \varphi(r_*)^2 \int_{r_*}^\infty s^{1-n} \exp\left(-\frac{s^2}{4}\right) \varphi(s)^{-2} ds \\ & = \frac{p+3}{2r_* \left\{ r_*^2 + 2(n-1) + l + (p+3) \left(\frac{r_*\varphi'(r_*)}{\varphi(r_*)} \right) \right\}} \end{aligned} \quad (3.2)$$

holds. Note that

$$2r_* \left\{ r_*^2 + 2(n-1) + l + (p+3) \left(\frac{r_*\varphi'(r_*)}{\varphi(r_*)} \right) \right\} > 0, \quad (3.3)$$

since the left-hand side of (3.2) is positive. Combining (3.1) with $r = r_*$ and (3.2), we obtain

$$\Phi'(r_*) = \frac{\Psi(r_*)}{2r_* \left\{ r_*^2 + 2(n-1) + l + (p+3) \left(\frac{r_* \varphi'(r_*)}{\varphi(r_*)} \right) \right\}},$$

where

$$\begin{aligned} \Psi(r) := & \frac{p-1}{2} r^4 + \left\{ (2n-1)p - 2n + 5 - \lambda(p+3)^2 + \frac{p-5}{2} l \right\} r^2 \\ & + \{2(n-1) + l\} \{(n-2)p - (n+2) - 2l\} \\ & - 2(p+3) \{r^2 + 2(n-1) + l\} \left(\frac{r \varphi'}{\varphi} \right) - (p+3)^2 \left(\frac{r \varphi'}{\varphi} \right)^2. \end{aligned} \quad (3.4)$$

In view of (3.3), we will investigate the sign of $\Psi(r_*)$ instead of $\Phi'(r_*)$. Using (2.1) and (2.2), it is easily seen that

$$\begin{cases} \lim_{r \rightarrow \infty} \Psi(r) = +\infty, \\ \lim_{r \rightarrow 0} \Psi(r) = \{2(n-1) + l\} \{(n-2)p - (n+2) - 2l\} < 0. \end{cases} \quad (3.5)$$

Moreover, we have the following lemma whose proof will be given at the end of this section.

Lemma 3.2 *Under the assumptions on n , l , p and λ in Theorem 1.1, there exists a unique number $\hat{r} \in (0, \infty)$ satisfying $\Psi(\hat{r}) = 0$ such that*

$$\Psi(r) < 0 \quad \text{in } [0, \hat{r}) \quad \text{and} \quad \Psi(r) > 0 \quad \text{in } (\hat{r}, \infty).$$

Recalling that the sign of $\Phi'(r_*)$ is equivalent to that of $\Psi(r_*)$, from Lemma 3.2 we get the sign of $\Phi'(r_*)$ as follows:

$$\begin{cases} \Phi'(r_*) < 0 & \text{if } r_* \in [0, \hat{r}), \\ \Phi'(r_*) = 0 & \text{if } r_* = \hat{r}, \\ \Phi'(r_*) > 0 & \text{if } r_* \in (\hat{r}, \infty). \end{cases} \quad (3.6)$$

First, we will show (i). If $q = \Phi(r)$ and $q = (p+3)/2$ cross in $(\hat{r}, \infty)$, then there exists a unique number r'_* in $(\hat{r}, \infty)$ such that $\Phi(r'_*) = (p+3)/2$ with $\Phi'(r'_*) > 0$. But it is impossible because of $\Phi(r) \rightarrow 2$ as $r \rightarrow \infty$. Therefore, $\Omega_1 = \emptyset$.

Moreover, if $q = \Phi(r)$ and $q = (p+3)/2$ cross in $[0, \hat{r})$, then there exists a unique number $r''_* \in [0, \hat{r})$ such that $\Phi(r''_*) = (p+3)/2$ with $\Phi'(r''_*) < 0$. Therefore, $\Omega_3 = \emptyset$ or

$\Omega_3 = \{r''_*\}$; so we conclude (iii). Statement (ii) is trivial in view of (3.6). ■

Thus we conclude that the relation between $q = \Phi(r)$ and $q = (p+3)/2$ in (r, q) -plane is one of the following:

- (a) $\Phi(r) > (p+3)/2$ in $[0, r''_*]$ and $\Phi(r) < (p+3)/2$ in (r''_*, ∞) ,
- (b) $\Phi(r) > (p+3)/2$ in $[0, \hat{r})$ and $\Phi(r) < (p+3)/2$ in $(\hat{r}, \infty)$,
- (c) $\Phi(r) > (p+3)/2$ in $[0, r''_*]$ and $\Phi(r) < (p+3)/2$ in $(r''_*, \infty) \setminus \hat{r}$.

(Here we used the notation introduced in the proof of Lemma 3.1.) Therefore, by putting $r^* := r''_*$ in cases (a) and (c) or $r^* = \hat{r}$ in case (b), (i) of Lemma 2.3 holds.

Proof of Lemma 3.2. In order to prove Lemma 3.2, we will investigate the sign of $\Psi'(\tilde{r})$ for $\tilde{r}$ satisfying $\Psi(\tilde{r}) = 0$. If we set

$$\begin{cases} X(r) := r\varphi'/\varphi; \\ A := -(p+3)^2; \\ B(r) := -2(p+3)\{r^2 + 2(n-1) + l\}; \\ C(r) := \frac{p-1}{2}r^4 + \left\{ (2n-1)p - 2n + 5 - \lambda(p+3)^2 + \frac{l}{2}(p-5) \right\} r^2 \\ \quad + \{2(n-1) + l\}\{(n-2)p - (n+2) - 2l\}, \end{cases}$$

then $\Psi(r)$ can be rewritten as

$$\Psi(r) \equiv AX(r)^2 + B(r)X(r) + C(r). \quad (3.7)$$

From (1.7), we can express $X'(r)$ in terms of $X(r)$ as

$$X'(r) = -\frac{1}{r}X(r)^2 - \frac{n-2}{r}X(r) - \frac{r}{2}X(r) - \lambda r. \quad (3.8)$$

So differentiating (3.7) and using (3.8), we obtain

$$\begin{aligned} \Psi'(r) = & -\frac{2A}{r}X(r)^3 - \left\{ \frac{2A(n-2) + B(r)}{r} + Ar \right\} X(r)^2 - \left\{ \frac{(n-2)B(r)}{r} \right. \\ & \left. - B'(r) + \left(2\lambda A + \frac{B(r)}{2} \right) r \right\} X(r) - \lambda B(r)r + C'(r). \end{aligned} \quad (3.9)$$

From $\Psi(\tilde{r}) = 0$, $X(\tilde{r})^2$ and $X(\tilde{r})^3$ can be replaced by

$$\begin{cases} X(\tilde{r})^2 = -\frac{B(\tilde{r})}{A}X(\tilde{r}) - \frac{C(\tilde{r})}{A}, \\ X(\tilde{r})^3 = \left(\frac{B(\tilde{r})^2}{A^2} - \frac{C(\tilde{r})}{A}\right)X(\tilde{r}) + \frac{B(\tilde{r})C(\tilde{r})}{A^2}. \end{cases} \quad (3.10)$$

Substituting (3.10) into (3.9), all the terms containing $X(\tilde{r})$ vanish and $\tilde{r}\Psi'(\tilde{r})$ turns out to be a polynomial in $\tilde{r}^2$ of degree 3:

$$\begin{aligned} (p+3)\tilde{r}\Psi'(\tilde{r}) &= \frac{(p+1)(p-1)}{2}\tilde{r}^6 + \eta(l, p, n, \lambda)\tilde{r}^4 + \kappa(l, p, n, \lambda)\tilde{r}^2 \\ &\quad + 2\{2(n-1) + l\}\{(n-2)p + n - 4 - l\}\{(n-2)p - (n+2) - 2l\} \end{aligned}$$

where

$$\eta(l, p, n, \lambda) := -\lambda p^3 + \left(3n - 5\lambda - 1 + \frac{l}{2}\right)p^2 - 3(\lambda - 2 + l)p - 3\left(n - 3\lambda - 1 + \frac{l}{2}\right),$$

and

$$\begin{aligned} \kappa(l, p, n, \lambda) &:= -3(p-1)l^2 + [(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4\lambda(p+3)^2]l \\ &\quad + 2(n-1)(p-1)\{-\lambda p^2 + 3(n-2\lambda-1)p + 3(n-3\lambda-1)\}. \end{aligned}$$

We want to determine the sign of $\Psi'(\tilde{r})$. Setting $x = \tilde{r}^2$, we have $(p+3)\tilde{r}\Psi'(\tilde{r}) = \Theta(x)$, where $\Theta(x)$ is a polynomial of degree 3 given by

$$\begin{aligned} \Theta(x) &:= \frac{(p+1)(p-1)}{2}x^3 + \eta(p, n, \lambda)x^2 + \kappa(p, n, \lambda)x \\ &\quad + 2\{2(n-1) + l\}\{(n-2)p + n - 4 - l\}\{(n-2)p - (n+2) - 2l\} \end{aligned}$$

for $x \in [0, \infty)$. Concerning the profile of $\Theta(x)$, we readily see

$$\begin{cases} \Theta(0) = 2\{2(n-1) + l\}\{(n-2)p + n - 4 - l\}\{(n-2)p - (n+2) - 2l\}, \\ \lim_{x \rightarrow \infty} \Theta(x) = +\infty. \end{cases} \quad (3.11)$$

(Here, we must note that $(n-2)p + n - 4 - l > 0$ holds if $n \geq 3$, $p > 1 + (2+l)/n$ and $-2 < l < n^2 - 2n - 2$.) What may happen in $(0, \infty)$ is clarified by the following lemmas.

Lemma 3.3 Suppose $n \geq 3$, $-2 < l < 1$, $1 + (2+l)/n < p < (n+2+2l)/(n-2)$ and set

$$\lambda^*(l) := \frac{(n-2)\{l^2 + 2(n+4)l + 4(2n+1)\}}{4(l+2)\{l + 2(n-1)\}}.$$

Then $\Theta(x)$ has exactly one zero in $(0, \infty)$ if $0 < \lambda < \lambda^*(l)$.

Proof. Note that it follows from $n^2 - 2n - 2 \geq 1$ for any $n \geq 3$ that $\Theta(0) < 0$ under the assumptions on n, l and p . Differentiating $\Theta(x)$, we obtain

$$\Theta'(x) = \frac{3}{2}(p+1)(p-1)x^2 + 2\eta(l, p, n, \lambda)x + \kappa(l, p, n, \lambda).$$

Now we will show $\eta(l, p, n, \lambda)$ is positive under the assumptions on n, l, p and λ . In fact, define

$$f_1(p) := \eta(l, p, n, \lambda), \quad p \in \left(1, \frac{n+2+2l}{n-2}\right)$$

then we observe

$$f_1(1) = 8 - 4l > 0 \quad (3.12)$$

and

$$\begin{aligned} (n-2)^3 f_1\left(\frac{n+2+2l}{n-2}\right) \\ = 2\{l+2(n-1)\}[(n-2)\{l^2+2(n+4)l+4(2n+1)\} \\ -4(l+2)\{l+2(n-1)\}\lambda] > 0. \end{aligned} \quad (3.13)$$

From (3.12) and (3.13), it is sufficient to show that $f_1(p)$ does not have any local minimum in $(1, (n+2+2l)/(n-2))$. So consider the sign of $f_1''(\tilde{p})$, where $\tilde{p}$ satisfies $f_1'(\tilde{p}) = 0$. Using

$$f_1'(p) = -3\lambda p^2 + 2\left(3n - 5\lambda - 1 + \frac{l}{2}\right)p - 3(\lambda - 2 + l)$$

and

$$f_1''(p) = -6\lambda p + 2\left(3n - 5\lambda - 1 + \frac{l}{2}\right),$$

we obtain

$$\begin{aligned} \tilde{p} f_1''(\tilde{p}) &= -6\lambda \tilde{p}^2 + 2\left(3n - 5\lambda - 1 + \frac{l}{2}\right)\tilde{p} \\ &= -3\lambda(\tilde{p}^2 - 1) + 3(l - 2) < 0. \end{aligned}$$

Thus, if $f_1(p)$ has an extremum, then it must be a local maximum and we conclude

$$\eta(l, p, n, \lambda) = f_1(p) \geq \min\left\{f_1(1), f_1\left(\frac{n+2+2l}{n-2}\right)\right\} > 0.$$

Since $\eta(l, p, n, \lambda) > 0$, the quadratic equation $\Theta'(x) = 0$ does not have two solutions in $(0, \infty)$; so $\Theta(x)$ has at most one extremum. Therefore, either $\Theta(x)$ is increasing in $[0, \infty)$, or $\Theta(x)$ has a local minimum at $\hat{x}_1 \in (0, \infty)$ such that $\Theta(x)$ is decreasing in $[0, \hat{x}_1)$ and increasing in $(\hat{x}_1, \infty)$. Thus from (3.11) and $\Theta(0) < 0$ we can conclude that $\Theta(x)$ has exactly one zero in $(0, \infty)$. ■

Here, note that $\lim_{l \rightarrow -2+0} \lambda^*(l) = +\infty$ and the following result which is easily shown.

Lemma 3.4 $\lambda^*(l)$ is decreasing in $(-2, \infty)$ and $\lim_{l \rightarrow \infty} \lambda^*(l) = \frac{n-2}{4} \left(< \frac{n}{2} \right)$.

So for $l \in (-2, 1)$, which satisfies $\lambda^*(l) < n/2$, and $\lambda \in (\lambda^*(l), n/2)$, the profile of $\Theta(x)$ has not been derived. First, we will prove the following result.

Lemma 3.5 Let $n = 3$. If $-2 < l < 2/3$, then $\Theta(x)$ has exactly one zero in $(0, \infty)$ for any $\lambda \in (0, 3/2)$ and $p \in (1 + (2+l)/3, 5+2l)$.

Proof. Since $\lambda^*(2/3) = 85/112 > 3/4$, for any $\lambda \in (0, 3/4)$, $\Theta(x)$ has exactly one zero in $(0, \infty)$ from Lemmas 3.3 and 3.4. So we suppose $3/4 \leq \lambda < 3/2$ and set

$$\begin{aligned} f_2(l) &:= \kappa(l, p, 3, \lambda) \\ &= -3(p-1)l^2 + \{3(p^2 - 10p - 7) + 4\lambda(p+3)^2\}l \\ &\quad + 4(p-1)\{-\lambda p^2 + 6(1-\lambda)p + 3(2-3\lambda)\}. \end{aligned}$$

Then we get

$$\begin{aligned} f_2(l) &= -3(p-1) \left\{ l - \frac{3(p^2 - 10p - 7) + 4\lambda(p+3)^2}{6(p-1)} \right\}^2 \\ &\quad + \frac{\{3(p^2 - 10p - 7) + 4\lambda(p+3)^2\}^2}{12(p-1)} \\ &\quad + 4(p-1)\{-\lambda p^2 + 6(1-\lambda)p + 3(2-3\lambda)\} \end{aligned}$$

and

$$\begin{aligned} \frac{3(p^2 - 10p - 7) + 4\lambda(p+3)^2}{6(p-1)} &\geq \frac{3(p^2 - 10p - 7) + 4 \cdot \frac{3}{4}(p+3)^2}{6(p-1)} \\ &= \frac{6(p-1)^2}{6(p-1)} \end{aligned}$$

$$\begin{aligned}
&= p - 1 \\
&> 1 + \frac{2+l}{3} - 1 \\
&= \frac{2+l}{3}.
\end{aligned}$$

Thus the axis of quadratic function $f_2(l)$ is positive for all $l > -2$. Moreover, since $f_2(0) < 0$ under the assumptions on p and λ with $l = 0$ (see Lemma 3.2 in [3]), we obtain $f_2(l)$ is negative in $(-2, 0]$. Furthermore, if $0 < l < 2/3$, then $p > 5/3$, the axis of $f_2(l)$ is greater than $2/3$, and

$$\begin{aligned}
\frac{3}{2}f_2\left(\frac{2}{3}\right) &= -6\lambda p^3 + (-26\lambda + 39)p^2 + (6\lambda - 32)p + 90\lambda - 55 \\
&= -(3p - 5) \left\{ 2\lambda(p + 3)^2 - 13p - 11 \right\} \\
&< -(3p - 5) \left\{ 2 \cdot \frac{3}{4}(p + 3)^2 - 13p - 11 \right\} \\
&= -(3p - 5) \left\{ \frac{3}{2} \left(p - \frac{4}{3} \right)^2 - \frac{1}{6} \right\} \\
&< -(3p - 5) \left\{ \frac{3}{2} \left(\frac{5}{3} - \frac{4}{3} \right)^2 - \frac{1}{6} \right\} \\
&= 0
\end{aligned}$$

holds. Therefore, we have $f_2(l)$ is also negative in $(0, 2/3)$. Hence, we have $f_2(l) < 0$, i.e., $\kappa(l, p, 3, \lambda) < 0$ under the assumptions on l , p and λ . Since $\kappa(l, p, 3, \lambda) < 0$, the quadratic equation $\Theta'(x) = 0$ has exactly one solution in $(0, \infty)$; so $\Theta(x)$ has a local minimum at $\hat{x}_2 \in (0, \infty)$ such that $\Theta(x)$ is decreasing in $[0, \hat{x}_2)$ and increasing in $(\hat{x}_2, \infty)$. Thus it follows from (3.11) that $\Theta(x)$ has a unique zero in $(0, \infty)$. ■

Next, we will show the following lemma.

Lemma 3.6 *Let $n \geq 4$. Then there exists some positive number $l^* = l^*(n) \in (0, 1)$ such that if $-2 < l < l^*$ and $1 + (2 + l)/n < p < (n + 2 + 2l)/(n - 2)$, $\Theta(x)$ has exactly one zero in $(0, \infty)$ for any $\lambda \in (0, n/2)$.*

Proof. Since it is easily seen

$$\lambda^*(1) = \frac{(n-2)(10n+13)}{12(2n-1)}$$

and

$$\frac{(n-2)(10n+13)}{12(2n-1)} > \frac{3(n-1)}{8} \quad \text{for } n \geq 4,$$

for any $\lambda \in (0, 3(n-1)/8)$, $\Theta(x)$ has exactly one zero in $(0, \infty)$ from Lemmas 3.3 and 3.4. So we suppose $3(n-1)/8 \leq \lambda < n/2$ and set $f_3(l) := \kappa(l, p, n, \lambda)$. Then we get

$$\begin{aligned} f_3(l) = & -3(p-1) \left\{ l - \frac{(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4\lambda(p+3)^2}{6(p-1)} \right\}^2 \\ & + \frac{\{(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4\lambda(p+3)^2\}^2}{12(p-1)} \\ & + 2(n-1)(p-1)\{-\lambda p^2 + 3(n-2\lambda-1)p + 3(n-3\lambda-1)\} \end{aligned}$$

and

$$\begin{aligned} & \frac{(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4\lambda(p+3)^2}{6(p-1)} \\ & \geq \frac{(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4 \cdot \frac{3(n-1)}{8}(p+3)^2}{6(p-1)} \\ & = \frac{(7n-9)p^2 - 6(n+1)p + 3(5n-11)}{12(p-1)} \\ & = \frac{1}{12} \left\{ (7n-9)(p-1) + 8(n-3) + \frac{16(n-3)}{p-1} \right\}. \end{aligned}$$

Thus the axis of $f_3(l)$ is positive for all $l > -2$. Moreover, since $f_3(0) < 0$ under the assumptions on n, p and λ with $l = 0$ (see Lemma 3.2 in [3]), we obtain $f_3(l)$ is negative in $(-2, 0]$. Furthermore, if $n \geq 4$ and $0 < l < 1$, then $1 < 1 + 2/n < p < \frac{n+4}{n-2} < 4$, the axis of $f_3(l)$ is greater than 1. In fact,

$$\begin{aligned} & \frac{(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4\lambda(p+3)^2}{6(p-1)} \\ & \geq \frac{1}{12} \left\{ (7n-9)(p-1) + 8(n-3) + \frac{16(n-3)}{p-1} \right\} \\ & > \frac{1}{12} \left\{ 8(n-3) + \frac{16(n-3)}{4-1} \right\} \\ & = \frac{10}{9}(n-3) > 1 \end{aligned}$$

holds. Therefore, there exists some positive number $l^* = l^*(n) \in (0, 1)$ such that $f_3(l)$ is negative in $(0, l^*)$. Hence, we have $f_3(l) < 0$, i.e., $\kappa(l, p, n, \lambda) < 0$ under the assumptions on n, l, p and λ . Since $\kappa(l, p, n, \lambda) < 0$, the quadratic equation $\Theta'(x) = 0$ has exactly

one solution in $(0, \infty)$; so $\Theta(x)$ has a local minimum at $\hat{x}_3 \in (0, \infty)$ such that $\Theta(x)$ is decreasing in $[0, \hat{x}_3)$ and increasing in $(\hat{x}_3, \infty)$. Thus it follows from (3.11) that $\Theta(x)$ has a unique zero in $(0, \infty)$. ■

Thus in view of Lemmas 3.3, 3.5, 3.6 and (3.11), there exists a unique number $x_0 \in (0, \infty)$ such that

$$\begin{cases} \Theta(x) < 0 & \text{in } (0, x_0), & \text{i.e. } \Psi'(\tilde{r}) < 0 & \text{if } \tilde{r} \in (0, \sqrt{x_0}), \\ \Theta(x_0) = 0, & & \text{i.e. } \Psi'(\tilde{r}) = 0 & \text{if } \tilde{r} = \sqrt{x_0}, \\ \Theta(x) > 0 & \text{in } (x_0, \infty), & \text{i.e. } \Psi'(\tilde{r}) > 0 & \text{if } \tilde{r} \in (\sqrt{x_0}, \infty). \end{cases}$$

Therefore, in view of (3.5), there exists a unique number $\hat{r}$ satisfying $\Psi(\hat{r}) = 0$ with $\hat{r} \geq \sqrt{x_0}$, such that $\Psi(r) < 0$ in $[0, \hat{r})$ and $\Psi(r) > 0$ in $(\hat{r}, \infty)$. This completes the proof of Lemma 3.2. ■

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