

Von Neumann-Jordan constant of generalized Banaś-Frączek spaces¹

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概要

バナッハ空間の von Neumann-Jordan 定数に関する最近の結果について報告する。特に, Banaś-Frączek 空間を含む 2 次元ノルム空間を導入し, Banach-Mazur 距離の概念を用いてこの空間における von Neumann-Jordan 定数を計算する。

Definition 1 ([2]). The von Neumann-Jordan (NJ-) constant of a Banach space X , denoted by $C_{\text{NJ}}(X)$, is the smallest constant C for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

holds for all $x, y \in X$ not both 0.

Definition 2 ([1, 10]). For $\lambda > 1$, the Banaś-Frączek space $\mathbb{R}_\lambda^2$ is the space $\mathbb{R}^2$ with the norm $|\cdot|_\lambda$ defined by

$$|(x, y)|_\lambda = \max \{ \lambda|x|, \|(x, y)\|_2 \},$$

where $\|\cdot\|_2$ is the ℓ_2 -norm on $\mathbb{R}^2$.

Yang-Wang [13] は幾何学的定数 $\gamma_X(t)$ を導入し, これを用いて ℓ_2 - ℓ_1 と ℓ_∞ - ℓ_1 における NJ 定数を計算した. この方法と同様にして Yang [10] は Banaś-Frączek space の NJ 定数を計算した.

Theorem 3 ([10]) For $\lambda > 1$,

$$C_{\text{NJ}}(\mathbb{R}_\lambda^2) = 2 - \frac{1}{\lambda^2}.$$

¹ *Keywords.* von Neumann-Jordan constant, Banach-Mazur distance

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本研究では, Banaś-Frączek space を含む次の 2 次元ノルム空間を導入し, Banach-Mazur distance を用いてこの空間の NJ 定数を計算する.

Definition 4. For $a \geq b \geq 1$ and $1 \leq p < \infty$, $\mathbb{R}_{a,b,p}^2$ is defined as $\mathbb{R}^2$ with the norm $\|\cdot\|$ on $\mathbb{R}^2$ by

$$\|(x, y)\| = \max\{a|x|, b|y|, \|(x, y)\|_p\},$$

where $\|\cdot\|_p$ is the ℓ_p -norm on $\mathbb{R}^2$.

Definition 5. For isomorphic Banach spaces X and Y , the Banach-Mazur distance between X and Y , denoted by $d(X, Y)$, is defined to be the infimum of $\|T\| \cdot \|T^{-1}\|$ taken over all bicontinuous linear operators T from X onto Y .

Lemma 6 ([4]). If X and Y are isomorphic Banach spaces, then

$$\frac{C_{\text{NJ}}(X)}{d(X, Y)^2} \leq C_{\text{NJ}}(Y) \leq C_{\text{NJ}}(X)d(X, Y)^2.$$

In particular, if X and Y are isometric, then $C_{\text{NJ}}(X) = C_{\text{NJ}}(Y)$.

Lemma 7 ([4]). Let $X = (X, \|\cdot\|)$ be a Banach space and let $X_1 = (X, \|\cdot\|_1)$, where $\|\cdot\|_1$ is an equivalent norm on X satisfying, for $\alpha, \beta > 0$,

$$\alpha\|x\| \leq \|x\|_1 \leq \beta\|x\|, \quad x \in X.$$

Then

$$\frac{\alpha^2}{\beta^2} C_{\text{NJ}}(X) \leq C_{\text{NJ}}(X_1) \leq \frac{\beta^2}{\alpha^2} C_{\text{NJ}}(X).$$

Definition 8. A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(|x|, |y|)\| = \|(x, y)\|$ for any $x, y \in \mathbb{R}$.

Lemma 7 を用いて, absolute norm の NJ 定数に関する公式を得る. 簡単のため $C_{\text{NJ}}((\mathbb{R}^2, \|\cdot\|))$ を $C_{\text{NJ}}(\|\cdot\|)$ とかく.

Theorem 9. Let $\|\cdot\|, \|\cdot\|_H$ be absolute norms on $\mathbb{R}^2$. Assume that the following hold:

(i) $(\mathbb{R}^2, \|\cdot\|_H)$ is an inner product space.

(ii) $\alpha\|(x, y)\|_H \leq \|(x, y)\| \leq \beta\|(x, y)\|_H$ for any $(x, y) \in \mathbb{R}^2$ (α, β are the best constants).

(iii) In (ii) it satisfies either $\alpha\|(1, 0)\|_H = \|(1, 0)\|$ and $\alpha\|(0, 1)\|_H = \|(0, 1)\|$, or $\beta\|(1, 0)\|_H = \|(1, 0)\|$ and $\beta\|(0, 1)\|_H = \|(0, 1)\|$.

Then

$$C_{\text{NJ}}(\|\cdot\|) = \frac{\beta^2}{\alpha^2}.$$

上の公式を用いて、 $\mathbb{R}_{a,b,p}^2$ の NJ 定数を計算する． $1/a^p + 1/b^p \leq 1$ のとき明らかに $C_{\text{NJ}}(\mathbb{R}_{a,b,p}^2) = 2$ ．また $a = b = 1$ のとき $\|(x, y)\| = \|(x, y)\|_p$ であるから $a > 1, 1/a^p + 1/b^p > 1$ の場合のみ考えればよい．

Definition 10. For $a \geq b \geq 1$, $\|\cdot\|_H$ is defined by $\|(x, y)\|_H = \|(ax, by)\|_2$.

$p \geq 2$ とする．Definition 4 と Definition 10 で定義した $\|\cdot\|$, $\|\cdot\|_H$ は absolute であり，さらに Theorem 9 の (i), (ii), (iii) の条件をみたす．Theorem 9 を適用することで次が得られる．

Theorem 11 ([5]) Let $a > 1, a \geq b \geq 1$ and $p \geq 2$ with $\frac{1}{a^p} + \frac{1}{b^p} > 1$.

(i) If $b \leq a(a^p - 1)^{\frac{p-2}{2p}}$, then

$$C_{\text{NJ}}(\mathbb{R}_{a,b,p}^2) = 1 + b^2 \left(1 - \frac{1}{a^p}\right)^{\frac{2}{p}}.$$

(ii) If $b > a(a^p - 1)^{\frac{p-2}{2p}}$, then

$$C_{\text{NJ}}(\mathbb{R}_{a,b,p}^2) = b^2 \left(1 + \left(\frac{a}{b}\right)^{\frac{2p}{p-2}}\right)^{1-\frac{2}{p}}.$$

In particular, $C_{\text{NJ}}(\mathbb{R}_{a,b,p}^2) = d(\mathbb{R}_{a,b,p}^2, H)^2$, where H is a two-dimensional inner product space.

この結果は Theorem 3 を含む．また， $p < 2$ の場合も同様の方法で計算することができる．

Theorem 12 Let $a > 1, a \geq b \geq 1$ and $p < 2$ with $\frac{1}{a^p} + \frac{1}{b^p} > 1$ and $a^{\frac{2p}{p-2}} + b^{\frac{2p}{p-2}} \leq 1$.

Then

$$C_{\text{NJ}}(\mathbb{R}_{a,b,p}^2) = 1 + b^2 \left(1 - \frac{1}{a^p}\right)^{\frac{2}{p}}.$$

In particular, $C_{\text{NJ}}(\mathbb{R}_{a,b,p}^2) = d(\mathbb{R}_{a,b,p}^2, H)^2$, where H is a two-dimensional inner product space.

Corollary 13 Let $a > 1, a \geq b \geq 1$ with $\frac{1}{a} + \frac{1}{b} > 1$ and $\frac{1}{a^2} + \frac{1}{b^2} \leq 1$. Then

$$C_{\text{NJ}}(\mathbb{R}_{a,b,1}^2) = 1 + b^2 \left(1 - \frac{1}{a}\right)^2.$$

In particular,

$$C_{\text{NJ}}(\mathbb{R}_{a,a,1}^2) = a^2 - 2a + 2.$$

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