
CONGRUENCE PROPERTIES OF ENDO-CLASSES AND THE LOCAL JACQUET–LANGLANDS CORRESPONDENCE

by

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These are notes from a lecture I gave at RIMS, Kyoto, for the conference *Analytic, geometric and p -adic aspects of automorphic forms and L -functions*, from 20 to 24 January, 2020. I wish to thank the organizers of this conference. These notes are intended to be a survey of recent results that have been published in [31].

Notation

Let us fix some notation:

- F is a non-Archimedean locally compact field of residual characteristic p ,
- $\overline{F}$ is a separable closure of F ,
- $\mathcal{W}_F$ is the Weil group of $\overline{F}/F$,
- $\mathcal{I}_F$ is the inertia subgroup of $\mathcal{W}_F$, that is its unique maximal compact subgroup,
- $\mathcal{P}_F$ is the wild inertia subgroup of $\mathcal{W}_F$, that is its unique maximal pro- p -subgroup.

1. Irreducible representations of the Weil group

Given an irreducible smooth complex representation σ of $\mathcal{W}_F$, its restriction to $\mathcal{P}_F$ decomposes into a direct sum of finitely many irreducible representations of $\mathcal{P}_F$. These irreducible representations form a single $\mathcal{W}_F$ -orbit under the conjugacy action, denoted $\mathcal{O}_F(\sigma)$. All representations in this orbit occur in the restriction of σ to $\mathcal{P}_F$ with the same multiplicity $m = m(\sigma)$.

Fix $\alpha \in \mathcal{O}_F(\sigma)$. Its stabilizer in $\mathcal{W}_F$ is equal to $\mathcal{W}_T$ for a uniquely determined tamely ramified, finite extension T of F contained in $\overline{F}$. The representation α extends (non-canonically) to an irreducible representation ρ of $\mathcal{W}_T$. The representation σ can then be written:

$$\sigma \simeq \mathrm{Ind}_{T/F}(\rho \otimes \tau)$$

where τ is an irreducible representation of $\mathcal{W}_T$ trivial on $\mathcal{P}_T = \mathcal{P}_F$, of dimension m , uniquely determined up to isomorphism, and where $\mathrm{Ind}_{T/F}$ denotes induction from $\mathcal{W}_T$ to $\mathcal{W}_F$.

2. Cuspidal representations of $\mathrm{GL}_n(F)$

On the other hand, consider cuspidal irreducible smooth complex representations of the group $G = \mathrm{GL}_n(F)$ for some $n \geq 1$. Bushnell and Kutzko have constructed in [8] an explicit family of pairs $(\mathbf{J}, \lambda)$, where $\mathbf{J}$ is an open, compact mod centre subgroup of G and λ is an irreducible representation of $\mathbf{J}$, such that:

- the compact induction of λ from $\mathbf{J}$ to G is irreducible and cuspidal for all pairs $(\mathbf{J}, \lambda)$,
- any cuspidal irreducible representation of G occurs this way, for a pair $(\mathbf{J}, \lambda)$, uniquely determined up to G -conjugacy.

These pairs have the following properties:

- (1) The group $\mathbf{J}$ has a unique maximal compact subgroup $\mathbf{J}^0$ and a unique maximal normal pro- p -subgroup $\mathbf{J}^1$.
- (2) The restriction of λ to $\mathbf{J}^1$ is a multiple of a single irreducible representation η .
- (3) The representation η extends (non-canonically) to a representation κ of $\mathbf{J}$, and λ is isomorphic to $\kappa \otimes \xi$ for a uniquely determined irreducible representation ξ of $\mathbf{J}$ trivial on $\mathbf{J}^1$.
- (4) The representation η is constructed as a Heisenberg representation from a character θ of a smaller open subgroup $H^1 \subseteq \mathbf{J}^1$ such that $\mathbf{J}^1/H^1$ is a finite-dimensional $\mathbb{F}_p$ -symplectic space.

We denote by $\mathcal{C}(G)$ the set of all characters θ obtained this way when one varies the pairs $(\mathbf{J}, \lambda)$, and by $\mathcal{C}(F)$ the union of $\mathcal{C}(\mathrm{GL}_n(F))$ for all $n \geq 1$. Bushnell and Henniart ([4]) equipped the set $\mathcal{C}(F)$ with an equivalence relation called *endo-equivalence*. Equivalence classes for this relation are called *endo-classes*. The set $\mathcal{E}(F)$ of endo-classes only depends on F .

Remark 2.1. — When one fixes n , two characters $\theta, \theta' \in \mathcal{C}(\mathrm{GL}_n(F))$ defined on H^1, H'^1 respectively are endo-equivalent if and only if there is a $g \in \mathrm{GL}_n(F)$ such that $\theta' = \theta^g$ on $H'^1 \cap H^{1g}$.

The endo-class of any character θ as above occurring in a cuspidal irreducible representation π of G will be denoted $\Theta(\pi)$.

3. The Ramification Theorem

We now consider the local Langlands correspondence $\pi \mapsto {}^L\pi$ (see [21, 17, 18]) between isomorphism classes of cuspidal irreducible representations of $\mathrm{GL}_n(F)$ and isomorphism classes of irreducible n -dimensional representations of $\mathcal{W}_F$.

Theorem 3.1 ([6]). — *Let π_1, π_2 be cuspidal irreducible representations of $\mathrm{GL}_{n_1}(F), \mathrm{GL}_{n_2}(F)$ respectively, and set $\sigma_1 = {}^L\pi_1$ and $\sigma_2 = {}^L\pi_2$. Then:*

$$\mathcal{O}_F(\sigma_1) \cap \mathcal{O}_F(\sigma_2) \neq \emptyset \iff \Theta(\pi_1) = \Theta(\pi_2).$$

4. Inner forms and the local Jacquet-Langlands correspondence

More generally, the classification of cuspidal representations by compact induction, as well as the notion of endo-classes, also works for inner forms of general linear groups ([2, 16, 26, 27,

28, 29, 3]). It is natural to ask whether there exists, for inner forms, an analogue of Bushnell–Henniart’s Ramification Theorem 3.1.

Let $H = \mathrm{GL}_m(D)$ be an inner form of $G = \mathrm{GL}_n(F)$, where D is a central division F -algebra of reduced degree d such that $md = n$. The local Jacquet–Langlands correspondence is a bijection $\pi \mapsto {}^{JL}\pi$ between the discrete series of H and G , characterized by a character identity at elliptic regular conjugacy classes ([19, 25, 13, 1]).

Given a discrete series representation π of H , it occurs as a subquotient of the parabolic induction of a cuspidal representation of the form:

$$\rho\chi_1 \otimes \cdots \otimes \rho\chi_r$$

where r is a divisor of m , ρ is a cuspidal irreducible representation of $\mathrm{GL}_{m/r}(D)$ and $\chi_1, \dots, \chi_r$ are unramified characters of $\mathrm{GL}_{m/r}(D)$. Define $\Theta(\pi) = \Theta(\rho)$.

Theorem 4.1 ([32, 31, 14]). — *The local Jacquet–Langlands correspondence preserves endo-classes.*

This result is one of the main steps in the explicit description of the local Jacquet–Langlands correspondence in terms of Bushnell–Kutzko’s simple types ([31, 14]).

In the sequel, I will explain a crucial step in the proof of this theorem. Surprisingly, it involves the *modular representation theory* of H .

5. Modular representations of p -adic groups

Fix a prime number ℓ different from p and an algebraic closure $\overline{\mathbb{F}}_\ell$ of $\mathbb{F}_\ell$, and consider smooth $\overline{\mathbb{F}}_\ell$ -representations of H . As in the complex case, one has Haar measures, normalized parabolic induction and restriction functors, cuspidal representations and uniqueness of the cuspidal support for irreducible representations. Here, as in the complex case, a representation is cuspidal if all its proper Jacquet modules are zero or, equivalently, if all its matrix coefficients are compactly supported mod centre. However, there is a crucial difference: an irreducible cuspidal representation may occur as a subquotient of a parabolically induced representation ([33] Corollaire 5).

Say an irreducible representation of H is supercuspidal when it does not occur as a subquotient of a parabolically induced representation. For irreducible representations of H , there is a well-behaved notion of supercuspidal support.

Theorem 5.1 ([23]). — *Let π be an irreducible $\overline{\mathbb{F}}_\ell$ -representation of H .*

(1) *There are a Levi subgroup M of H and an irreducible supercuspidal representation ρ of M such that π occurs as a subquotient of the parabolic induction of ρ to H with respect to any parabolic subgroup with Levi component M .*

(2) *The pair (M, ρ) is unique up to H -conjugacy.*

Remark 5.2. — Unlike uniqueness of the cuspidal support, Theorem 5.1 is non-trivial. Uniqueness of the supercuspidal support also holds for special linear p -adic groups [9] and small unramified unitary groups [20], but is false in general (see [15, 12] for a counterexample in the symplectic group Sp_8).

Based on the well-defined notion of supercuspidal support for the group H , we have the following decomposition theorem.

Theorem 5.3 ([30]). — *The category $\text{Rep}_{\overline{\mathbb{F}}_\ell}(H)$ of smooth $\overline{\mathbb{F}}_\ell$ -representations of H decomposes into blocks, which correspond bijectively to inertial classes of supercuspidal pairs of H .*

6. Reduction mod ℓ of representations over ℓ -adic numbers

Now fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field of ℓ -adic numbers and consider smooth $\overline{\mathbb{Q}}_\ell$ -representations of H . Let $\overline{\mathbb{Z}}_\ell$ denote the ring of algebraic integers of $\overline{\mathbb{Q}}_\ell$. An irreducible representation π of H on a $\overline{\mathbb{Q}}_\ell$ -vector space V is said to be *integral* if it has an H -stable $\overline{\mathbb{Z}}_\ell$ -lattice $L \subseteq V$. We have the following p -adic Brauer-Nesbitt principle.

Proposition 6.1 ([34] I.9.6). — *The $\overline{\mathbb{F}}_\ell$ -representation $L \otimes \overline{\mathbb{F}}_\ell$ has finite length. Its semi-simplification only depends on the isomorphism class of π , and not on the choice of L .*

The semi-simplification of $L \otimes \overline{\mathbb{F}}_\ell$ will be denoted $r_\ell(\pi)$ and called the reduction mod ℓ of π . Two integral irreducible representations are said to be *congruent* if they have the same reduction mod ℓ .

7. Congruence properties of the local Jacquet-Langlands correspondence

Now replace $\mathbb{C}$ by $\overline{\mathbb{Q}}_\ell$ in Paragraphs 3 and 4 by fixing a field isomorphism $\iota : \mathbb{C} \rightarrow \overline{\mathbb{Q}}_\ell$. Since cuspidal representations are characterized by the properties of the support of their matrix coefficients, base change from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$ preserves cuspidality. One thus gets a local Langlands correspondence between isomorphism classes of cuspidal irreducible $\overline{\mathbb{Q}}_\ell$ -representations of $\text{GL}_n(F)$ and isomorphism classes of irreducible n -dimensional $\overline{\mathbb{Q}}_\ell$ -representations of W_F . The congruence properties of this local Langlands correspondence have been studied by Vignéras, Bushnell–Henniart and Dat ([35, 7, 10]).

For the local Jacquet–Langlands correspondence, one first has to check that the set of $\overline{\mathbb{Q}}_\ell$ -representations of H coming from a discrete series complex representation by base change from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$ does not depend on the choice of ι . This comes from the fact that discrete series complex representations can be characterized as Langlands quotients of certain induced representations.

Theorem 7.1 ([11, 24, 31]). — *Let π, π' be integral discrete series $\overline{\mathbb{Q}}_\ell$ -representations of H .*

- (1) *${}^{JL}\pi$ and ${}^{JL}\pi'$ are integral, and they are congruent if and only if π, π' are congruent.*
- (2) *$r_\ell({}^{JL}\pi)$ and $r_\ell({}^{JL}\pi')$ are in the same block of $\text{Rep}_{\overline{\mathbb{F}}_\ell}(G)$ if and only if $r_\ell(\pi)$ and $r_\ell(\pi')$ are in the same block of $\text{Rep}_{\overline{\mathbb{F}}_\ell}(H)$.*

8. Linked representations

Now go back to complex representations.

Definition 8.1. — Let π, π' be discrete series complex representations of H . They are:

(1) *1-linked* if there are a prime number $\ell \neq p$ and an isomorphism of fields $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ such that $r_\ell(\iota^* \pi)$ and $r_\ell(\iota^* \pi')$ are in the same block of $\text{Rep}_{\overline{\mathbb{F}}_\ell}(H)$.

(2) *linked* if there are discrete series representations $\pi_0, \dots, \pi_r$ of H such that $\pi_0 = \pi$, $\pi_r = \pi'$ and π_i, π_{i-1} are 1-linked for all $i \in \{1, \dots, r\}$.

Any two linked discrete series representations of H have the same endo-class. The following theorem says that the converse is true.

Theorem 8.2 ([31]). — *Two complex discrete series representations of H are linked if and only if they have the same endo-class.*

Putting all these results together, we get:

Corollary 8.3. — *Let π, π' be discrete series complex representations of H . Then π, π' have the same endo-class if and only if ${}^{JL}\pi, {}^{JL}\pi'$ have the same endo-class.*

It follows that, given a discrete series complex representation π of H , the endo-class $\Theta({}^{JL}\pi)$ only depends on $\Theta(\pi)$. This defines a map from the set of endo-classes of discrete series representation of H to that of endo-classes of discrete series representation of G , and one verifies that this map is bijective. Moreover, varying n and the inner form, one gets a bijective map:

$$j_F : \Theta \mapsto {}^{JL}\Theta$$

from $\mathcal{E}(F)$ to itself. Theorem 4.1 is then equivalent to saying that this map is the identity.

9. The restriction map

Let K be a finite, tamely ramified extension of F contained in $\overline{F}$. Associated with it, there is the set $\mathcal{E}(K)$ of K -endo-classes. It corresponds, via the Ramification Theorem 3.1, to the set of $\mathcal{W}_K$ -conjugacy classes of irreducible representations of $\mathcal{P}_K = \mathcal{P}_F$. Bushnell-Henniart ([4]) have defined a surjective map:

$$\rho_{K/F} : \mathcal{E}(K) \rightarrow \mathcal{E}(F)$$

called the *restriction map*. It corresponds, through the local Langlands correspondence, to passing from $\mathcal{W}_K$ -conjugacy classes to $\mathcal{W}_F$ -conjugacy classes of irreducible representations of $\mathcal{P}_F$.

10. The last step

Given an endo-class $\Theta \in \mathcal{E}(F)$, there exists an $m \geq 1$ such that the group $H = \text{GL}_m(D)$ has a cuspidal representation π with the following properties:

- (1) the endo-class of π is equal to Θ ,
- (2) the transfer ${}^{JL}\pi$ is a cuspidal representation of $G = \text{GL}_n(F)$, with $n = md$.

Let K be the unique finite unramified extension of F contained in $\overline{F}$ of degree the torsion number of π , that is the number of unramified characters χ of H such that $\pi\chi \simeq \pi$. This number only depends on n and Θ . Choose an endo-class $\Psi \in \mathcal{E}(K)$ whose restriction to $\mathcal{E}(F)$ is equal to Θ .

Following Bushnell-Henniart's analysis of the traces of π and ${}^{JL}\pi$ at certain well chosen elliptic regular conjugacy elements in terms of types ([5]), one get the following result:

Proposition 10.1 ([5] 6, [31] 7, [14] 4). — *One has $j_F\Theta = \rho_{K/F}(j_K\Psi)$.*

Fix an F -embedding of K in $M_m(D)$, and let H_K be the centralizer of K in H . Define G_K similarly. Let ρ be a cuspidal representation of H_K of endo-class Ψ . The endo-class of its transfer to G_K is thus $j_K\Psi$.

Dotto's idea ([14]) is to choose the integer $m \geq 1$ so that G_K and H_K are K -isomorphic. (For instance, choose m large enough so that G_K and H_K are both split over K .) We thus may identify G_K and H_K so that ρ is isomorphic to its transfer to G_K . This gives us $j_K\Psi = \Psi$. Applying the restriction map and Proposition 10.1, we get the desired identity $j_F\Theta = \rho_{K/F}(\Psi) = \Theta$.

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