

Majorization と作用素不等式

(Majorization and operator inequalities)

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In this paper we deal with bounded self-adjoint operators or Hermitian matrices. Let's start with the definition of an o.m. function. Let f be a real valued continuous function on an interval I . The functional calculus by f induces a non-linear mapping on $H_n(I)$, which is the set of all Hermitian matrices on n -dimensional space. If the mapping preserves the order, then f is called a o.m. function. We denote the set of all o.m. by $\mathbf{P}(I)$, and the subset of non-negative functions by $\mathbf{P}_+(I)$. So a power function with a exponent between 0 and 1 belongs to $\mathbf{P}_+$ on $[0, \infty)$; The inequality induced from this is called **Löwner-Heinz inequality**.

It seemed that only one mapping was considered so far. I tried to compare two mappings. At first We noticed that for $0 \leq A, B$

$$A^2 \leq B^2 \implies (A+1)^2 \leq (B+1)^2,$$

but the converse is not valid. We posed a problem by myself to seek a pair of u, v s.t.

$$0 \leq A, B, u(A) \leq u(B) \implies v(A) \leq v(B).$$

And We first considered the case both u and v are polynomials with positive coefficients.

1 A New Majorization

To study systematically We defined the set of the inverses of o.m. functions. If the left extreme point a is finite, then these two sets are identical by natural extension. Also we considered the set of a function whose logarithm is o.m. And we introduced the concept of a new majorization as follows:

h is said to be majorized by k and denoted by

$$h \preceq k$$

if $J \subset I$, $h \circ k^{-1} \in \mathbf{P}(k(J))$.

This definition is equivalent with

$$k(A) \leq k(B) \implies h(A) \leq h(B).$$

Löwner-Heinz inequality says for $0 < a \leq 1 \leq \beta$

$$t^a \preceq t \preceq t^\beta \quad ([0, \infty)).$$

We list several properties of the majorization.

several properties

(i) $k^\alpha \preceq k^\beta$ for any increasing function

$$k(t) \geq 0 \text{ and } 0 < \alpha \leq \beta;$$

(ii) (transitive) $g \preceq h$, $h \preceq k \implies g \preceq k$;

(iii) (invariant for homeomorphism) if τ is an increasing function whose range is the domain of k , then

$$h \preceq k \iff h \circ \tau \preceq k \circ \tau;$$

(iv) if the range of k is $[0, \infty)$ and $h, k \geq 0$, then

$$h \preceq k \implies h^2 \preceq k^2;$$

Remark: Consider t and $t - 1$ on $1 \leq t < \infty$.

$$t - 1 \preceq t \text{ but } (t - 1)^2 \not\preceq t^2.$$

(v) if the ranges of k, h are $[0, \infty)$, then

$$h \preceq k, \quad k \preceq h \iff h = ck + d$$

for real numbers $c > 0, d$.

Remark: The range condition is indispensable: in fact, $t \preceq \frac{t}{1+t}, \frac{t}{1+t} \preceq t$ on $[0, \infty)$.

The next lemma is very significant for our study, so We named it.

Lemma 1.1 (Product lemma)

Suppose $-\infty \leq a < b \leq \infty$,

$0 \leq h(t), 0 \leq g(t)$ on $[a, b)$.

If the product $h(t)g(t)$ is increasing and the range is $[0, \infty)$ (or $(0, \infty)$ if $a = -\infty$),

then

$$g \preceq hg \implies h \preceq hg.$$

Moreover

$$\psi_1(h)\psi_2(g) \preceq hg \quad \text{for } \psi_1, \psi_2 \in \mathbf{P}_+[0, \infty).$$

This lemma is subtle; so we give some examples.

$$\diamond 1 \preceq t \text{ on } [0, \infty), \quad t \preceq 1 + t^2 \text{ on } [0, \infty).$$

$$\text{but, } t \not\preceq t(1 + t^2) \text{ on } [0, \infty).$$

$\diamond t \preceq t + 1 [0, \infty)$.

but, $t^2 \not\preceq (1 + t)^2 [0, \infty)$.

Now we are in the position to state the main theorem.

Theorem 1.2 (Product theorem)

Suppose $-\infty \leq a < b \leq \infty$. $[a, b)$ denotes $(-\infty, b)$ if $a = -\infty$. Then

$$\mathbb{LP}_+[a, b) \cdot \mathbf{P}_+^{-1}[a, b) \subset \mathbf{P}_+^{-1}[a, b),$$

$$\mathbf{P}_+^{-1}[a, b) \cdot \mathbf{P}_+^{-1}[a, b) \subset \mathbf{P}_+^{-1}[a, b).$$

Further, let $h_i(t) \in \mathbf{P}_+^{-1}[a, b)$ for $1 \leq i \leq m$,

and let $g_j(t) \in \mathbb{LP}_+[a, b)$ for $1 \leq j \leq n$.

Then for $\psi_i, \phi_j \in \mathbf{P}_+[0, \infty)$

$$\prod_{i=1}^m h_i(t) \prod_{j=1}^n g_j(t) \in \mathbf{P}_+^{-1}[a, b),$$

$$\prod_{i=1}^m \psi_i(h_i) \prod_{j=1}^n \phi_j(g_j) \preceq \prod_{i=1}^m h_i \prod_{j=1}^n g_j.$$

It is easy to see the following result is the special case of the above.

Corollary 1.3 Ando[1]

$$f(t) \in \mathbf{P}_+[0, \infty) \Rightarrow tf(t) \in \mathbf{P}_+^{-1}[0, \infty).$$

He proved this by successive approximation. We could get the above theorem by using successive approximation too. $\mathbf{P}_+^{-1}[a, b)$ is closed in the sense that if a limit point of $\mathbf{P}_+^{-1}[a, b)$ is increasing and the range is $[0, \infty)$, then it belongs to $\mathbf{P}_+^{-1}[a, b)$. However we can construct a sequence of functions in this set which converges to $(t - 1)_+$.

2 Polynomials

Let's get back to the original problem. Now we can reach at the solution to the problem.

For non-increasing sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^m$,

$$u(t) := \prod_{i=1}^n (t - a_i) \quad (t \geq a_1),$$

$$v(t) := \prod_{i=1}^m (t - b_i) \quad (t \geq b_1).$$

Lemma 2.1 Suppose $v \preceq u$ for u and v .

Then $m \leq n$.

Theorem 2.2 Suppose $m \leq n$.

$$\sum_{i=1}^k b_i \leq \sum_{i=1}^k a_i \quad (1 \leq k \leq m) \implies v \preceq u.$$

Recall the classical definition of submajorization for two sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^m$. If they satisfies the above condition, it is said that $\{a_i\}_{i=1}^n$ submajorizes $\{b_i\}_{i=1}^m$.

Corollary 2.3 Let $\{p_n\}_{n=0}^\infty$ be a sequence of orthonormal polynomials with the positive leading coefficient. Consider the restricted part of p_n to $[a_n, \infty)$, where a_n is the maximal zero of p_n . Then

$$p_{n-1} \preceq p_n.$$

As to a polynomial with imaginary zeros, we can get similar result:

Theorem 2.4

$$u(t) := \prod_{i=1}^n (t - a_i) \quad (t \geq a_1),$$

$$w(t) := \prod_{j=1}^m (t - \alpha_j) \quad (\Re \alpha_1 \leq t < \infty),$$

where $\Re \alpha_1 \geq \Re \alpha_2 \geq \dots \geq \Re \alpha_m, m \leq n$. Then

$$\sum_{j=1}^k \Re \alpha_j \leq \sum_{j=1}^k a_j \quad (1 \leq k \leq m) \implies w \preceq u.$$

Theorem 2.5 Let $p(t)$ be a real polynomial with a positive leading coefficient such that $p(0) = 0$ and zeros of p are all in $\{z: \Re z \leq 0\}$. Let $q(t)$ be a factor of $p(t)$. Then

$$p(\sqrt{t})^2 \in \mathbb{P}_+^{-1}[0, \infty), \quad q(t)^2 \preceq p(t)^2,$$

that is

$$p(A)^2 \leq p(B)^2 \quad (0 \leq A, B) \Rightarrow A^2 \leq B^2, \quad q(A)^2 \leq q(B)^2.$$

Furthermore, if $p(0) = p'(0) = 0$, then

$$p(\sqrt{t}) \in \mathbb{P}_+^{-1}[0, \infty), \quad q(t) \preceq p(t),$$

that is

$$p(A) \leq p(B) \quad (0 \leq A, B) \Rightarrow A^2 \leq B^2, \quad q(A) \leq q(B).$$

We was asked by S. Pereverzev and U. Tautenhahn if $t^\alpha e^{-t^{-\beta}} \in \mathcal{P}_+^{-1}(0, \infty)$.

It is clear that $t^\alpha e^{-t^{-\beta}} \rightarrow 0$ as $t \rightarrow +0$ for $\alpha, \beta > 0$.

Proposition 2.6 For $0 < \beta \leq \alpha$

$$t^\alpha \preceq t^\alpha e^{-t^{-\beta}}.$$

Moreover, if $1 \leq \alpha$, then

$$t^\alpha e^{-t^{-\beta}} \in \mathcal{P}_+^{-1}(0, \infty).$$

3 Operator Inequalities

Theorem 3.1 Let $h(t) \in \mathbf{P}_+^{-1}[a, b)$, $g(t) \in \mathbf{LP}_+[a, b)$ and $\tilde{h}(t) \geq 0$ on $[a, b)$.

Suppose

$$\tilde{h} \preceq h.$$

Then the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$ belongs to $\mathbf{P}_+[0, \infty)$, and satisfies

$$a \leq A \leq B < b \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}, \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leq g(B)^{\frac{1}{2}}\tilde{h}(A)g(B)^{\frac{1}{2}}. \end{cases}$$

Furthermore, if $\tilde{h} \in \mathbf{P}_+[a, b)$, then

$$a \leq A \leq B < b \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geq \tilde{h}(A)g(A), \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leq \tilde{h}(B)g(B). \end{cases}$$

Proposition 3.2 Let $h(t) \in \mathbf{P}_+^{-1}[a, b)$, $g(t) \in \mathbf{LP}_+[a, b)$. If $0 < \alpha < 1$, $h(t)^\alpha g(t)^{\alpha-1} \preceq h(t)$, then

$$0 \leq A \leq B \Rightarrow \begin{cases} (g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}})^\alpha \geq g(A)^{\frac{1}{2}}h(B)^\alpha g(B)^{\alpha-1}g(A)^{\frac{1}{2}}, \\ (g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})^\alpha \leq g(B)^{\frac{1}{2}}h(A)^\alpha g(A)^{\alpha-1}g(B)^{\frac{1}{2}}. \end{cases}$$

Furthermore, if $h(t)^\alpha g(t)^{\alpha-1} \in \mathbf{P}_+[a, b)$, then

$$a \leq A \leq B < b \Rightarrow \begin{cases} (g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}})^\alpha \geq (h(A)g(A))^\alpha, \\ (g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})^\alpha \leq (h(B)g(B))^\alpha. \end{cases}$$

Corollary 3.3 Let $f(t) \in \mathbf{P}_+[0, \infty)$. Suppose $p, r, \alpha > 0$ and $s \geq 0$ satisfy $1 \leq p$, $r(s-1) \leq p$, $\alpha \leq \frac{1+r}{p+s+r}$.

Then

$$0 \leq A \leq B \Rightarrow \begin{aligned} (A^{\frac{r}{2}} B^p f(B)^s A^{\frac{r}{2}})^{\alpha} &\geq (A^{\frac{r}{2}} A^p f(A)^s A^{\frac{r}{2}})^{\alpha}, \\ (B^{\frac{r}{2}} B^p f(B)^s B^{\frac{r}{2}})^{\alpha} &\geq (B^{\frac{r}{2}} A^p f(A)^s B^{\frac{r}{2}})^{\alpha}. \end{aligned}$$

Example Let $f(t) \in \mathbf{P}_+[0, \infty)$. Suppose $p, r > 0$, $0 < \alpha \leq \frac{1+r}{p+1+r}$. Then

$$0 \leq A \leq B \Rightarrow \begin{cases} (A^{\frac{r}{2}} B^p f(B) A^{\frac{r}{2}})^{\alpha} \geq (A^{\frac{r}{2}} A^p f(A) A^{\frac{r}{2}})^{\alpha}, \\ (B^{\frac{r}{2}} B^p f(B) B^{\frac{r}{2}})^{\alpha} \geq (B^{\frac{r}{2}} A^p f(A) B^{\frac{r}{2}})^{\alpha}. \end{cases}$$

Suppose $p, r > 0$, $0 < \alpha \leq \frac{1+r}{p+r}$. Then

$$0 \leq A \leq B \Rightarrow \begin{cases} (A^{\frac{r}{2}} f(A)^{\frac{1}{2}} B^p f(A)^{\frac{1}{2}} A^{\frac{r}{2}})^{\alpha} \geq (A^{\frac{r}{2}} f(A)^{\frac{1}{2}} A^p f(A)^{\frac{1}{2}} A^{\frac{r}{2}})^{\alpha}, \\ (B^{\frac{r}{2}} f(B)^{\frac{1}{2}} B^p f(B)^{\frac{1}{2}} B^{\frac{r}{2}})^{\alpha} \geq (B^{\frac{r}{2}} f(B)^{\frac{1}{2}} A^p f(B)^{\frac{1}{2}} B^{\frac{r}{2}})^{\alpha}. \end{cases}$$

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