

Trace identity for parabolic elements of $SL(2, \mathbb{C})$, II

Dedicated to Professor Hiroshige Shiga on the occasion of his sixtieth birthday

By

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Abstract

Let $\mathcal{P}$ be the set of all parabolic elements in $SL(2, \mathbb{C})$ with trace -2 . If P_1 and P_2 in $\mathcal{P}$ do not commute, then the complex lambda length between P_1 and P_2 is the trace of a matrix $Q \in SL(2, \mathbb{C})$ satisfying $Q^2 = -P_1 P_2$, which is determined uniquely up to sign. For each n -gon $(P_1, P_2, \dots, P_n)$ in $\mathcal{P}$ consider the tuples $(Q_1, Q_2, \dots, Q_n)$ with $Q_i^2 = -P_i P_{i+1}$ with $P_{n+1} = P_1$. The tuples are classified into tuples of $(-)$ -system and tuples of $(+)$ -system. Suppose that $(P_1, \dots, P_n)$ is divided into subpolygons $(P_1, P_2, \dots, P_m)$ and $(P_1, P_m, P_{m+1}, \dots, P_n)$, and R_m and $S_m \in SL(2, \mathbb{C})$ with $R_m^2 = -P_m P_1$, $S_m^2 = -P_1 P_m$ and $\text{tr} R_m = \text{tr} S_m$ are given. We show that if $(Q_1, \dots, Q_{m-1}, R_m)$ and $(S_m, Q_m, \dots, Q_n)$ are $(-)$ -systems, then $(Q_1, Q_2, \dots, Q_n)$ is also a $(-)$ -system.

§ 1. Introduction and the main result

This paper is a continuation of [4] which established the “ideal Ptolemy identity” for complex λ -lengths introduced in [2] and [3] following Penner’s paper [5]. We define

$$\mathcal{P} = \{P \in SL(2, \mathbb{C}) : P \text{ is parabolic with } \text{tr} P = -2\}.$$

Note that $\mathcal{P}$ is the conjugacy class of

$$(1.1) \quad \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

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and hence two matrices in $\mathcal{P}$ are conjugate to each other in $SL(2, \mathbb{C})$. If two elements P_1 and $P_2 \in \mathcal{P}$ do not commute, then there exists a square root Q of $-P_1P_2$, that is, a matrix in $SL(2, \mathbb{C})$ such that

$$(1.2) \quad Q^2 = -P_1P_2.$$

Q is determined up to sign, satisfies $\text{tr}(P_1P_2) = 2 - (\text{tr}Q)^2$ and also

$$(1.3) \quad P_2 = Q^{-1}P_1Q, \text{ and } Q^{-1}P_1 \text{ and } Q^{-1}P_2 \text{ are elliptic of order 2.}$$

(Here the order of an elliptic A in $SL(2, \mathbb{C})$ means the order of the Möbius transformation $A(z)$.) In order to see this, it suffices to consider the normalized pair

$$P_1 = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -1 & 0 \\ \lambda & -1 \end{pmatrix}$$

with $\lambda \neq 0$. Then Q must be of the form

$$Q = \pm \begin{pmatrix} \sqrt{\lambda} & -1/\sqrt{\lambda} \\ \sqrt{\lambda} & 0 \end{pmatrix}.$$

With this we can verify (1.3) and also

$$(1.4) \quad \text{tr}Q \neq 0.$$

In what follows the diagram

$$(1.5) \quad P_1 \xrightarrow{Q} P_2$$

means that P_1 and $P_2 \in \mathcal{P}$ do not commute and $Q^2 = -P_1P_2$.

Definition 1.1. A cycle $(P_1, P_2, \dots, P_n)$, $P_{n+1} = P_1$, of elements in $\mathcal{P}$ is called an n -gon if P_i and P_j do not commute for $i \neq j$. If, in particular, $n = 3$ or 4 , then it is called a *triangle* or *quadrangle*, respectively. Two n -gons $(P_1, P_2, \dots, P_n)$ and $(R_1, R_2, \dots, R_n)$ are *congruent* if there exists $T \in SL(2, \mathbb{C})$ such that $R_j = T^{-1}P_jT$ for $j = 1, \dots, n$.

Let $(P_1, \dots, P_n)$ be an n -gon in $\mathcal{P}$. Then there exists a square root Q_i of $-P_iP_{i+1}$ for $i = 1, 2, \dots, n$. Since from (1.3)

$$P_2 = Q_1^{-1}P_1Q_1, \quad P_3 = Q_2^{-1}P_2Q_2, \dots, \quad P_1 = Q_n^{-1}P_nQ_n,$$

$Q_1Q_2 \cdots Q_n$ commutes with P_1 and hence $\text{tr}Q_1Q_2 \cdots Q_n$ is either -2 or $+2$.

Definition 1.2. $(Q_1, Q_2, \dots, Q_n)$ is called a $(-)$ -system if $\text{tr}Q_1Q_2 \cdots Q_n = -2$ and a $(+)$ -system if $\text{tr}Q_1Q_2 \cdots Q_n = +2$.

Let $(P_1, P_2, \dots, P_n)$ be an n -gon and Q_j be such that $P_j \xrightarrow{Q_j} P_{j+1}$ for $j = 1, \dots, n$. If $2 < m < n$, then the “diagonal” $P_1 P_m$ divides the n -gon into an m -gon $(P_1, P_2, \dots, P_m)$ and an $(n - m + 1)$ -gon $(P_1, P_m, P_{m+1}, \dots, P_n)$. Choose R_m and $S_m \in SL(2, \mathbb{C})$ such that

$$P_m \xrightarrow{R_m} P_1, \quad P_1 \xrightarrow{S_m} P_m,$$

and that $\text{tr} R_m = \text{tr} S_m$. So $S_m = P_1 R_m P_1^{-1}$. The main objective of this paper is to prove

Theorem 1.1. If two among $(Q_1, Q_2, \dots, Q_{m-1}, R_m)$, $(S_m, Q_m, Q_{m+1}, \dots, Q_n)$ and $(Q_1, Q_2, \dots, Q_n)$ are $(-)$ -systems, then so is the rest.

In [4] we showed this theorem for $n = 4$ and $m = 3$. In this case, if both of (Q_1, Q_2, R_3) and (S_3, Q_3, Q_4) are $(-)$ -systems, then (Q_1, Q_2, Q_3, Q_4) is also a $(-)$ -system. We choose R_2 and S_2 so that

$$P_2 \xrightarrow{R_2} P_4, \quad P_4 \xrightarrow{S_2} P_2,$$

and that $\text{tr} R_2 = \text{tr} S_2$. See Figure 1. If (Q_1, R_2, Q_4) is a $(-)$ -system, then from Theorem 1.1, (Q_2, Q_3, S_2) is also a $(-)$ -system. In this situation the following “ideal Ptolemy identity” holds ([4, Theorem 0.1])

$$(1.6) \quad \text{tr} R_2 \text{tr} R_3 = \text{tr} Q_1 \text{tr} Q_3 + \text{tr} Q_2 \text{tr} Q_4.$$

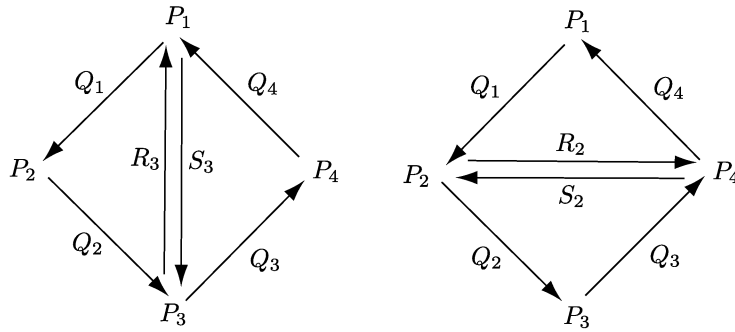


Figure 1. A decomposition of a quadrangle into triangles

Theorem 1.1 follows immediately from

Lemma 1.1. With the notation as above the following identity holds:

$$(1.7) \quad (\text{tr} Q_1 Q_2 \cdots Q_{m-1} R_m)(\text{tr} S_m Q_m \cdots Q_n) = -2 \text{tr} Q_1 Q_2 \cdots Q_n.$$

We prove (1.7) in Section 3.

Remark 1.1. Let $\bar{S}$ be an oriented closed surface of genus g and $P = \{x_1, \dots, x_n\}$ a non-empty set of distinct points on $\bar{S}$. Let $S = \bar{S} - P$. We assume that $2g - 2 + n > 0$. Let $\mathcal{R}(S)$ denote the space of all conjugacy classes of faithful representations $\rho : \pi_1(S) \rightarrow SL(2, \mathbb{C})$ such that if δ is the homotopy class of a loop which goes around a puncture x_j once, then $\rho(\delta) \in \mathcal{P}$. Let $\Delta = \{c_1, c_2, \dots, c_d\}$, where $d = 6g - 6 + 3n$, be an arbitrary ideal triangulation of S (see [5]). Let $c = c_i \in \Delta$ and suppose that x_j and x_k are the end points of c . Choose a point y of c . We define δ_1 to be the loop which goes from y to x_j along c and turns around x_j in the positive direction and goes back to y along c . We define δ_2 in the same way for x_k . Choose an arc δ_0 from the base point of $\pi_1(S)$ to y . Let $[\rho] \in \mathcal{R}(S)$. Then homotopy classes of $\delta_0 \delta_1 \delta_0^{-1}$ and $\delta_0 \delta_2 \delta_0^{-1}$ determine two elements $P_1 = \rho(\delta_0 \delta_1 \delta_0^{-1})$ and $P_2 = \rho(\delta_0 \delta_2 \delta_0^{-1})$ in $\mathcal{P}$. Since ρ is faithful, P_1 and P_2 do not commute. Choose Q_i so that $P_1 \xrightarrow{Q_i} P_2$. The value

$$\lambda_i = \lambda(c_i, \rho) = \text{tr} Q_i.$$

depends only on the class $[\rho]$ and the homotopy class of c_i . This value λ_i is called in [2] and [3] the *complex λ -length* of c_i associated to $[\rho]$. The positive branch of λ_i restricted to the Fuchsian representation space of $\pi_1(S)$ coincides with the λ -length (for a special choice of horocycles around punctures) introduced by Penner [5].

Since λ_i is determined up to sign, the tuple $(\lambda_1, \dots, \lambda_d)$ defines a map $\underline{\Lambda}_\Delta : \mathcal{R}(S) \rightarrow (\mathbb{C}/\{\pm 1\})^d$. If it is restricted to, for example, the subspace $\mathcal{QF}$ of quasifuchsian representations, which is simply connected, the map $\underline{\Lambda}_\Delta$ can be lifted to a holomorphic injection Λ_Δ of $\mathcal{QF}$ into $\mathbb{C}^d$, and it is possible to choose a lift Λ_Δ so that $\lambda_1, \dots, \lambda_d$ satisfy the condition that (Q_i, Q_j, Q_k) are $(-)$ -systems for all triangles (c_i, c_j, c_k) in Δ , see [3] for details. By using (1.6) we can show just as in [5] that, for two ideal triangulations Δ_1 and Δ_2 , the coordinate change between $\Lambda_{\Delta_1}(\mathcal{QF})$ and $\Lambda_{\Delta_2}(\mathcal{QF})$ is a rational transformation. Thus the faithful representation of the mapping class group of S by a group of rational transformations for its action on the decorated Teichmüller space ([5]) is naturally extended to its action on $\mathcal{QF}$.

§ 2. Trace identities

We shall use repeatedly the following basic trace identities which hold for matrices in $SL(2, \mathbb{C})$ (see [1, 3.4]):

$$(2.1) \quad \text{tr} Y^{-1} X Y = \text{tr} X,$$

$$(2.2) \quad \text{tr} X Y + \text{tr} X Y^{-1} = \text{tr} X \text{tr} Y,$$

From (2.1), $\text{tr} X_1 X_2 \cdots X_n = \text{tr} X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(n)}$ for any cyclic permutation σ on $\{1, 2, \dots, n\}$. So (2.2) yields

$$(2.3) \quad \text{tr} X Y Z = \text{tr} Y \text{tr} X Z - \text{tr} X Y^{-1} Z$$

for X, Y and $Z \in SL(2, \mathbb{C})$. The following trace identities are proved in [2, Proposition 1.1] and [4, Lemma 1.3], respectively.

Lemma 2.1. If A, B, C and $D \in SL(2, \mathbb{C})$ are such that $\text{tr}ABCD = -2$, then

$$\begin{aligned} & (\text{tr}AB + \text{tr}CD)(\text{tr}BC + \text{tr}AD) \\ &= (\text{tr}A + \text{tr}BCD)(\text{tr}C + \text{tr}ABD) + (\text{tr}B + \text{tr}ACD)(\text{tr}D + \text{tr}ABC). \end{aligned} \quad (2.4)$$

Lemma 2.2. Let $X, Y_1, \dots, Y_{n+1} \in SL(2, \mathbb{C})$, where $n \geq 1$. If $\text{tr}Y_1 = \dots = \text{tr}Y_{n+1}$, then

$$\begin{aligned} & \sum_{\epsilon_1, \dots, \epsilon_n \in \{0,1\}} (-1)^{\epsilon_1 + \dots + \epsilon_n} \text{tr}XY_1^{\epsilon_1}Y_2^{\epsilon_1+\epsilon_2} \dots Y_n^{\epsilon_{n-1}+\epsilon_n}Y_{n+1}^{\epsilon_n+1} \\ (2.5) \quad &= \sum_{\epsilon_1, \dots, \epsilon_n \in \{0,1\}} (-1)^{\epsilon_1 + \dots + \epsilon_n} \text{tr}XY_1^{\epsilon_1+1}Y_2^{\epsilon_1+\epsilon_2} \dots Y_n^{\epsilon_{n-1}+\epsilon_n}Y_{n+1}^{\epsilon_n}. \end{aligned}$$

Lemma 2.3. Let $X \in SL(2, \mathbb{C})$ and $P_1, \dots, P_n \in \mathcal{P}$ with $n \geq 2$. Then

$$\begin{aligned} & \sum_{\epsilon_1, \dots, \epsilon_n \in \{0,1\}} \text{tr}XP_1^{\epsilon_1}P_2^{\epsilon_2} \dots P_n^{\epsilon_n} \\ (2.6) \quad &= \sum_{\epsilon_1, \dots, \epsilon_n \in \{0,1\}} (-1)^{\epsilon_1 + \dots + \epsilon_{n-1}+1} \text{tr}XP_1^{1+\epsilon_1}P_2^{\epsilon_1+\epsilon_2} \dots P_n^{\epsilon_{n-1}+\epsilon_n}. \end{aligned}$$

Proof. If $n = 2$, then by using (2.3) and $\text{tr}P_1 = \text{tr}P_2 = -2$, we can deform the right hand side of (2.6) to the left hand side as follows:

$$\begin{aligned} & \sum_{\epsilon_1, \epsilon_2 \in \{0,1\}} (-1)^{\epsilon_1+1} \text{tr}XP_1^{1+\epsilon_1}P_2^{\epsilon_1+\epsilon_2} = -\text{tr}XP_1 + \text{tr}XP_1^2P_2 - \text{tr}XP_1P_2 + \text{tr}XP_1^2P_2^2 \\ &= -\text{tr}XP_1 + (-2\text{tr}XP_1P_2 - \text{tr}XP_2) - \text{tr}XP_1P_2 + (-2\text{tr}XP_1^2P_2 - \text{tr}XP_1^2) \\ &= -\text{tr}XP_1 - \text{tr}XP_2 - 3\text{tr}XP_1P_2 \\ &\quad + (-2(-2\text{tr}XP_1P_2 - \text{tr}XP_2) + 2\text{tr}XP_1 + \text{tr}X) \\ &= \text{tr}X + \text{tr}XP_1 + \text{tr}XP_2 + \text{tr}XP_1P_2. \end{aligned}$$

We prove (2.6) for $n > 2$ by induction. We divide the sum in the right hand side into the sum for $\epsilon_1 = 0$ and that for $\epsilon_1 = 1$. Then it equals

$$\begin{aligned} & \sum_{\epsilon_2, \dots, \epsilon_n \in \{0,1\}} (-1)^{\epsilon_2 + \dots + \epsilon_{n-1}+1} \text{tr}XP_1P_2^{-1}P_2^{1+\epsilon_2}P_3^{\epsilon_2+\epsilon_3} \dots P_n^{\epsilon_{n-1}+\epsilon_n} \\ & - \sum_{\epsilon_2, \dots, \epsilon_n \in \{0,1\}} (-1)^{\epsilon_2 + \dots + \epsilon_{n-1}+1} \text{tr}XP_1^2P_2^{1+\epsilon_2}P_3^{\epsilon_2+\epsilon_3} \dots P_n^{\epsilon_{n-1}+\epsilon_n}. \end{aligned}$$

We assume that (2.6) holds for $n - 1$ and we apply it to $P_2, \dots, P_n$ and X replaced by $XP_1P_2^{-1}$ and XP_1^2 . Then the last term equals

$$(2.7) \quad \sum_{\epsilon_2, \dots, \epsilon_n \in \{0,1\}} \text{tr} XP_1 P_2^{-1} P_2^{\epsilon_2} \cdots P_n^{\epsilon_n} - \sum_{\epsilon_2, \dots, \epsilon_n \in \{0,1\}} \text{tr} XP_1^2 P_2^{\epsilon_2} \cdots P_n^{\epsilon_n}.$$

Let $Y = P_2^{\epsilon_2} P_3^{\epsilon_3} \cdots P_n^{\epsilon_n}$. From (2.3) $\text{tr} XP_1 P_2^{-1} Y = -\text{tr} XP_1 P_2 Y - 2\text{tr} XP_1 Y$ and $\text{tr} XP_1^2 Y = -2\text{tr} XP_1 Y - \text{tr} XY$. Then we have with $Z = P_3^{\epsilon_3} \cdots P_n^{\epsilon_n}$

$$\begin{aligned} & \sum_{\epsilon_2 \in \{0,1\}} \text{tr} XP_1 P_2^{-1} (P_2^{\epsilon_2} Z) - \sum_{\epsilon_2 \in \{0,1\}} \text{tr} XP_1^2 (P_2^{\epsilon_2} Z) \\ &= - \sum_{\epsilon_2 \in \{0,1\}} \text{tr} XP_1 P_2 P_2^{\epsilon_2} Z + \sum_{\epsilon_2 \in \{0,1\}} \text{tr} XP_2^{\epsilon_2} Z \\ &= -\text{tr} XP_1 P_2 Z - \text{tr} XP_1 P_2^2 Z + \text{tr} XZ + \text{tr} XP_2 Z \\ &= -\text{tr} XP_1 P_2 Z - (-2\text{tr} XP_1 P_2 Z - \text{tr} XP_1 Z) + \text{tr} XZ + \text{tr} XP_2 Z \\ &= \text{tr} XZ + \text{tr} XP_1 Z + \text{tr} XP_2 Z + \text{tr} XP_1 P_2 Z. \end{aligned}$$

Summing the last term over $\epsilon_3, \dots, \epsilon_n$, we obtain the left hand side of (2.6). Thus (2.6) holds for all n . $\square$

Lemma 2.4. Let $P_1, P_2 \in \mathcal{P}$ and $X, Y \in SL(2, \mathbb{C})$. Then

$$(2.8) \quad \sum_{\epsilon_1, \epsilon_2 \in \{0,1\}} \text{tr} P_2^{\epsilon_1} P_1^{\epsilon_2} Y \cdot \sum_{\epsilon_3, \epsilon_4 \in \{0,1\}} \text{tr} P_1^{\epsilon_3} P_2^{\epsilon_4} X = (\text{tr} P_1 P_2 - 2) \sum_{\epsilon_1, \epsilon_2 \in \{0,1\}} \text{tr} P_1^{\epsilon_1} Y P_2^{\epsilon_2} X.$$

Proof. We can substitute $A = P_1$, $B = P_1^{-1} X P_1$, $C = P_1^{-1} X^{-1} Y^{-1}$ and $D = Y P_2$ into (2.4), because $\text{tr} ABCD = \text{tr} P_2 = -2$. We have

$$\begin{aligned} \text{tr} A + \text{tr} BCD &= \text{tr} P_1 + \text{tr} P_1^{-1} P_2 \\ &= \text{tr} P_1 + (-2\text{tr} P_1 - \text{tr} P_1 P_2) = -\text{tr} P_1 - \text{tr} P_1 P_2. \end{aligned}$$

Likewise we obtain

$$\begin{aligned} \text{tr} A + \text{tr} BCD &= 2 - \text{tr} P_1 P_2, & \text{tr} B + \text{tr} ACD &= -\text{tr} X - \text{tr} X P_2, \\ \text{tr} C + \text{tr} ABD &= \text{tr} X P_1 Y + \text{tr} X P_1 Y P_2, & \text{tr} D + \text{tr} ABC &= \text{tr} Y + \text{tr} Y P_2, \\ \text{tr} AB + \text{tr} CD &= -\text{tr} X P_1 - \text{tr} X P_1 P_2, & \text{tr} BC + \text{tr} AD &= \text{tr} P_1 Y + \text{tr} P_1 Y P_2. \end{aligned}$$

Therefore (2.4) in this case equals

$$\begin{aligned} & (\text{tr} X P_1 + \text{tr} X P_1 P_2)(\text{tr} P_1 Y + \text{tr} P_1 Y P_2) \\ &= (\text{tr} P_1 P_2 - 2)(\text{tr} X P_1 Y + \text{tr} X P_1 Y P_2) + (\text{tr} X + \text{tr} X P_2)(\text{tr} Y + \text{tr} Y P_2). \end{aligned} \quad (2.9)$$

Substituting $P_1^{-1}Y$ to Y in this equation, we obtain

$$\begin{aligned}
 & (\operatorname{tr}XP_1 + \operatorname{tr}XP_1P_2)(\operatorname{tr}Y + \operatorname{tr}YP_2) \\
 &= (\operatorname{tr}P_1P_2 - 2)(\operatorname{tr}XY + \operatorname{tr}XYP_2) + (\operatorname{tr}X + \operatorname{tr}XP_2)(\operatorname{tr}P_1^{-1}Y + \operatorname{tr}P_1^{-1}YP_2). \\
 &= (\operatorname{tr}P_1P_2 - 2)(\operatorname{tr}XY + \operatorname{tr}XYP_2) \\
 (2.10) \quad &+ (\operatorname{tr}X + \operatorname{tr}XP_2)(-2\operatorname{tr}Y - \operatorname{tr}P_1Y - 2\operatorname{tr}YP_2 - \operatorname{tr}P_1YP_2).
 \end{aligned}$$

By adding (2.9) and (2.10) we obtain (2.8).

§ 3. Proof of the main theorem

Let $(P_1, P_2, \dots, P_n)$ be an n -gon in $\mathcal{P}$, where $n \geq 4$, and $Q_i \in SL(2, \mathbb{C})$ be such that $P_i \xrightarrow{Q_i} P_{i+1}$ for $i = 1, 2, \dots, n$.

Lemma 3.1.

$$(3.1) \quad \operatorname{tr}Q_1\operatorname{tr}Q_2 \cdots \operatorname{tr}Q_n\operatorname{tr}Q_1 \cdots Q_n = \sum_{\epsilon_1, \dots, \epsilon_n \in \{0,1\}} 2\operatorname{tr}P_1^{\epsilon_1}P_2^{\epsilon_2} \cdots P_n^{\epsilon_n}.$$

Proof. By (2.2) we have with $X_{n-1} = Q_1 \cdots Q_{n-1}$

$$\operatorname{tr}Q_n\operatorname{tr}Q_1 \cdots Q_n = \operatorname{tr}X_{n-1}Q_n^2 + \operatorname{tr}X_{n-1}Q_nQ_n^{-1} = \operatorname{tr}X_{n-1}Q_n^2 + \operatorname{tr}X_{n-1}$$

and then with $X_{n-2} = Q_1 \cdots Q_{n-2}$

$$\begin{aligned}
 & \operatorname{tr}Q_{n-1}\operatorname{tr}Q_n\operatorname{tr}Q_1 \cdots Q_n \\
 &= (\operatorname{tr}Q_n^2X_{n-2}Q_{n-1}^2 + \operatorname{tr}Q_n^2X_{n-2}) + (\operatorname{tr}X_{n-2}Q_{n-1}^2 + \operatorname{tr}X_{n-2}) \\
 &= \sum_{\epsilon_{n-1}, \epsilon_n \in \{0,1\}} \operatorname{tr}X_{n-2}Q_{n-1}^{2\epsilon_{n-1}}Q_n^{2\epsilon_n}..
 \end{aligned}$$

By proceeding in this manner we have

$$\operatorname{tr}Q_1\operatorname{tr}Q_2 \cdots \operatorname{tr}Q_n\operatorname{tr}Q_1 \cdots Q_n = \sum_{\epsilon_1, \dots, \epsilon_n \in \{0,1\}} \operatorname{tr}Q_1^{2\epsilon_1}Q_2^{2\epsilon_2} \cdots Q_n^{2\epsilon_n}.$$

Thus

$$\begin{aligned}
 & \operatorname{tr}Q_1\operatorname{tr}Q_2 \cdots \operatorname{tr}Q_n\operatorname{tr}Q_1 \cdots Q_n \\
 &= \sum_{\epsilon_1, \dots, \epsilon_n \in \{0,1\}} \operatorname{tr}(-P_1P_2)^{\epsilon_1}(-P_2P_3)^{\epsilon_2} \cdots (-P_nP_1)^{\epsilon_n} \\
 &= \sum_{\epsilon_1, \dots, \epsilon_n \in \{0,1\}} (-1)^{\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n} \operatorname{tr}P_1^{\epsilon_n + \epsilon_1}P_2^{\epsilon_1 + \epsilon_2} \cdots P_n^{\epsilon_{n-1} + \epsilon_n}
 \end{aligned}$$

We divide the last sum into the sum for $\epsilon_n = 0$ and the sum for $\epsilon_n = 1$ and apply (2.5) to the second term by setting $X = P_1$ and $Y_i = P_i$ for $i = 1, \dots, n$. Then we obtain

$$\begin{aligned}
 & \sum_{\epsilon_1, \dots, \epsilon_{n-1} \in \{0,1\}} (-1)^{\epsilon_1 + \dots + \epsilon_{n-1}} \text{tr} P_1^{\epsilon_1} P_2^{\epsilon_1 + \epsilon_2} \dots P_n^{\epsilon_{n-1}} \\
 & + \sum_{\epsilon_1, \dots, \epsilon_{n-1} \in \{0,1\}} (-1)^{1 + \epsilon_1 + \dots + \epsilon_{n-1}} \text{tr} P_1^{1 + \epsilon_1} P_2^{\epsilon_1 + \epsilon_2} \dots P_n^{\epsilon_{n-1} + 1} \\
 & = \sum_{\epsilon_1, \dots, \epsilon_{n-1} \in \{0,1\}} (-1)^{\epsilon_1 + \dots + \epsilon_{n-1}} \text{tr} P_1^{\epsilon_1} P_2^{\epsilon_1 + \epsilon_2} \dots P_n^{\epsilon_{n-1}} \\
 (3.2) \quad & + \sum_{\epsilon_1, \dots, \epsilon_{n-1} \in \{0,1\}} (-1)^{1 + \epsilon_1 + \dots + \epsilon_{n-1}} \text{tr} P_1^{2 + \epsilon_1} P_2^{\epsilon_1 + \epsilon_2} \dots P_n^{\epsilon_{n-1}}.
 \end{aligned}$$

Let $Y = P_2^{\epsilon_1 + \epsilon_2} \dots P_n^{\epsilon_{n-1}}$. Then from (2.3)

$$\text{tr} P_1^{\epsilon_1} Y - \text{tr} P_1^{2 + \epsilon_1} Y = 2 \text{tr} P_1^{1 + \epsilon_1} Y + 2 \text{tr} P_1^{\epsilon_1} Y.$$

Taking the sum over $\epsilon_1, \dots, \epsilon_{n-1}$ we see that (3.2) equals

$$\begin{aligned}
 & \sum_{\epsilon_1, \dots, \epsilon_{n-1} \in \{0,1\}} (-1)^{\epsilon_1 + \dots + \epsilon_{n-1}} 2 \text{tr} P_1^{1 + \epsilon_1} P_2^{\epsilon_1 + \epsilon_2} \dots P_n^{\epsilon_{n-1}} \\
 & + \sum_{\epsilon_1, \dots, \epsilon_{n-1} \in \{0,1\}} (-1)^{\epsilon_1 + \dots + \epsilon_{n-1}} 2 \text{tr} P_1^{\epsilon_1} P_2^{\epsilon_1 + \epsilon_2} \dots P_n^{\epsilon_{n-1}}.
 \end{aligned}$$

We apply (2.5) to the first term in this expression, then it equals

$$\begin{aligned}
 & \sum_{\epsilon_1, \dots, \epsilon_{n-1} \in \{0,1\}} (-1)^{\epsilon_1 + \dots + \epsilon_{n-1}} 2 \text{tr} P_1^{\epsilon_1} P_2^{\epsilon_1 + \epsilon_2} \dots P_n^{\epsilon_{n-1} + 1} \\
 & + \sum_{\epsilon_1, \dots, \epsilon_{n-1} \in \{0,1\}} (-1)^{\epsilon_1 + \dots + \epsilon_{n-1}} 2 \text{tr} P_1^{\epsilon_1} P_2^{\epsilon_1 + \epsilon_2} \dots P_n^{\epsilon_{n-1}} \\
 & = \sum_{\epsilon_1, \dots, \epsilon_n \in \{0,1\}} (-1)^{\epsilon_1 + \epsilon_2 + \dots + \epsilon_{n-1}} 2 \text{tr} P_1^{\epsilon_1} P_2^{\epsilon_1 + \epsilon_2} \dots P_n^{\epsilon_{n-1} + \epsilon_n}
 \end{aligned}$$

Let $a_{(1,2,\dots,n)}$ denote the last expression. Then by dividing the sum in it into the sum for $\epsilon_1 = 0$ and the sum for $\epsilon_1 = 1$,

$$\begin{aligned}
 a_{(1,2,\dots,n)} & = \sum_{\epsilon_2, \dots, \epsilon_n \in \{0,1\}} (-1)^{\epsilon_2 + \dots + \epsilon_{n-1}} 2 \text{tr} P_2^{\epsilon_2} P_3^{\epsilon_2 + \epsilon_3} \dots P_n^{\epsilon_{n-1} + \epsilon_n} \\
 & + \sum_{\epsilon_2, \dots, \epsilon_n \in \{0,1\}} (-1)^{1 + \epsilon_2 + \dots + \epsilon_{n-1}} 2 \text{tr} P_1 P_2^{1 + \epsilon_2} P_3^{\epsilon_2 + \epsilon_3} \dots P_n^{\epsilon_{n-1} + \epsilon_n}.
 \end{aligned}$$

From (2.6) follows

$$(3.3) \quad a_{(1,2,\dots,n)} = a_{(2,3,\dots,n)} + \sum_{\epsilon_2, \dots, \epsilon_n \in \{0,1\}} 2 \text{tr} P_1 P_2^{\epsilon_2} P_3^{\epsilon_3} \dots P_n^{\epsilon_n}.$$

We have

$$\begin{aligned}
a_{((n-1)n)} &= \sum_{\epsilon_{n-1}, \epsilon_n \in \{0,1\}} (-1)^{\epsilon_{n-1}} 2\mathrm{tr} P_{n-1}^{\epsilon_{n-1}} P_n^{\epsilon_{n-1} + \epsilon_n} \\
&= 2\mathrm{tr} I + 2\mathrm{tr} P_n - 2\mathrm{tr} P_{n-1} P_n - 2\mathrm{tr} P_{n-1} P_n^2 \\
&= 2\mathrm{tr} I + 2\mathrm{tr} P_n - 2\mathrm{tr} P_{n-1} P_n - 2(-2\mathrm{tr} P_{n-1} P_n - \mathrm{tr} P_{n-1}) \\
&= 2\mathrm{tr} I + 2\mathrm{tr} P_{n-1} + 2\mathrm{tr} P_n + 2\mathrm{tr} P_{n-1} P_n,
\end{aligned}$$

where I is the unit matrix. From this and (3.3) we can obtain (3.1) by induction on n . $\square$

Now we prove the identity (1.7) in Lemma 1.1 from which Theorem 1.1 is easily obtained. From (3.1) we see that

$$(\mathrm{tr} Q_1 \cdots \mathrm{tr} Q_{m-1} \mathrm{tr} R_m)(\mathrm{tr} Q_1 \cdots Q_{m-1} R_m) \cdot (\mathrm{tr} S_m \mathrm{tr} Q_m \cdots \mathrm{tr} Q_n)(\mathrm{tr} S_m Q_m \cdots Q_n)$$

equals

$$\sum_{\eta_m, \eta_1, \epsilon_2, \dots, \epsilon_{m-1} \in \{0,1\}} 2\mathrm{tr} P_m^{\eta_m} P_1^{\eta_1} P_2^{\epsilon_2} \cdots P_{m-1}^{\epsilon_{m-1}} \cdot \sum_{\epsilon_1, \epsilon_m, \dots, \epsilon_n \in \{0,1\}} 2\mathrm{tr} P_1^{\epsilon_1} P_m^{\epsilon_m} P_{m+1}^{\epsilon_{m+1}} \cdots P_n^{\epsilon_n}$$

By replacing P_2 , X and Y in (2.8) by P_m , $P_{m+1}^{\epsilon_{m+1}} \cdots P_n^{\epsilon_n}$ and $P_2^{\epsilon_2} \cdots P_{m-1}^{\epsilon_{m-1}}$, respectively, we see that the last expression equals

$$\begin{aligned}
&4(\mathrm{tr} P_1 P_m - 2) \sum_{\epsilon_1, \dots, \epsilon_n \in \{0,1\}} P_1^{\epsilon_1} P_2^{\epsilon_2} \cdots P_n^{\epsilon_n} \\
&= -2(\mathrm{tr} R_m)^2 \mathrm{tr} Q_1 \mathrm{tr} Q_2 \cdots \mathrm{tr} Q_n \mathrm{tr} Q_1 Q_2 \cdots Q_n.
\end{aligned}$$

Here we used $-\mathrm{tr} P_1 P_m = \mathrm{tr} R_m^2 = (\mathrm{tr} R_m)^2 - 2$ and (3.1). Since $\mathrm{tr} R_m = \mathrm{tr} S_m$ and none of $\mathrm{tr} R_m$, $\mathrm{tr} Q_1, \dots, \mathrm{tr} Q_n$ are non-zero (see (1.4)), we obtain (1.7).

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