

Existence of non-topological solutions to a nonlinear elliptic equation arising in self-dual Chern-Simons-Higgs theory

Kazuhiro Kurata, Kazuyuki Matsuda

1 Introduction

In this paper I will report our recent studies on existence of 0-vortex non-topological solutions to a nonlinear elliptic equation arising in self-dual Chern-Simons-Higgs theory in a general background metric. For physical backgrounds for the Chern-Simons-Higgs theory, see [HKP], [JW], and [Du].

As in Schiff[Sc], the energy for static states in the (2+1)-dimensional relativistic Abelian Chern-Simons Higgs theory under the background metric $g = \text{diag}(1, -k(x), -k(x))$ is defined as follows:

$$E = \int \left\{ (|D_1\phi|^2 + |D_2\phi|^2) + \frac{\kappa^2 F_{12}^2}{4k(x)|\phi|^2} + k(x)V(|\phi|) \right\} dx,$$

where ϕ is a complex scalar field, $A_\mu (\mu = 1, 2)$ is a vector field, $D_\mu\phi = (\partial_\mu - iA_\mu)\phi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\kappa > 0$ is a coupling constant, and $k(x)$ is a positive function. If we take the special Higgs potential

$$V(|\phi|) = (1/\kappa^2)|\phi|^2(|\phi|^2 - 1)^2.$$

Then we have the following formula under certain decay assumptions to some quantities.

$$E = \int \left\{ |(D_1 \pm iD_2)\phi|^2 + \left| \frac{\kappa F_{12}}{2\sqrt{k(x)}\phi} \pm \frac{\sqrt{k(x)}}{\kappa} \phi^* (|\phi|^2 - 1) \right|^2 \right\} dx \pm \int F_{12} dx. \quad (1)$$

Here, ϕ^* is the complex conjugate of ϕ . Let $\Phi \equiv \int F_{12} dx$ be the total magnetic flux and fix it. Then, it follows that (ϕ, A_μ) is the global minimizer of E if and only if (ϕ, A_μ) satisfies

$$(D_1 \pm iD_2)\phi = 0, \quad (2)$$

$$F_{12} \pm \frac{2k(x)}{\kappa^2} |\phi|^2 (|\phi|^2 - 1) = 0. \quad (3)$$

As in self-dual models in many gauge theory, the study of this system can be reduced to the one of an certain second order scalar nonlinear elliptic equation. For simplicity, consider 0-vortex solutions, i.e. $\phi \neq 0$. Writing $\phi = he^{i\omega} = e^{(1/2)u+i\omega}$, (2) yields

$$A_i = -\partial_i \omega \pm \epsilon_{ij} \partial_j (\log h).$$

Thus, $F_{12} = -\{\partial_1(\partial_1 h/h) + \partial_2(\partial_2 h/h)\} = -\Delta(\log h)$. Therefore by (3) we obtain

$$\Delta u = \frac{4k(x)}{\kappa^2} e^u (e^u - 1). \quad (4)$$

We use the notation $\lambda = 4/\kappa^2$ throughout this paper.

Remark 1 (3) in the case of plus sign yields

$$\Phi = \int F_{12} dx = \frac{1}{2} \int \frac{4k(x)}{\kappa^2} e^u (1 - e^u) dx. \quad (5)$$

Later, we will find solutions to (4) (i.e. 0-vortex non-topological solution) in the form $u = u_0 + w$, $u_0 = \log(1 + |x|^2)^{\alpha/2}$, where w satisfies $\int \Delta w dx = 0$ and $w(x)$ tends to a constant at infinity. Since $\int \Delta u_0 dx = 2\pi\alpha$, we have the relation $\Phi = -\alpha\pi$. So, to prescribe the total magnetic flux Φ is equivalent to prescribe the number α .

From the energy finiteness of solutions, the following two type of solution can be considered: the one is $|\phi| \rightarrow 1$ as $|x| \rightarrow \infty$ (which is called topological solutions), the other is $|\phi| \rightarrow 0$ as $|x| \rightarrow \infty$ (which is called non-topological solutions). Hence, for solution u to (4), we call u is topological iff $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$; u is non-topological iff $u(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$. We also say ϕ is a N -Vortex solution ($N \in 0 \cup \mathbb{N}$) for prescribed points $\{p_i\}_{i=1}^l$, if

$$|\phi| \sim c_i |x - p_i|^{n_i} (|x - p_i| \rightarrow 0), c_i > 0, N = \sum_{i=1}^l n_i.$$

For a N -vortex solution, the system (2)-(3) can be reduced into the study of

$$\Delta v = \frac{4k(x)}{\kappa^2} e^{u_0+v} (e^{u_0+v} - 1) + 4 \sum_{i=1}^l n_i \frac{1}{(1 + |x - p_i|^2)^2},$$

where $u_0(x) = -\sum_{i=1}^l n_i \log(1 + |x - p_i|^{-2})$ (see e.g. [SpYa2]), with the asymptotic behaviour $v(x) \rightarrow 0$ for a topological solution, $v(x) \rightarrow -\infty$ for a non-topological solution, respectively. We recall several known results on existence of non-topological and topological solutions. First, for the case $g = \text{diag}(1, -1, -1)$ (i.e., $k(x) \equiv 1$), existence of arbitrary N -vortex topological solution was shown by Wang[Wa], Spruck-Yang[SpYa2], and Φ , Q (total charge), E are all quantized:

$$\Phi = 2\pi N, Q = 2\pi N\kappa, E = 2\pi N.$$

They also proved the asymptotic behaviour $(|\phi|^2 - 1), |F_{12}| \sim O(e^{-(c/|\kappa|)|x|})$ at infinity. The existence of radially symmetric N -vortex non-topological solutions was proved by Spruck-Yang[SpYa1], Chen-Hastings-McLeod-Yang, especially in [CHMY] they showed that for every $\beta > 2N + 4, N \geq 0$, there exist a solution s.t. $|\phi|^2, |F_{12}| \sim O(|x|^{-\beta}), |D_j \phi|^2 = O(|x|^{-(2+\beta)})$ at infinity. In this case, we have

$$\Phi = 2\pi N + \pi\beta, Q = \Phi\kappa, E = \Phi.$$

Next, Schiff studied self-dual Chern-Simon-Higgs theory in a general background metric and proved that for the case $4k(|x|)/\kappa^2 = \beta^2/|x|^2, \beta > 0$, $u(|x|) = -\log(\lambda|x|^\beta + 1), \lambda > 0$ is a solution of (4) (β -vortex non-topological solution). In [CHMY] they also studied for certain $k(x) = k(|x|)$ the uniqueness of N -vortex topological radially symmetric solution for the prescribed N ; existence of N -vortex non-topological radially symmetric solutions for certain range for β .

The purpose of this paper is to study (4) for certain general, not necessarily radially symmetric, $k(x)$ and show existence of 0-vortex non-topological solutions via a variational method or a fixed point theorem.

2 Main Results

Throughout this paper, we assume $k(x) \not\equiv 0$ and $k(x)$ is a non-negative Hölder continuous function. Our main result is as follows.

Theorem 1 Suppose $k(x)$ satisfies $k(x) = O(1/|x|^l)$ as $|x| \rightarrow \infty$ for some $l > 2$. Let $-4 < \alpha < \min(0, l - 4)$. Then there exists a constant $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ (4) has a solution u satisfying

$$\lim_{|x| \rightarrow \infty} (u(x) - \alpha \log |x|) = C_0$$

for some constant C_0 .

Remark 2 Actually, we can prove the following: there exists a critical parameter $\lambda_c \geq (-8\pi\alpha) / \int k(x) dx$ s.t. there exists a solution for every $\lambda > \lambda_c$ and no solution for $0 < \lambda < \lambda_c$. We can prove this result by combining a subsolution-supersolution method with Theorem 1. It will be published in elsewhere. Our method can be applied to obtain a 1-vortex non-topological solution under certain conditions.

We can show Theorem 1 via a variational method based on several results on the weighted Sobolev spaces $W_{s,\delta}^2$ (see [Mc]). (4) has some similarity to the Gauss curvature equation, but a difficult problem to determine the sign of the Lagrange multiplier occurs due to the nonlinearity in (4). We overcome this difficulty by using the idea of Caffarelli and Yang, in [CaYa] they employed their idea to periodic problem.

Theorem 2 Suppose $k(x)$ satisfies $k(x) = O(1/|x|^l)$ as $|x| \rightarrow \infty$ for some $l > 2$. Fix $\lambda > 0$ in (4). Then, there exists sufficiently small constant $\alpha_0 > 0$ s.t. for any $\alpha \in (-\alpha_0, 0)$, (4) has a solution u which satisfies

$$\alpha \log |x| - C_1 \leq u(x) \leq \alpha \log |x| + C_2$$

near infinity, where C_1, C_2 are positive constants.

Theorem 2 is proved by using the Leray-Schauder's fixed point theorem on a weighted Sobolev space $W_{s,\delta}^p$. In this paper, we only give a sketch of the proof of Theorem 1 (see [Ku] for the details). See [Ma] for the proof of Theorem 2.

We can also show an existence theorem under somewhat mild condition for the decay on $k(x)$ via subsolution-supersolution method (see [Ma]) for certain α . However, to author's knowledge, it is an open problem to obtain an existence theorem for non-topological solutions under slow decay general (not radially symmetric) background metric.

Remark 3 Recently, the results on the periodic problem in [CaYa] is sharp-end by Tarantello[Ta]. As an analogy to [Ta], we have a following conjecture: there exist two solutions $\underline{u}_\lambda, \bar{u}_\lambda$ s.t. $\underline{u}_\lambda \rightarrow 0$ as $\lambda \rightarrow +\infty$, $\bar{u}_\lambda = w_\lambda + c_\lambda$, c_λ is a constant, which satisfy $w_\lambda \rightarrow w_0$ in $\mathcal{H}$ and $c_\lambda \rightarrow -\infty$ as $\lambda \rightarrow +\infty$. Moreover, w_0 satisfies the following equation:

$$-\Delta w_0 = (-2\pi\alpha) \frac{k(x)e^{u_0+w_0}}{\int_{\mathbf{R}^2} k e^{u_0+w_0} dx} - f, \quad \int w_0 f dx = 0,$$

where $f = -\Delta u_0$ and $u_0 = \log(1 + |x|^2)^{\alpha/2}$.

3 Preliminaries

In this section, we recall several known results on weighted Sobolev spaces and Moser-Trudinger's inequality and Poincaré's inequality adapted in this setting. The weighted Sobolev spaces $W_{s,\delta}^2$ are defined as the closure of C_0^∞ with respect to the norm:

$$\|u\|_{W_{s,\delta}^2}^2 = \sum_{|\beta| \leq s} \|(1 + |x|)^{|\beta|+\delta} |D_x^\beta u|\|_{L^2}^2$$

We use the notation $L_\delta^2 = W_{0,\delta}^2$. The following results are well-known (see e.g. [Mc]).

- (i) If $s' > s, \delta' > \delta$, then we have a compact embedding: $W_{s',\delta'}^2 \subset W_{s,\delta}^2$.
- (ii) If $s > 1, \delta > -1$, then $W_{s,\delta}^2 \subset C_0(\mathbf{R}^2)$.
- (iii) $u \in L_\delta^2, \Delta u \in L_{\delta+2}^2$ implies $u \in W_{2,\delta}^2$.
- (iv) Let $-1 < \delta < 0$. Then $\Delta : W_{2,\delta}^2 \rightarrow L_{\delta+2}^2$ is the bijection to the range $\{f \in L_{\delta+2}^2; \int f dx = 0\}$. We also need the following two technical lemmas.

Lemma 1 Let $d\mu = h(x) dx$ with $h(x) \sim (1 + |x|)^{-(2+\epsilon)}, \epsilon > 0$, and $0 < \beta < \min(4\pi, 2\pi\epsilon)$. Then we have

$$\int e^{\alpha|\nu|} d\mu \leq C \exp\left(\frac{\alpha^2}{4\beta} \|\nabla \nu\|_{L^2}^2\right), \quad \nu \in \widetilde{\mathcal{H}}.$$

Here $\mathcal{H}$ is the closure of C_0^∞ w.r.t. $\|\nu\|_{\mathcal{H}}^2 = \int |\nabla \nu|^2 dx + \int \nu^2 d\mu$ and $\widetilde{\mathcal{H}} = \{\nu \in \mathcal{H}; \int \nu d\mu = 0\}$.

Lemma 2 Let $\eta > 0$ and $\nu \in \widetilde{\mathcal{H}}$. Then there exists a constant $C = C(\eta)$ such that

$$\|\nu\|_{L_{-1-\eta}^2}^2 \leq C(\eta) \|\nabla \nu\|_{L^2}^2.$$

4 Sketch of The Proof of Theorem 1

Let $\alpha < 0$ and take $u_0(x) = \log(1 + |x|^2)^{\alpha/2}$. Then

$$f(x) \equiv -\Delta u_0(x) = \frac{-2\alpha}{(1 + r^2)^2} (\geq 0).$$

Consider the measure $d\mu = f(x) dx$. Then we have

$$\int d\mu = \int f(x) dx = -2\pi\alpha.$$

Let $\mathcal{H}$ be the closure of C_0^∞ w.r.t. the norm $\|\nu\|_{\mathcal{H}}^2 = \{\int |\nabla w|^2 dx + \int w^2 d\mu < +\infty\}$ and $\widetilde{\mathcal{H}} = \{w \in \mathcal{H}; \int w d\mu = 0\}$. Now $u = w + u_0$ is a solution to (4) iff w satisfies

$$\Delta w + \lambda k(x) e^{u_0+w} (1 - e^{u_0+w}) = f.$$

We will find a solution w in the class $\mathcal{H}$. Decompose $w \in \mathcal{H}$ into $w = \nu + c$, $\nu \in \widetilde{\mathcal{H}}$ with a constant c . Assume

$$\int \Delta w dx = 0.$$

ν and c should satisfy

$$e^{2c} \int k(x) e^{2(u_0+\nu)} dx - e^c \int k(x) e^{u_0+\nu} dx + (-2\pi\alpha)/\lambda = 0.$$

Thus the following condition is necessary:

$$\left(\int k(x) e^{u_0+\nu} dx\right)^2 + \frac{8\pi\alpha}{\lambda} \int k(x) e^{2(u_0+\nu)} dx \geq 0. \quad (6)$$

Let $\mathcal{H}_* = \{\nu \in \widetilde{\mathcal{H}}; \nu \text{ satisfies the condition above}\}$. Define the constant $c = c(\nu)$ as follows:

$$e^c = \frac{\int k(x) e^{u_0+\nu} dx + \sqrt{\left(\int k(x) e^{u_0+\nu} dx\right)^2 + \frac{8\pi\alpha}{\lambda} \int k(x) e^{2(u_0+\nu)} dx}}{2 \int k(x) e^{2(u_0+\nu)} dx}. \quad (7)$$

Then consider the following minimizing problem:

$$\sigma = \inf_{\nu \in \mathcal{H}_*} I(\nu), \quad (8)$$

where

$$I(\nu) = \int \frac{1}{2} |\nabla \nu|^2 + \frac{\lambda}{2} k(x) e^{2(u_0+\nu+c)} - \lambda k(x) e^{u_0+\nu+c} dx - 2\pi\alpha c. \quad (9)$$

Lemma 3 *There exists a constant $c = c(\alpha)$ s.t.*

$$I(\nu) \geq -(-2\pi\alpha) \log \lambda - c(\alpha), \nu \in \partial\mathcal{H}_*.$$

Next take λ_0 sufficiently large s.t.

$$\left(\int k(x)e^{u_0} dx\right)^2 + \frac{8\pi\alpha}{\lambda_0} \int k(x)e^{2u_0} dx > 0.$$

Then 0 belongs to the interior of $\mathcal{H}_*$ for every $\lambda \geq \lambda_0$. On the other hand,

Lemma 4 *Taking λ_0 sufficiently large if necessary, there exist positive constants $C_j = C_j(\alpha)$, $j = 1, 2$ s.t.*

$$I(0) \leq -C_1\lambda + C_2$$

for $\lambda \geq \lambda_0$.

Therefore, we have, taking λ_0 sufficiently large if necessary again,

$$I(0) < -1 + I(\nu), \nu \in \partial\mathcal{H}_*$$

for $\lambda \geq \lambda_0$. Hence, if there exists a minimizer ν_0 to the minimizing problem, ν_0 belongs to the interior of $\mathcal{H}_*$.

Lemma 5 *There exist positive constants δ, C s.t.*

$$I(\nu) \geq \delta \|\nabla \nu\|_{L^2}^2 - C, \nu \in \mathcal{H}_*.$$

Our assumption $l > 2$ need here! This lemma and the compactness of the embedding $\widetilde{\mathcal{H}} \hookrightarrow L^2(d\mu)$, we can conclude the existence of the minumizer ν_0 , which belongs to the interior of $\mathcal{H}_*$. Note $\mathcal{H} \hookrightarrow W_{1,-1-\epsilon}^2$ for every $\epsilon < 1$.

Since ν_0 belongs to the interior of $\mathcal{H}_*$, we have

$$\langle I'(\nu_0), \phi \rangle = 0, \phi \in \widetilde{\mathcal{H}}.$$

Using $\int \phi f dx = 0$ and the definition of $c = c(\nu_0)$, this implies

$$\int \nabla \nu_0 \cdot \nabla \psi dx + \int \{\lambda k(x)(e^{2(u_0+\nu_0+c)} - e^{u_0+\nu_0+c}) + f\} \psi dx = 0$$

for every $\psi \in \mathcal{H}$. Thus $U(x) = u_0 + \nu_0 + c(\nu_0)$ satisfies $-\Delta U + \lambda k(x)e^U(e^U - 1) = 0$. Finally, since $\mathcal{H} \hookrightarrow C_0$, we have $\nu_0(x) \rightarrow 0$ as $|x| \rightarrow \infty$. This concludes the desired result. $\square$

References

- [CaYa] L.A.Caffarelli, Y.Yang, Vortex condensation in the Chern-Simons Higgs Model: An existence theorem, *Comm.Math.Phys.*, 168(1995), 321–336.
- [Ca] M.Cantor, Elliptic operators and the decomposition of tensor fields, *Bull.Amer.Math.Soc.(N.S.)*5(1981), 235–262.
- [CHMY] X.Chen, S.Hastings, J.B.McLeod, Y.Yang, A nonlinear elliptic equation arising from gauge field theory and cosmology, *Proc. Roy.Soc. Lond. A* 446(1994), 453–478.
- [Du] G. Dunne, *Self-Dual Chern-Simons Theories*, Lecture Notes in Physics 36, 1995, Springer.
- [HKP] J.Hong, Y.Kim, P.Y.Pac, Multivortex solutions of the Abelian Chern-Simons -Higgs theory, *Phys.Rev.Lett.*, 64(1990),2230–2233.
- [JW] R.Jackiw, E.J.Weinberg, Self-dual Chern-Simons vortices, *Phys.Rev.Lett.*, 64(1990), 2234–2237.
- [Ku] K.Kurata, Existence of non-topological solutions for a nonlinear elliptic equation arising from Chern-Simons Higgs theory in a general background metric, Preprint(1997).
- [Ma] K.Matsuda, Existence and symmetry of solutions for some nonlinear elliptic equations, (in Japanese) Master Thesis(1997), Tokyo Metropolitan University.
- [Mc] R.C.McOwen, Conformal metrics in $\mathbf{R}^2$ with prescribed Gaussian curvature and positive total curvature, *Indiana Univ.Math.J.*, 34(1985), 97–104.
- [Sc] J.Schiff, Integrability of Chern-Simons-Higgs and Abelian Higgs vortex equations in a background metric, *J.Math.Phys.*, 32(1991), 753–761.
- [SpYa1] J.Spruck, Y.Yang, The existence of non-topological solitons in the self-dual Chern-Simons theory, *Comm. Math. Phys.*, 149(1992), 361–376.
- [SpYa2] J.Spruck, Y.Yang, Topological solutions in the self-dual Chern-Simons theory;existence and approximation, *Ann. Inst. H. Poincaré*, 12(1995), 75–97.

[Ta] G. Tarantello, Multiple condensate solutions for the Chern-Simons-Higgs theory, J.Math.Phys., 37(1996), 3769–3796.

[Wa] R.Wang, The existence of Chern-Simons vortices, Comm. Math. Phy., 137(1991), 587–597.

Address:

Department of Mathematics

Tokyo Metropolitan University

Minami-Ohsawa 1-1, Hachioji-shi

Tokyo 192-03, JAPAN

kurata@math.metro-u.ac.jp