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Von Neumann-Jordan constant and some geometrical constants of Banach spaces

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In this note some recent results of the authors are announced concerning von Neumann-Jordan (NJ-) constant, non-square (or James) constant, and normal structure coefficient for a Banach space.

A sequence of results on the NJ-constant of a Banach space X, we denote it by $C_{NJ}(X)$, has been recently obtained by the first and third authors, etc. ([8, 9, 10, 11, 12, 14]; refer to [2, 7] for classical results). Their concerns were/are as follows:

(i) Determine or estimate $C_{NJ}(X)$ for various X.

(ii) What informations does $C_{NJ}(X)$ give about X?

Here we discuss the following question raised by the second author:

(iii) What is the relation between $C_{NJ}(X)$ and some other geometrical constants of X?

In particular we estimate $C_{NJ}(X)$ with the non-square (or James) constant $J(X)$, and also the normal structure coefficient $N(X)$ with $C_{NJ}(X)$. An estimate for $J(X^*)$ with $J(X)$ is given as well.

The *von Neumann-Jordan (NJ-) constant* for a Banach space X (Clarkson [2]), $C_{NJ}(X)$, is the smallest constant for which

$$(1)\qquad \frac{1}{C} \leqq \frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} \leqq C \qquad \forall (x,\ y) \neq (0,\ 0).$$

The *non-square* (or *James*) *constant* of X (Gao-Lau [3]) is defined by

$$(2)\qquad J(X) := \sup_{x,\, y \in S_X} \min\{\|x+y\|,\ \|x-y\|\},$$

where S_X stands for the unit sphere of X. We recall some notions related with J(X):

(i) X is called *uniformly convex* ([1]) if for any ε $(0<\varepsilon<2)$ there exists a $\delta>0$ such that

$$(3)\qquad \|x-y\| \geqq \varepsilon \quad (x,\ y \in S_X) \quad \Longrightarrow \quad \|(x+y)/2\| \leqq 1-\delta.$$

(ii) X is called *uniformly non-square* (James [6]) if there exists a $\delta>0$ $(0<\delta<1)$ such that

$$(4)\qquad \|(x-y)/2\| > 1-\delta \quad (x,\ y \in S_X) \quad \Longrightarrow \quad \|(x+y)/2\| \leqq 1-\delta.$$

The difference between (3) and (4) is clear: In (3) we can let $\varepsilon \to 0$. On the contrary, in (4) we cannot do it, that is, we can only get the same conclusion as (3) for x, $y \in S_X$ apart from each other to some extent.

(iii) The *modulus of convexity* of X ([1]) is defined by

$$\delta_X(\varepsilon) := \inf\{1-\|(x+y)/2\|\ ;\ \|x-y\| \geqq \varepsilon,\ x,\ y \in S_X\}.$$

Now, (4) is reformulated as

$$\min\{\|x+y\|,\ \|x-y\|\} \leqq 2(1-\delta);$$

thus we understand the above definition (2) of the non-square constant J(X) as a sort of modulus of non-squareness of X. Gao and Lau [3] showed that

$$J(X) = \sup\{\varepsilon>0;\ \delta_X(\varepsilon) \leqq 1-\varepsilon/2\}.$$

1. Comparison of NJ- and James constant

We compare some known facts on NJ- and James constants:

(i) For any Banach space X

$$1 \leqq C_{NJ}(X) \leqq 2,$$

$$\sqrt{2} \leqq J(X) \leqq 2 \quad (\dim X \geqq 2)$$

(ii) X: a Hilbert space $\Leftrightarrow$ $C_{NJ}(X) = 1$,

X: a Hilbert space $\Longrightarrow$ $J(X) = \sqrt{2}$

(iii) X: uniformly non-square $\Leftrightarrow$ $C_{NJ}(X) < 2$ (Takahashi-Kato[14])

$\Leftrightarrow$ $J(X) < 2$ (clear by definition)

(iv) Let $1 \leqq p \leqq 2$, $1/p + 1/p' = 1$. Then

(5) $$C_{NJ}(L_p) = C_{NJ}(L_{p'}) = 2^{2/p-1},$$

(6) $$J(L_p) = J(L_{p'}) = 2^{1/p}.$$

2. Relation between $C_{NJ}(X)$ and $J(X)$

Theorem 1. For any Banach space X

(7) $$\frac{1}{2}J(X)^2 \leqq C_{NJ}(X) \leqq \frac{J(X)^2}{(J(X)-1)^2 + 1}.$$

Remarks. According to the facts stated in the preceding section, equality occurs in (7) with several spaces:

(i) $\frac{1}{2}J(L_p)^2 = C_{NJ}(L_p)$; the same is true for $W_p^k(\Omega)$ (Sobolev space), c_p (space of p-Schatten class operators) and $L_p(L_q)$ (L_q-valued L_p-space), etc.

(ii) For a Hilbert space H, $\frac{1}{2}J(H)^2 = C_{NJ}(H) = 1.$

(iii) If X is not uniformly non-square,

$$\frac{1}{2}J(X)^2 = C_{NJ}(X) = \frac{J(X)^2}{(J(X)-1)^2+1} = 2.$$

3. Relation between J(X) and J(X*)

For the dual space X^* it is known that $C_{NJ}(X^*)=C_{NJ}(X)$, whereas $J(X^*)\neq J(X)$ in general. In [4] Gao and Lau ask what relation $J(X)$ and $J(X^*)$ have. We have the following

Theorem 2. For any Banach space X

$$2J(X) - 2 \leqq J(X^*) \leqq \frac{J(X)}{2} + 1.$$

Remark. If X is not uniformly non-square,

$$2J(X) - 2 = J(X^*) = \frac{J(X)}{2} + 1 = 2.$$

Corollary. X^* is uniformly non-square if and only if X is so.

This result seems not to have appeared in literature.

4. NJ-constant and normal structure of Banach spaces

A Banach space X is said to have *normal structure* provided for any bounded convex subset K of X with diam $K > 0$, its radius $r(K)$ is less than diam K, that is,

$$r(K) < \operatorname{diam} K.$$

If there exists some c $(0<c<1)$ such that

$$r(K) \leqq c\cdot\operatorname{diam} K, \tag{8}$$

X is said to have *uniform normal structure*. The smallest c ($0 < c \leqq 1$) satisfying (8) for all K (bounded convex) with diam $K > 0$, is called the *normal structure coefficient* of X and denoted by N(X): Clearly $0 \leqq N(X) \leqq 1$; and X has uniform normal structure if and only if $N(X) < 1$. These notions are strongly connected with the fixed point property. X is said to have *fixed point property* (*FPP*) (for non-expansive mappings) provided for any non-empty bounded convex subset K of X, every non-expansive mapping $T: K \to K$ has a fixed point. It is known ([5]) that (i) If X is reflexive and has the normal structure, then X has FPP; (ii) If X has the uniform normal structure, then X is reflexive, whence X has FPP.

Now, Gao and Lau [4] showed that:

If $J(X) < 3/2$, then X has the uniform normal structure.

Prus [13] gave more precisely the following estimate for N(X) by J(X): For any Banach space X,

$$N(X) \leqq \frac{1}{J(X)+1-\{(J(X)+1)^2-4\}^{1/2}}. \tag{9}$$

Note that the estimate (9) implies that if $J(X) < 3/2$ then $N(X) < 1$. (One should also note here that the definition of N(X) in Prus [11] is the reciprocal of our N(X).) We present the following estimate for N(X) by NJ-constant:

Theorem 3. For any Banach space X

$$N(X) \leqq \left\{C_{NJ}(X) - \frac{1}{4}\right\}^{1/2}. \tag{10}$$

Theorem 4. Let $C_{NJ}(X) < 5/4$. Then X, as well as X*, has the uniform normal structure; and hence X (X*) has the fixed point property.

Indeed, the above estimate (10) impies that if $C_{NJ}(X)<5/4$, then $N(X)<1$. The assertion for X^* is a consequence of the fact that $C_{NJ}(X^*) = C_{NJ}(X)$.

Remarks. (i) For the spaces with $C_{NJ}(X) = \frac{1}{2} J(X)^2$ (recall Remarks after Theorem 1), Gao and Lau's condition $J(X)<3/2$ is rewritten as $C_{NJ}(X)<9/8$; thus our condition $C_{NJ}(X)<5/4=10/8$ is weaker than theirs in this case.

(ii) The normal structure is not inherited by dual spaces ([4; esp. p. 63]).

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