

Parametric extensions of Shannon inequality and its reverse one in Hilbert space operators via characterizations of operator concave functions

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Abstract. We shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators. Let $p \in [0, 1]$ and also let $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^n A_j \sharp_p B_j \leq I$. Then

$$\begin{aligned} \sum_{j=1}^n S_{p+1}(A_j|B_j) &\geq \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + (I - \sum_{j=1}^n A_j \sharp_p B_j) \right] \log \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + (I - \sum_{j=1}^n A_j \sharp_p B_j) \right] \\ &\geq \log \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + (I - \sum_{j=1}^n A_j \sharp_p B_j) \right] \geq \sum_{j=1}^n S_p(A_j|B_j) \geq -\log \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + (I - \sum_{j=1}^n A_j \sharp_p B_j) \right] \\ &\geq -\left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + (I - \sum_{j=1}^n A_j \sharp_p B_j) \right] \log \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + (I - \sum_{j=1}^n A_j \sharp_p B_j) \right] \geq \sum_{j=1}^n S_{p-1}(A_j|B_j) \end{aligned}$$

where $S_q(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^q(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$ for $A > 0$, $B > 0$ and any real number q and $A \natural_q B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^qA^{\frac{1}{2}}$ for $A > 0$, $B > 0$ and any real number q .

In particular, if $\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I$, then

$$\begin{aligned} \sum_{j=1}^n S_2(A_j|B_j) &\geq \left[\sum_{j=1}^n B_j A_j^{-1} B_j \right] \log \left[\sum_{j=1}^n B_j A_j^{-1} B_j \right] \geq \log \left[\sum_{j=1}^n B_j A_j^{-1} B_j \right] \geq \sum_{j=1}^n S_1(A_j|B_j) \geq 0 \\ &\geq \sum_{j=1}^n S(A_j|B_j) \geq -\log \left[\sum_{j=1}^n A_j B_j^{-1} A_j \right] \geq -\left[\sum_{j=1}^n A_j B_j^{-1} A_j \right] \log \left[\sum_{j=1}^n A_j B_j^{-1} A_j \right] \geq \sum_{j=1}^n S_{-1}(A_j|B_j) \end{aligned}$$

where $S(A|B) = S_0(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$ which is the relative operator entropy of $A > 0$ and $B > 0$.

Our results can be considered as parametric extensions of the following celebrated Shannon inequality ([7],[9] and [233 p. 1]) which is very useful and so famous in information theory. Let $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ be two probability vectors. Then

$$0 \geq \sum_{j=1}^n a_j \log b_j - \sum_{j=1}^n a_j \log a_j \text{ (see inequalities (2.4) of Corollary 2.4).}$$

§1 Introduction

First the Shannon inequality asserts: *Let $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ be two probability vectors. Then*

$$(1.1) \quad 0 \geq \sum_{j=1}^n a_j \log \frac{b_j}{a_j}.$$

We remark that $0 \geq \sum_{j=1}^n a_j \log \frac{b_j}{a_j}$ in (1.1) is equivalent to $D = \sum_{j=1}^n a_j \log \frac{a_j}{b_j} \geq 0$ which is the original number type Shannon inequality and this D is called "divergence" in [7] and [9].

In this paper we shall state parametric extensions of Shannon inequality and its reverse one in Hilbert space operators.

A bounded linear operator T on a Hilbert space H is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

Definition 1.1. $S_q(A|B)$ for $A > 0$, $B > 0$ and any real number q is defined by

$$S_q(A|B) = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

We recall that $S_0(A|B) = A^{\frac{1}{2}} (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} = S(A|B)$ is the relative operator entropy in [2] and $S(A|I) = -A \log A$ is the usual operator entropy in [8].

Definition 1.2. $A \natural_q B$ for $A > 0$ and $B > 0$ and any real number q is defined by

$$A \natural_q B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q A^{\frac{1}{2}}$$

and $A \natural_p B$ for $p \in [0, 1]$ just coincides with $A \sharp_p B$ which is well known as p -power mean.

We remark that $S_1(A|B) = -S(B|A)$ and moreover $S_q(A|B) = -S_{1-q}(B|A)$ for any q .

Following after Definition 1.1, The original Shannon inequality can be expressed as follows:

$$0 \geq \sum_{j=1}^n a_j \log \frac{b_j}{a_j} = \sum_{j=1}^n a_j^{\frac{1}{2}} (\log a_j^{-\frac{1}{2}} b_j a_j^{-\frac{1}{2}}) a_j^{\frac{1}{2}} = \sum_{j=1}^n S(a_j|b_j).$$

Consequently $0 \geq \sum_{j=1}^n S(a_j|b_j)$ in the original Shannon inequality can be extended to

$$0 \geq \sum_{j=1}^n S(A_j|B_j) \text{ in operator version case (2.4) of Corollary 2.4, so that the form of (1.1)}$$

is convenient for operator type extension. We can summarize the following contrast:

The original Shannon inequality

and its reverse one

$$0 \geq \sum_{j=1}^n a_j \log \frac{b_j}{a_j} \geq -\log \sum_{j=1}^n \frac{a_j^2}{b_j}.$$

for $a_j, b_j > 0$ with $1 = \sum_{j=1}^n a_j = \sum_{j=1}^n b_j$.

The operator version Shannon inequality

and its reverse one

$$0 \geq \sum_{j=1}^n S(A_j|B_j) \geq -\log \sum_{j=1}^n A_j B_j^{-1} A_j.$$

for $A_j, B_j > 0$ with $I = \sum_{j=1}^n A_j = \sum_{j=1}^n B_j$.

§2 Parametric extensions of operator reverse type Shannon inequality derived from two operator concave functions $f_1(t) = \log t$ and $f_2(t) = -t \log t$

Firstly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function $f(t) = \log t$.

Theorem 2.1. Let $p \in [0, 1]$ and also let $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^n A_j \sharp_p B_j \leq I$, where I means the identity operator on H . Then

$$\begin{aligned} (2.1) \quad & \log \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + t_0 (I - \sum_{j=1}^n A_j \sharp_p B_j) \right] - \log t_0 (I - \sum_{j=1}^n A_j \sharp_p B_j) \\ & \geq \sum_{j=1}^n S_p(A_j|B_j) \\ & \geq -\log \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + t_0 (I - \sum_{j=1}^n A_j \sharp_p B_j) \right] + \log t_0 (I - \sum_{j=1}^n A_j \sharp_p B_j) \end{aligned}$$

for fixed real number $t_0 > 0$, where $S_p(A|B)$ is defined in Definition 1.1 and $A \natural_q B$ is defined in Definition 1.2.

Secondly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function $f(t) = -t \log t$.

Theorem 2.2. Let $p \in [0, 1]$ and also let $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^n A_j \sharp_p B_j \leq I$, where I means the identity operator on H . Then

$$(2.2) \quad \sum_{j=1}^n S_{p+1}(A_j|B_j)$$

$$\geq \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + t_0 \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] \log \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + t_0 \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] \\ - t_0 \log t_0 \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \quad \text{for fixed real number } t_0 > 0,$$

and

$$(2.2') \quad \sum_{j=1}^n S_{p-1}(A_j | B_j) \\ \leq - \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + t_0 \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] \log \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + t_0 \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] \\ + t_0 \log t_0 \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \quad \text{for fixed real number } t_0 > 0,$$

where $S_q(A|B)$ is defined in Definition 1.1 and $A \natural_q B$ is defined in Definition 1.2.

We shall state the following result which can be shown by combining Theorem 2.1 with Theorem 2.2.

Corollary 2.3. Let $p \in [0, 1]$ and also let $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^n A_j \sharp_p B_j \leq I$, where I means the identity operator on H . Then

$$(2.3) \quad \sum_{j=1}^n S_{p+1}(A_j | B_j) \\ \geq \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] \log \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] \\ \geq \log \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] \\ \geq \sum_{j=1}^n S_p(A_j | B_j) \\ \geq - \log \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] \\ \geq - \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] \log \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] \\ \geq \sum_{j=1}^n S_{p-1}(A_j | B_j)$$

where $S_q(A|B)$ is defined in Definition 1.1 and $A \natural_q B$ is defined in Definition 1.2.

Corollary 2.3 easily implies the following result which can be considered as *operator version of Shannon inequality and its reverse one*.

Corollary 2.4. *Let $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H . If $\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I$, then*

$$\begin{aligned}
 (2.4) \quad \sum_{j=1}^n S_2(A_j|B_j) &\geq \left[\sum_{j=1}^n B_j A_j^{-1} B_j \right] \log \left[\sum_{j=1}^n B_j A_j^{-1} B_j \right] \geq \log \left[\sum_{j=1}^n B_j A_j^{-1} B_j \right] \\
 &\geq \sum_{j=1}^n S_1(A_j|B_j) \geq 0 \geq \sum_{j=1}^n S(A_j|B_j) \\
 &\geq -\log \left[\sum_{j=1}^n A_j B_j^{-1} A_j \right] \geq - \left[\sum_{j=1}^n A_j B_j^{-1} A_j \right] \log \left[\sum_{j=1}^n A_j B_j^{-1} A_j \right] \\
 &\geq \sum_{j=1}^n S_{-1}(A_j|B_j).
 \end{aligned}$$

Remark 2.1. We recall $S_q(A|B)$ for $A > 0$, $B > 0$ and any real number q as follows:

$$S_q(A|B) = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

By an easy calculation we have

$$\frac{d}{dq} [S_q(A|B)] = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q [\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}]^2 A^{\frac{1}{2}} \geq 0,$$

so that $S_q(A|B)$ is an increasing function of q , and it is interesting to point out that the decreasing order of the positions of $\sum_{j=1}^n S_2(A_j|B_j)$, $\sum_{j=1}^n S_1(A_j|B_j)$, $\sum_{j=1}^n S(A_j|B_j)$, and $\sum_{j=1}^n S_{-1}(A_j|B_j)$ in (2.4) of Corollary 2.4 is quite reasonable since $\sum_{j=1}^n S(A_j|B_j) = \sum_{j=1}^n S_0(A_j|B_j)$.

§3 Propositions needed to give proofs of the results in §2

By careful scrutinizing nice proofs in [5, Theorem 2.1] and [4, Theorem], we have the following parallel result to [5, Theorem 2.1].

Proposition 3.1. *If f is a continuous, real function on an interval J , the following conditions are equivalent:*

(i) f is operator concave.

$$(ii) f(C^*AC + t_0(I - C^*C)) \geq C^*f(A)C + f(t_0)(I - C^*C)$$

for operator C with $\|C\| \leq 1$ and self-adjoint operator A with $\sigma(A) \subseteq J$ and for fixed real number $t_0 \in J$.

$$(iii) f\left(\sum_{j=1}^n C_j^* A_j C_j + t_0(I - \sum_{j=1}^n C_j^* C_j)\right) \geq \sum_{j=1}^n C_j^* f(A_j) C_j + f(t_0)(I - \sum_{j=1}^n C_j^* C_j)$$

for operators C_j with $\sum_{j=1}^n C_j^* C_j \leq I$ and self-adjoint operators A_j with $\sigma(A_j) \subseteq J$ for $j = 1, 2, \dots, n$ and for fixed real number $t_0 \in J$.

$$(iv) f\left(\sum_{j=1}^n C_j^* A_j C_j\right) \geq \sum_{j=1}^n C_j^* f(A_j) C_j$$

for operators C_j with $\sum_{j=1}^n C_j^* C_j = I$ and self-adjoint operators A_j with $\sigma(A_j) \subseteq J$ for $j = 1, 2, \dots, n$, where $n \geq 2$.

$$(v) f(PAP + t_0(I - P)) \geq Pf(A)P + f(t_0)(I - P)$$

for projection P and self-adjoint operator A with $\sigma(A) \subseteq J$ and for fixed real number $t_0 \in J$.

Corollary 3.2. If f is continuous operator concave function on the half open interval $[0, \alpha)$ to $[0, \alpha)$ with $\alpha \leq \infty$, then

$$\begin{aligned} f\left(\sum_{j=1}^n C_j^* A_j C_j\right) &\geq \sum_{j=1}^n C_j^* f(A_j) C_j + f(0)(I - \sum_{j=1}^n C_j^* C_j) \\ &\geq \sum_{j=1}^n C_j^* f(A_j) C_j \end{aligned}$$

for operators C_j with $\sum_{j=1}^n C_j^* C_j \leq I$ and self-adjoint operators A_j with $\sigma(A_j) \subseteq [0, \alpha)$ for $j = 1, 2, \dots, n$.

We recall the following obvious Proposition 3.3.

Proposition 3.3. Let $A > 0$ and $B > 0$. Then

(i) $A \natural_{-1} B = AB^{-1}A$, (ii) $A \natural_2 B = BA^{-1}B$, (iii) $A \natural_0 B = A$, (iv) $A \natural_1 B = B$, and

(v) $A \log A \geq \log A$ for any $A > 0$.

Remark 3.1. If (i') f is continuous operator concave on J containing 0 and $f(0) \geq 0$, then the following (ii') holds by (i) and (ii) of Proposition 5.1

$$(ii') \quad f(C^*AC) \geq C^*f(A)C + f(0)(I - C^*C) \geq C^*f(A)C$$

for operator C with $\|C\| \leq 1$ and self-adjoint operator A with $\sigma(A) \subseteq J$ since $f(0) \geq 0$ and $I - C^*C \geq 0$.

As " f is continuous operator concave function and $f(0) \geq 0$ " just essentially corresponds to " f is continuous operator convex function and $f(0) \leq 0$ " in (i) of [5, Theorem 2.1], it turns out that Proposition 3.1 is essentially shown under an additional condition $f(0) \geq 0$ in [5, Theorem 2.1], *briefly speaking, Proposition 3.1 with $f(0) \geq 0$ becomes Theorem 2.1 in [5].*

Remark 3.2. It is shown in [6, Theorem 6] that if f is operator monotone function, (iv) of Proposition 3.1 holds. Also Corollary 3.2 implies that if f is an operator monotone function on the half open interval $[0, \alpha)$ to $[0, \alpha)$ with $\alpha \leq \infty$, then $f(\sum_{j=1}^n C_j^* A_j C_j) \geq \sum_{j=1}^n C_j^* f(A_j) C_j$ for operators C_j with $\sum_{j=1}^n C_j^* C_j \leq I$ and self-adjoint operators A_j with $\sigma(A_j) \subseteq [0, \alpha)$ for $j = 1, 2, \dots, n$, which is shown in [6, Corollary 7], because f is operator concave on $[0, \alpha)$ to $[0, \alpha)$ with $\alpha \leq \infty$ if and only if f is operator monotone on $[0, \alpha)$ to $[0, \alpha)$ with $\alpha \leq \infty$.

Addendum. After we have written this manuscript, we know that quite similar results to Proposition 5.1 are shown in the following recent paper: F.Hansen and G.K.Pedersen, Jensen's operator inequality, Bull. London Math. Soc., **35**(2003), 553-564.

This paper will appear elsewhere with complete proofs.

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