

On the Woronowicz's twisted product construction of quantum groups, with comments on related cubic Hecke algebra. *

JANUSZ WYSOCZAŃSKI †

Institute of Mathematics, Wrocław University
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Abstract

We study the construction of compact quantum groups, based on the method invented by Woronowicz [SLW3], which uses a *twisted determinant*. As an example Woronowicz considered the function $S_N \ni \sigma \mapsto \text{inv}(\sigma)$, where $\text{inv}(\sigma)$ is the number of inversions in the permutation σ . Our twisted determinant is related to the function $S_N \ni \sigma \mapsto c(\sigma)$, where $c(\sigma)$ is the number of cycles in a permutation σ . For $N = 3$ it gave the quantum group $U_q(2)$. Here we show how the construction works if $N = 4$. We also describe the cubic Hecke algebra, associated with the quantum group $U_q(2)$.

1 Introduction

In [SLW3] Woronowicz provided a general method for constructing compact matrix quantum groups. The method depends on finding an N^N -element array $E = (E_{i_1, \dots, i_N})_{i_1, \dots, i_N=1}^N$ of complex numbers, called *twisted determinant*, which is (left and right) non-degenerate. Theorem 1.4 of [SLW3] says that if a C^* -algebra $\mathcal{A}$, is generated by N^2 elements u_{jk} which satisfy the unitarity condition:

$$\sum_{r=1}^N u_{jr}^* u_{rk} = \delta_{jk} I = \sum_{r=1}^N u_{jr} u_{rk}^*$$

and the following twisted determinant condition:

$$\sum_{k_1, \dots, k_N=1}^N u_{j_1 k_1} \dots u_{j_N k_N} E_{k_1, \dots, k_N} = E_{j_1, \dots, j_N} I$$

*Research partially supported by the European Commission Marie Curie Host Fellowship for the Transfer of Knowledge "Harmonic Analysis, Nonlinear Analysis and Probability" MTKD-CT-2004-013389, the Polish Ministry of Science's research grants 1P03A01330 and N N201 270735

†e-mail: jwys@math.uni.wroc.pl

and if the array E is non-degenerate, then $(\mathcal{A}, u)$ is a compact matrix quantum group, where $u = (u_{jk})_{j,k=1}^N$. Woronowicz described the following example. For $\mu \in (0, 1]$, he defined

$$E_{i_1, \dots, i_N} = (-\mu)^{\text{inv}(\sigma)} \quad \text{if } \sigma = \begin{pmatrix} 1 & 2 & \dots & N \\ i_1 & i_2 & \dots & i_N \end{pmatrix} \in S_N$$

is a permutation (S_N denotes the set of permutations of $\{1, 2, \dots, N\}$) and $E_{i_1, \dots, i_N} = 0$ otherwise. Here, for a permutation $\sigma \in S_N$, $\text{inv}(\sigma)$ is the number of inversions of σ , which is the number of pairs (j, k) such that $j < k$ and $i_j = \sigma(j) > \sigma(k) = i_k$. Then as $(\mathcal{A}, u)$ one gets the quantum group $S_\mu U(N)$, called the *twisted* $SU(N)$ group.

In [W3] we considered another array E for $N = 3$, related to the number of cycles in a permutation. It was defined for a parameter $0 < q < 1$ as follows:

$$E(i, j, k) = \begin{cases} (-q)^{3-c(i,j,k)} & \text{if } \{i, j, k\} = \{1, 2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

Here $c(i, j, k)$ is the number of cycles of the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$$

(which makes sense if and only if $\{i, j, k\} = \{1, 2, 3\}$). Then, following the Woronowicz's scheme, we obtained a quantum group, which turned out to be $U_q(2)$, the quantum deformation of the unitary 2×2 group. Moreover, the construction provided a description of it as a *twisted product* of its quantum subgroups

$$U_q(2) = SU_q(2) \ltimes_\sigma U(1)$$

with the $*$ -isomorphism $\sigma : \mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow \mathcal{A}_2 \otimes \mathcal{A}_1$ given by

$$\sigma(1 \otimes v) = v \otimes 1, \quad \sigma(a \otimes v^k) = v^k \otimes a, \quad \sigma(c \otimes v^k) = v^{k-1} \otimes c.$$

The natural continuation of the construction given in [W3], was investigating the cases $N \geq 4$. However, as shall see below, after some tiresome computations it turned out that for $N = 4$ (and thus also for all $N \geq 4$) the quantum group we obtain (via the Woronowicz's theorem) is classical abelian.

Regarding the quantum group $U_q(2)$, we shall present also a construction of a cubic Hecke algebra. In [SLW3] Woronowicz showed that there are Hecke algebras associated with the quantum groups $SU_q(N)$, for every $N \in \mathbb{N}$, $N \geq 2$. The Hecke algebra $H_{q,n}$ described the intertwining operators for the n^{th} tensor power of the fundamental representation of the group. In this note we shall show similar construction for $U_q(2)$. The construction depends on defining an operator $\alpha : \mathbb{C}^3 \otimes \mathbb{C}^3 \mapsto \mathbb{C}^3 \otimes \mathbb{C}^3$, which satisfies the Yang-Baxter equation (3.1). The operator is not self-adjoint (contrary to the $SU_q(N)$ cases), although its square is so ($\alpha^2 = (\alpha^*)^2$). Nevertheless, it satisfies a generalization of the Hecke equation, namely $(\alpha^2 - I)(\alpha + q^2 I) = 0$ (see (4.1)). Therefore the operators $h_j := I_j \otimes \alpha \otimes I_{n-j-2}$, defined for $j = 1, \dots, n-2$, generate a *cubic Hecke algebra* (Theorem 4.3).

The paper is organized as follows. In Section 2 we give the computation showing the generalization of our $U_q(2)$ construction, for $N = 4$. Then, in Section 3, we give the construction of the operator α , and show that it satisfies the Yang-Baxter equation. The last Section 4, contains the construction of the cubic Hecke algebra, associated with $U_q(2)$. In particular, we show there that α satisfies the cubic equation.

2 The construction associated with E

Let $N_4 = \{(i, j, k, l) : \{i, j, k, l\} \subset \{1, 2, 3, 4\}\}$, let $E : N_4 \mapsto \mathbb{C}$ be zero outside $S_4 \subset N_4$, where the inclusion is given by $(i, j, k, l) \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix}$ if $\{i, j, k, l\} = \{1, 2, 3, 4\}$, and, for $0 < q < 1$, let the (non-zero) values of E (with the notation $E((i, j, k, l)) = E_{ijkl}$) be given by the function

$$S_4 \ni \sigma \mapsto (-q)^{4 - c(\sigma)}.$$

Explicitely, it can be written in the following way:

$$\begin{array}{llllll} E_{1234} = 1 & E_{1243} = -q & E_{1324} = -q & E_{1342} = q^2 & E_{1423} = q^2 & E_{1432} = -q \\ E_{2134} = -q & E_{2143} = q^2 & E_{2314} = q^2 & E_{2341} = -q^3 & E_{2413} = -q^3 & E_{2431} = q^2 \\ E_{3124} = q^2 & E_{3142} = -q^3 & E_{3214} = -q & E_{3241} = q^2 & E_{3412} = q^2 & E_{3421} = -q^3 \\ E_{4123} = -q^3 & E_{4132} = q^2 & E_{4213} = q^2 & E_{4231} = -q & E_{4312} = -q^3 & E_{4321} = q^2 \end{array} \quad (2.1)$$

The function $S_4 \ni \sigma \mapsto 4 - c(\sigma) = t(\sigma)$ counts the *number of transpositions* in σ . It follows from [SLW3], Theorem 4.1, that this way we obtain a compact quantum group $(\mathcal{A}, \mathbf{u})$, where $\mathcal{A}$ is the C^* -algebra generated by 16 matrix elements $\{u_{jk} : 1 \leq j, k \leq 4\}$ of $\mathbf{u}$, which satisfy the unitarity condition:

$$\sum_{r=1}^4 u_{jr}^* u_{rk} = \delta_{jk} I = \sum_{r=1}^4 u_{jr} u_{rk}^* \quad (2.2)$$

and the twisted determinant condition:

$$\sum_{i,j,k,l=1}^4 u_{\alpha i} u_{\beta j} u_{\gamma k} u_{\delta l} E_{ijkl} = E_{\alpha\beta\gamma\delta} I \quad (2.3)$$

for each $\{\alpha, \beta, \gamma, \delta\} \subset \{1, 2, 3, 4\}$. The matrix $\mathbf{u} = (u_{jk})_{j,k=1}^4$ is the fundamental unitary co-representation of the quantum group. In our case the co-representation $\mathbf{u} = (u_{kl})_{k,l=1}^4$ is reducible by the following reason. The operator $P = (E^* \otimes I)(I \otimes E)$, which acts on $\mathbb{C}^4$, intertwines the fundamental representation with itself: $(P \otimes I)\mathbf{u} = \mathbf{u}(P \otimes I)$. Moreover, P has a diagonal matrix for the standard basis of $\mathbb{C}^4$: $P = \text{diag}\{c_1, c_2, c_3, c_4\}$, with $c_j = \sum_{\alpha, \beta, \gamma} E_{j\alpha\beta\gamma} E_{\alpha\beta\gamma j}$, and therefore $c_1 = c_4 = -(5q^3 + q^5)$, $c_2 = c_3 = -(2q^3 + 4q^5)$. Hence, for $q \neq 0, -1, 1$, which shall be the case in the sequel, $c_1 \neq c_2$, so P is not a multiple of the identity operator I . The condition $(P \otimes I)\mathbf{u} = \mathbf{u}(P \otimes I)$ is equivalent to $c_j \cdot u_{jk} = c_k \cdot u_{jk}$ for all natural numbers $1 \leq j, k \leq 4$. This yields $u_{12} = u_{21} = 0, u_{13} = u_{31} = 0, u_{24} = u_{42} = 0, u_{34} = u_{43} = 0$, and therefore

$$\mathbf{u} = \begin{pmatrix} u_{11} & 0 & 0 & u_{14} \\ 0 & u_{22} & u_{23} & 0 \\ 0 & u_{32} & u_{33} & 0 \\ u_{41} & 0 & 0 & u_{44} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & x & y & 0 \\ 0 & z & w & 0 \\ c & 0 & 0 & d \end{pmatrix}. \quad (2.4)$$

This yields the decomposition of u decomposes into two irreducible subrepresentations

$$\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus \begin{pmatrix} x & y \\ z & w \end{pmatrix}. \quad (2.5)$$

Substitution in (2.3) of appropriate sequences $(\alpha, \beta, \gamma, \delta)$ gives the following relations between the generators of the C^* -algebra $\mathcal{A}$ (the associated sequence is left of the relation):

$$\begin{aligned} (1423) \quad I &= (ad - qbc)(xw - q^{-1}yz) & (1) \quad (4123) \quad I &= (da - q^{-1}cb)(xw - q^{-1}yz) & (2) \\ (1432) \quad I &= (ad - qbc)(wx - qzy) & (3) \quad (4132) \quad I &= (da - q^{-1}cb)(wx - qzy) & (4) \\ (2314) \quad I &= (xw - q^{-1}yz)(ad - qbc) & (5) \quad (2341) \quad I &= (xw - q^{-1}yz)(da - q^{-1}cb) & (6) \\ (3214) \quad I &= (wx - qzy)(ad - qbc) & (7) \quad (3241) \quad I &= (wx - qzy)(da - q^{-1}cb) & (8) \end{aligned}$$

Let $W = ad - qbc$ and $V = xw - q^{-1}yz$, then the above relation give $VW = I = WV$ and also $W = da - q^{-1}cb$, $V = wx - qzy$. Hence these relations are pairwise equivalent: $(1) \Leftrightarrow (5)$, $(2) \Leftrightarrow (6)$, $(3) \Leftrightarrow (7)$ and $(4) \Leftrightarrow (8)$. The operators V , W , being the inverse of each other, are twisted determinants for the two matrix co-representations:

$$W = \det_q \begin{pmatrix} a & b \\ c & d \end{pmatrix}, V = \det_{q^{-1}} \begin{pmatrix} x & y \\ z & w \end{pmatrix}. \quad (2.6)$$

Let us observe here that a change of order in the basis for $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ gives us the matrix $\begin{pmatrix} w & z \\ y & x \end{pmatrix}$ which satisfies the same relations and for which the twisted determinant is

$$\det_q \begin{pmatrix} w & z \\ y & x \end{pmatrix} = wx - qzy = V. \quad (2.7)$$

Using the invertibility of W and V one can easily get the following relations:

$$\begin{aligned} (1123) \quad ab &= qba & (9) \quad (2214) \quad cd &= qdc & (10) \\ (4423) \quad yx &= qxy & (11) \quad (3314) \quad wz &= qzw & (12) \end{aligned}$$

In addition, the relations (2.2) can be written as:

$$\begin{aligned} I &= aa^* + bb^* & (13) \quad I &= cc^* + dd^* & (14) \\ I &= a^*a + c^*c & (15) \quad I &= b^*b + d^*d & (16) \\ 0 &= a^*b + c^*d & (17) \quad 0 &= ca^* + db^* & (18) \end{aligned}$$

and

$$\begin{aligned} I &= xx^* + yy^* & (19) \quad I &= zz^* + ww^* & (20) \\ I &= x^*x + z^*z & (21) \quad I &= y^*y + w^*w & (22) \\ 0 &= x^*y + z^*w & (23) \quad 0 &= zx^* + wy^* & (24) \end{aligned}$$

Multiplication of (16) from the left by a^* and using (9) and then (17) gives the equation $d^*W = a$, or, equivalently, $d = V^*a^*$. On the other hand, multiplication (15) from the right by d and using (10) and (17) gives $d = a^*W$. These two combined ensure also that $W^*a = aV$. Similarly, by multiplying (16) from the right by c and using (10) and then (17) one gets $b^*W = -qc$, or equivalently, $b^* = -qcV$. Then, multiplying (15) from the right by b and using (9) and (17) one obtains $b = -qc^*W$. These two yield also $cV = W^*c$. Therefore we have

$$d = V^*a^* = a^*W, \quad b = -qV^*c^* = -qc^*W; \quad (2.8)$$

$$x = w^*V = W^*w^*, \quad z = -qy^*V = -qW^*y^*. \quad (2.9)$$

There are also other relations obtained from (2.3). They are listed in the following, with the associated sequences $(\alpha\beta\gamma\delta)$ on the left-hand side:

$$(2143) \quad I = x(ad - qbc)w - qy(ad - q^{-1}bc)z \quad (25)$$

$$(2413) \quad I = x(da - q^{-1}cb)w - q^{-1}y(da - qcb)z \quad (26)$$

$$(3142) \quad I = w(ad - q^{-1}bc)x - q^{-1}z(ad - qbc)y \quad (27)$$

$$(3412) \quad I = w(da - qcb)x - qz(da - q^{-1}cb)y \quad (28)$$

and

$$(1234) \quad I = a(xw - qyz)d - qb(xw - qyz)c \quad (29)$$

$$(4231) \quad I = d(xw - qyz)a - q^{-1}c(xw - qyz)b \quad (30)$$

$$(1324) \quad I = a(wx - q^{-1}zy)d - qb(wx - q^{-1}zy)c \quad (31)$$

$$(4321) \quad I = d(wx - q^{-1}zy)a - q^{-1}c(wx - q^{-1}zy)b \quad (32)$$

From now on we shall assume the following additional relation:

$$V = W^* \quad (2.10)$$

meaning that the twisted determinants are unitary operators. This yields that we are dealing with the quantum groups $U_q(2)$ (for the generators a, b, c, d) and another copy of $U_q(2)$ (for the generators w, y, z, x). This assumption is also necessary to allow the technical procedure used in [W3].

Let us substitute (2.8) into the (1) - (32). In (1) - (8) we do the substitution in one of the bracket and put V or V^* for the other. Thus for each equation we get two:

$$\begin{aligned} VaV^*a^* + q^2c^*c &= 1 \quad (1'a) & w^*VwV^* + yy^* &= 1 \quad (1'b) \\ a^*a + VcV^*c^* &= 1 \quad (2'a) & w^*VwV^* + yy^* &= 1 \quad (2'b) \\ VaV^*a^* + q^2c^*c &= 1 \quad (3'a) & ww^* + q^2y^*VyV^* &= 1 \quad (3'b) \\ a^*a + VcV^*c^* &= 1 \quad (4'a) & ww^* + q^2y^*VyV^* &= 1 \quad (4'b) \end{aligned} \quad (2.11)$$

We see that $(1'a) \Leftrightarrow (3'a)$, $(2'a) \Leftrightarrow (4'a)$, $(1'b) \Leftrightarrow (2'b)$ and $(3'b) \Leftrightarrow (4'b)$. For (9) - (12) we obtain:

$$\begin{aligned} cVa^* &= qa^*cV \quad (9') & aVc^* &= qc^*aV \quad (10') \\ yw^*V &= qw^*Vy \quad (11') & wy^*V &= qy^*Vw \quad (12') \end{aligned} \quad (2.12)$$

The relation (13) - (18) give:

$$\begin{aligned} aa^* + q^2V^*c^*cV &= 1 \quad (13') & cc^* + V^*a^*aV &= 1 \quad (14') \\ a^*a + c^*c &= 1 \quad (15') & aa^* + q^2cc^* &= 1 \quad (16') \\ aVc &= qcVa \quad (17') & Vca^* &= qa^*cV \quad (18') \end{aligned} \quad (2.13)$$

and for (19) - (24) we get:

$$\begin{aligned} w^*w + yy^* &= 1 \quad (19') & ww^* + q^2y^*y &= 1 \quad (20') \\ ww^* + q^2yy^* &= 1 \quad (21') & w^*w + y^*y &= 1 \quad (22') \\ wy &= qyw \quad (23') & wy^* &= qy^*w \quad (24') \end{aligned} \quad (2.14)$$

Let us first deal with the relations (2.14) involving w and y . Comparing (19') with (21') one gets easily that y is normal: $yy^* = y^*y$. Comparing (3'b) with (20') gives

$$y^*Vy = y^*yV \quad (2.15)$$

and (1'b) with (19') yield

$$w^*Vw = w^*wV. \quad (2.16)$$

Putting (24') into (11') gives

$$w^*yV = w^*Vy. \quad (2.17)$$

Multiplying both sides of (2.16) this from the left by w provides $ww^*yV = ww^*Vy$. Similarly, multiplying (2.14) from the right by y gives $yy^*Vy = yy^*yV$. Adding these two side by side yields

$$Vy = yV. \quad (2.18)$$

In a similar manner one gets

$$Vw = wV. \quad (2.19)$$

This requires putting (24') into (12') to get $y^*wV = y^*Vw$ which is then multiplied from the left by q^2y and added side by side to $ww^*Vw = ww^*wV$, which is obtained from (2.15). These can be collected together as the following relations:

$$\begin{aligned} w^*w + y^*y &= 1 & ww^* + q^2yy^* &= 1 \\ wy &= qyw & wy^* &= qy^*w \\ yy^* &= y^*y \\ wV &= Vw & yV &= Vy \end{aligned} \quad (2.20)$$

The fundamental co-representation is thus $\begin{pmatrix} w^*V & y \\ -qy^*V & w \end{pmatrix}$ and the above relations define the C^* -algebra of $U_q(2)$, and V is the $(-q)^{-1}$ -determinant.

Let us now work with the relations for a and c . From (4') and (15') one deduces that $cVc^* = c^*cV$. Then, multiplying (9') from the right by a one gets $cVaa^* = qa^*cVa$. The left-hand side of this can be transformed as follows (using (15')):

$$cVaa^* = cV(1 - c^*c) = cV - (cVc^*)c = cV - c^*cVc.$$

For the right-hand side one can use (17') and then (15') to get:

$$qa^*cVa = a^*aVc = (1 - cc^*)Vc = Vc - c^*cVc.$$

It follows from these two that $cV = Vc$, and also $c^*V = Vc^*$, since V is unitary. Using this combined with (14') and (15') one obtains $cc^* = c^*c$, so c is normal. Then from (10') follows $ac^* = qc^*a$. Comparing (1'a) with (16') one concludes $aVa^* = aa^*V$. Then, multiplication of (17') by c^* from the right gives $aVcc^* = qcVac^*$. The left-hand side of this is $aV - aa^*Va$. The right-hand side of this can be transformed, with the help of the above relations, into:

$$qcVac^* = q^2cVc^*a = q^2c^*cVa = Va - aa^*Va.$$

Hence one concludes $aV = Va$, and also $a^*V = Va^*$. Therefore the above relations may be written as follows:

$$\begin{aligned} a^*a + c^*c &= 1 & aa^* + q^2cc^* &= 1 \\ ac &= qca & ac^* &= qc^*a \\ cc^* &= c^*c \\ aV &= Va & cV &= Vc \end{aligned} \quad (2.21)$$

For $N = 4$ we have more nontrivial relations between a, c, w, y given by (2.3) then in the case $N = 3$, since, for example the sequence $(1, 1, 2, 2)$ gives a nontrivial relation here, and gave trivial relation there. Let us write them as follows, indicating the associated sequence $(\alpha, \beta, \gamma, \delta)$ on the left-hand side of it and successive numbering on the right-hand side of it. In the first set of equations we put elements from the same C^* -subalgebra outside, and the other inside.

$$\begin{aligned} (1231) \quad a(xw - qyz)b &= qb(xw - qyz)a & (33) \\ (1321) \quad a(wx - \frac{1}{q}zy)b &= qb(wx - \frac{1}{q}zy)a & (34) \\ (4234) \quad c(xw - qyz)d &= qd(xw - qyz)c & (35) \\ (4324) \quad c(wx - \frac{1}{q}zy)d &= qd(wx - \frac{1}{q}zy)c & (36) \\ (2142) \quad x(ad - qbc)y &= qy(ad - \frac{1}{q}bc)x & (37) \\ (2412) \quad y(da - qcb)x &= qx(da - \frac{1}{q}cb)y & (38) \\ (3143) \quad z(ad - qbc)w &= qw(ad - \frac{1}{q}bc)z & (39) \\ (3413) \quad w(da - qcb)z &= qz(da - \frac{1}{q}cb)w & (40) \\ (1224) \quad axyd &= qbxyc, \quad ayxd = qbyxc & (41) \\ (4221) \quad cxyb &= qdxya, \quad cyxb = qdyxa & (42) \\ (1334) \quad azwd &= qbzwc, \quad awzd = qbwzc & (43) \\ (4331) \quad czwb &= qdzwa, \quad cwzb = qdwza & (44) \\ (2113) \quad yabz &= 0 = ybaz & (45) \\ (3112) \quad wabx &= 0 = wbar & (46) \\ (2443) \quad ydcz &= 0 = ycdz & (47) \\ (3442) \quad wdcx &= 0 = wcdx & (48) \end{aligned} \quad (2.22)$$

In the second set of equations we have alternating sequences of elements from different C^* -subalgebras.

$$\begin{aligned} (1243) \quad axdw - qaydz - qbxcw + q^2bycz &= I & (49) \\ (4213) \quad dxaw - qdyaz - \frac{1}{q}cxbw + cybz &= I & (50) \\ (1342) \quad awdx - \frac{1}{q}azdy - qbwcx + bzcy &= I & (51) \\ (4312) \quad dwa x - \frac{1}{q}dzay - \frac{1}{q}cwbx + \frac{1}{q^2}czby &= I & (52) \\ (2134) \quad xawd - qxbwc - qyazd + q^2ybzc &= I & (53) \\ (3124) \quad waxd - qwbxc - \frac{1}{q}zayd + zbyc &= I & (54) \\ (2431) \quad xdwa - \frac{1}{q}xcwb - qydza + yczb &= I & (55) \\ (3421) \quad wdx a - \frac{1}{q}wcxb - \frac{1}{q}zdya + \frac{1}{q^2}zcyb &= I & (56) \end{aligned} \quad (2.23)$$

Computing

$$\begin{aligned} xw - qyz &= V - (1 - q^2)yy^*V \\ wx - \frac{1}{q}zy &= V + (1 - q^2)yy^*V \\ ad - \frac{1}{q}bc &= V^* + (1 - q^2)cc^*V^* \\ da - qcb &= V^* - (1 - q^2)cc^*V^* \end{aligned} \quad (2.24)$$

and substituting these into (2.22) one obtains

$$\begin{aligned}
a y y^* c^* &= q c^* y y^* a & (33'), (34') \\
c y y^* a^* &= q a^* y y^* c & (35'), (36') \\
w^* y &= q y c c^* w^* & (37'), (39') \\
y c c^* w^* &= 0 & (38') \\
w c c^* y^* &= 0 & (40') \\
a w^* y a^* + q^2 c^* w^* y c &= 0 & (41'), (43') \\
a^* w^* y a + c w^* y c^* &= 0 & (42'), (44') \\
y a c^* y^* &= 0 & (45') \\
w a c^* w^* &= 0 & (46') \\
y a^* c y^* &= 0 & (47') \\
w a^* c w^* &= 0 & (48')
\end{aligned} \tag{2.25}$$

Unfortunately, (37') combined with (38') give

$$w^* y = 0$$

and it follows from (2.20) that $y = 0$. To see this let us observe that $w w^* y y^* + q^2 y y^* y y^* = y y^*$ implies $q^2 (y y^*)^2 = y y^*$, and hence, by induction, $q^{2n} (y y^*)^{n+1} = y y^*$ for any positive integer $n \in \mathbb{N}$. This yields that the spectral radius $r(y y^*) = \lim_n \|(y y^*)^n\|^{\frac{1}{n}}$ satisfies $r(y y^*) = q^{-2} > 1$. However, it follows from the description of the irreducible representations of the relations (2.20) (see [W3]) that $\|y\| \leq 1$, so that $r(y y^*) \leq 1$. This is a contradiction, except $y = 0$.

Then $xw = V = wx$ and $xx^* = 1 = x^*x$, $ww^* = 1 = w^*w$, so that x, w are unitary. Moreover $x = w^*V$, so that for the fundamental co-representation eventually we get $\begin{pmatrix} w^*V & 0 \\ 0 & w \end{pmatrix}$. In a similar manner one gets that

$$a^*c = 0$$

and hence $c = 0$. Substitution of these to (2.23) gives

$$a w a^* w^* = 1 = a^* w^* a w.$$

If we set $t := aw$ and $s := wa$, then $tt^* = 1 = t^*t$, $ss^* = 1 = s^*s$ and $ts^* = 1 = s^*t$. Therefore $t = s$, which gives $aw = wa$.

These computations show that the C^* -algebra of the constructed quantum group is generated by three commuting unitaries a, w, V , so it is isomorphic to $C(\mathbb{T}) \otimes C(\mathbb{T}) \otimes C(\mathbb{T})$. Therefore, the quantum group we consider is in fact the classical group $U(1) \times U(1) \times U(1)$.

3 The Yang-Baxter operator associated with $U_q(2)$

In the next two Sections we are going to show a construction of a cubic Hecke algebra associated with the quantum group $U_q(2)$. In [W3] we gave a construction of the quantum group $U_q(2)$, in which the crucial role is played by the function counting the number of cycles in permutations from the symmetric group S_3 . Namely, by considering the function $S_3 \ni \sigma \mapsto (-q)^{3 - c(\sigma)}$, where $c(\sigma)$ is the number of cycles and $q > 0$, we constructed the following array:

$$\begin{aligned}
E_{1,2,3} &= 1 & E_{1,3,2} &= E_{2,1,3} = E_{3,2,1} = -q \\
E_{2,3,1} &= E_{3,1,2} = q^2 & E_{i,j,k} &= 0 \text{ if } \{i, j, k\} \subsetneq \{1, 2, 3\}
\end{aligned}$$

This array defines an operator ρ on $\mathbb{C}^3 \otimes \mathbb{C}^3$ by

$$\rho : \mathbb{C}^3 \otimes \mathbb{C}^3 \ni (a, b) \mapsto \sum_{i,j,k=1}^3 E_{i,j,k} E_{k,a,b}(i, j) \in \mathbb{C}^3 \otimes \mathbb{C}^3, \quad (3.26)$$

where (a, b) denotes in short the standard basis element $\varepsilon_a \otimes \varepsilon_b$. In particular $\varepsilon_1 = (1, 0, 0)$, $\varepsilon_2 = (0, 1, 0)$ and $\varepsilon_3 = (0, 0, 1)$.

The definition of E implies that (3.26) simplifies to

$$\rho(a, b) = E_{a,b,k} E_{k,a,b}(a, b) + E_{b,a,k} E_{k,a,b}(b, a), \quad \text{where } \{a, b, k\} = \{1, 2, 3\} \quad (3.27)$$

for $a \neq b$ and $a, b = 1, 2, 3$. If $a = b$ then we get $\rho(a, a) = 0$. The formulas can be written explicitly as follows.

$$\begin{aligned} \rho(1, 2) &= E_{1,2,3} E_{3,1,2}(1, 2) + E_{2,1,3} E_{3,1,2}(2, 1) = q^2(1, 2) + q^3(2, 1) \\ \rho(2, 1) &= E_{2,1,3} E_{3,2,1}(2, 1) + E_{1,2,3} E_{3,2,1}(1, 2) = q^2(2, 1) + q(1, 2) \\ \rho(1, 3) &= E_{1,3,2} E_{2,1,3}(1, 3) + E_{3,1,2} E_{2,1,3}(3, 1) = q^2(1, 3) + q^3(3, 1) \\ \rho(3, 1) &= E_{3,1,2} E_{2,3,1}(3, 1) + E_{1,3,2} E_{2,3,1}(1, 3) = q^4(3, 1) + q^3(1, 3) \\ \rho(2, 3) &= E_{2,3,1} E_{1,2,3}(2, 3) + E_{3,2,1} E_{1,2,3}(3, 2) = q^2(2, 3) + q(3, 2) \\ \rho(3, 2) &= E_{3,2,1} E_{1,3,2}(3, 2) + E_{2,3,1} E_{1,3,2}(2, 3) = q^2(3, 2) + q^3(2, 3) \end{aligned}$$

Therefore, the operator $\alpha := I_2 - \frac{1}{q^2} \rho$ acts as: $\alpha(a, a) = (a, a)$ for $a = 1, 2, 3$ and

$$\begin{aligned} \alpha(1, 2) &= -q(2, 1) \\ \alpha(1, 3) &= -q(3, 1) \\ \alpha(3, 2) &= -q(2, 3) \\ \alpha(2, 1) &= -q^{-1}(1, 2) \\ \alpha(2, 3) &= -q^{-1}(3, 2) \\ \alpha(3, 1) &= (1 - q^2)(3, 1) - q(1, 3) \end{aligned} \quad (3.28)$$

This operator is not self-adjoint, but $\alpha^2 = (\alpha^2)^*$ is so, since

$$\begin{aligned} \alpha^2(1, 2) &= (2, 1) \\ \alpha^2(2, 1) &= (2, 1) \\ \alpha^2(2, 3) &= (3, 2) \\ \alpha^2(3, 2) &= (2, 3) \\ \alpha^2(1, 3) &= q^2(1, 3) - q(1 - q^2)(3, 1) \\ \alpha^2(3, 1) &= (1 - q^2 + q^4)(3, 1) - q(1 - q^2)(1, 3) \end{aligned} \quad (3.29)$$

The first important property of α is that it is a Yang-Baxter operator.

Proposition 3.1 *The operator α satisfies the Yang-Baxter equation*

$$(\alpha \otimes I)(I \otimes \alpha)(\alpha \otimes I) = (I \otimes \alpha)(\alpha \otimes I)(I \otimes \alpha). \quad (3.30)$$

Proof: Let $L = (\alpha \otimes I)(I \otimes \alpha)(\alpha \otimes I)$ be the left-hand side and $P = (I \otimes \alpha)(\alpha \otimes I)(I \otimes \alpha)$ be the right-hand side of (3.30). We have to show that $L(a, b, c) = P(a, b, c)$ for every $a, b, c \in \{1, 2, 3\}$ (with the notation: $(a, b, c) = \varepsilon_a \otimes \varepsilon_b \otimes \varepsilon_c$). This requires checking 27 cases. It is clear that $L(a, a, a) = (a, a, a) = P(a, a, a)$ for any $a = 1, 2, 3$. The direct calculation provides the following formulas for the other cases.

$$\begin{aligned}
L(3, 2, 3) &= (3, 2, 3) = P(3, 2, 3) \\
L(2, 3, 2) &= (2, 3, 2) = P(2, 3, 2) \\
L(1, 2, 1) &= (1, 2, 1) = P(1, 2, 1) \\
L(2, 1, 2) &= (2, 1, 2) = P(2, 1, 2) \\
L(1, 2, 3) &= -q(3, 2, 1) = P(1, 2, 3) \\
L(1, 3, 2) &= -q^3(2, 3, 1) = P(1, 3, 2) \\
L(2, 1, 3) &= -q^{-1}(3, 1, 2) = P(3, 1, 2) \\
L(3, 3, 2) &= q^2(2, 3, 3) = P(3, 3, 2) \\
L(2, 2, 3) &= q^2(3, 2, 2) = P(2, 2, 3) \\
L(3, 2, 2) &= q^2(2, 2, 3) = P(3, 2, 2) \\
L(1, 1, 3) &= q^2(3, 1, 1) = P(1, 1, 3) \\
L(1, 3, 3) &= q^2(3, 3, 1) = P(1, 3, 3) \\
L(1, 1, 2) &= q^2(2, 1, 1) = P(1, 1, 2) \\
L(1, 2, 2) &= q^2(2, 2, 1) = P(1, 2, 2) \\
L(2, 3, 3) &= q^{-2}(3, 3, 2) = P(2, 3, 3) \\
L(2, 1, 1) &= q^{-2}(1, 1, 2) = P(2, 1, 1) \\
L(2, 2, 1) &= q^{-2}(1, 2, 2) = P(2, 2, 1)
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
L(3, 2, 1) &= (1 - q^2)(3, 2, 1) - q(1, 2, 3) = P(3, 2, 1) \\
L(3, 1, 2) &= q^2(1 - q^2)(2, 3, 1) - q^3(2, 1, 3) = P(3, 1, 2) \\
L(2, 3, 1) &= q^{-2}(1 - q^2)(3, 1, 2) - q^{-1}(1, 3, 2) = P(2, 3, 1) \\
L(1, 3, 1) &= -q(1 - q^2)(3, 1, 1) + q^2(1, 3, 1) = P(1, 3, 1) \\
L(3, 1, 3) &= -q(1 - q^2)(3, 3, 1) + q^2(3, 1, 3) = P(3, 1, 3)
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
L(3, 1, 1) &= (1 - q^2)(3, 1, 1) - q(1 - q^2)(1, 3, 1) + q^2(1, 1, 3) = P(3, 1, 1) \\
L(3, 3, 1) &= (1 - q^2)(3, 3, 1) - q(1 - q^2)(3, 1, 3) + q^2(1, 3, 3) = P(3, 3, 1)
\end{aligned} \tag{3.33}$$

From these formulas the Proposition follows. $\square$

4 The cubic Hecke algebra associated with $U_q(2)$

The second important property of the operator α is that, even though it is not a Hecke operator, it does satisfy a cubic equation, and thus it generates a *cubic Hecke algebra*. This notion has been introduced by Funar in [F], where the cubic equation $\alpha^3 - I = 0$ was considered.

Proposition 4.1 *The operator α satisfies the cubic equation:*

$$(\alpha^2 - I)(\alpha + q^2 I) = 0. \tag{4.34}$$

Proof: From the formulas (3.28), defining α it follows that it acts on the following subspaces by simple matricial formulas.

1. On the span of $(1, 2), (2, 1)$ as $\beta := \begin{pmatrix} 0 & \frac{-1}{q} \\ -q & 0 \end{pmatrix}$
2. On the span of $(2, 3), (3, 2)$ as $\beta^* := \begin{pmatrix} 0 & -q \\ \frac{-1}{q} & 0 \end{pmatrix}$
3. On the span of $(1, 3), (3, 1)$ as $\gamma := \begin{pmatrix} 0 & -q \\ -q & 1 - q^2 \end{pmatrix}$
4. As identity on every (a, a) with $a = 1, 2, 3$.

It is strightforward to see that $\beta^2 - I = 0 = (\beta^*)^2 - I$. On the other hand, since

$$\gamma^2 = \begin{pmatrix} q^2 & -q(1 - q^2) \\ -q(1 - q^2) & 1 - q^2 + q^4 \end{pmatrix},$$

we obtain

$$(\gamma^2 - I)(\gamma + q^2 I) = (q^2 - 1) \begin{pmatrix} 1 & q \\ q & q^2 \end{pmatrix} \begin{pmatrix} q^2 & -q \\ -q & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore both β and γ satisfy the equation (4.34), so the α does. $\square$

Let us define the elements

$$h_j := I_j \otimes \alpha \otimes I_{n-j-2} \quad \text{for } j = 1, \dots, n-2, \quad (4.35)$$

where I_k denotes the identity map on $(\mathbb{C}^N)^{\otimes k}$. Then by Propositions 3.1 and 4.1 the elements $h_1, \dots, h_n$ generate a cubic Hecke algebra, associated with the quantum group $U_q(2)$.

Definition 4.2 *The algebra $\mathcal{H}_{q,n}(2)$ generated by the elements h_j , $j = 1, 2, \dots, n$ defined by (4.35) will be called the **cubic Hecke algebra** associated with the quantum group $U_q(2)$.*

The basic properties of this algebra are summarized in the following.

Theorem 4.3 *The generators $\{h_j : 1 \leq j \leq n\}$ of $\mathcal{H}_{q,n}(2)$ satisfy:*

$$\begin{aligned} h_j h_{j+1} h_j &= h_{j+1} h_j h_{j+1} & \text{for } j = 1, \dots, n-1, \\ h_j h_k &= h_k h_j & \text{for } |j - k| \geq 2, \\ ((h_j)^2 - 1)(h_j + q^2) &= 0 & \text{for } j = 1, \dots, n. \end{aligned} \quad (4.36)$$

The role of the Hecke algebra in the study of $SU_q(N)$ was that it was the intertwining algebra of the tensor powers of the fundamental co-representation. In [W3] the irreducible co-representations of $U_q(2)$ have been described, but it is not clear if the description is complete. So, it is still to be checked whether $\mathcal{H}_{q,n}(2)$ plays the same role as in $SU_q(N)$.

References

- [F] L. FUNAR *On the quotients of cubic Hecke algebras*, Comm.Math.Phys. **173** (1995), 513–558.
- [K] H. T. KOELINK, *On $\ast$ -representations of the Hopf $\ast$ -algebra associated with the quantum group $U_q(n)$* , Compositio Math. **77** (1991), 199–231.
- [M-HR] F. MÜLLER-HOISSEN, C. REUTEN *Bicovariant differential calculi on $GL_{p,q}(2)$ and quantum subgroups*, J. Phys. A **26** (1993), 2955 – 2975.
- [PoM] P. PODLEŚ, E. MÜLLER, *Introduction to quantum groups*, Rev. Math. Phys. **10** (1998), no. 4, 511–551.
- [PoW] P. PODLEŚ, S.L. WORONOWICZ *Quantum deformation of Lorentz group*, Comm. Math. Phys. **130** (1990), 381–431.
- [PuW] W. PUSZ, S. L. WORONOWICZ, *Representations of quantum Lorentz group on Gelfand spaces*, Rev. Math. Phys. Vol. **12**, No. 12 (2000), 1551 – 1625.
- [Ta] M. TAKEUCHI, *A two-parameter quantization of $GL(n)$* , Proc. Japan Acad. Ser. A Math. Sci **66** no. 5 (1990), 112 – 114.
- [SLW1] S.L. WORONOWICZ, *Twisted $SU(2)$ group. An example of non-commutative differential calculus*, Publ. RIMS, Kyoto Univ. **23** (1987), 117–181.
- [SLW2] S.L. WORONOWICZ, *Compact Matrix Pseudogroups*, Comm. Math. Phys. **111** (1987), 613–665
- [SLW3] S.L. WORONOWICZ, *Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups*, Invent. Math. **93** (1988), 35–76.
- [SLW4] S.L. WORONOWICZ, *Compact quantum groups*, Symétries quantiques (Les Houches, Session LXIV), (1995), 845 – 884; in "Quantum symmetries" (A. Connes, K. Gawędzki, J. Zinn-Justin, Eds.) (1998) North-Holland, Amsterdam.
- [SLW5] S.L. WORONOWICZ, *From multiplicative unitaries to quantum groups*, Internat. J. Math. **7** (1996), 127 – 149.
- [W1] J. WYSOCZAŃSKI, *A construction of compact matrix quantum groups and description of the related C^* -algebras*, Infinite dimensional analysis and quantum probability theory (Japanese) (Kyoto, 2000). Su-rikaiseikikenkyu-sho Ko-kyu-roku No. 1227 (2001), 209–217.
- [W2] J. WYSOCZAŃSKI, *Unitary representations for twisted product of matrix quantum groups*, Trends in infinite-dimensional analysis and quantum probability (Japanese) (Kyoto, 2001). Su-rikaiseikikenkyu-sho Ko-kyu-roku No. 1278 (2002), 188–193.
- [W3] J. WYSOCZAŃSKI, *Twisted product structure and representation theory of the quantum group $U_q(2)$* , Reports on Math. Phys. **54** No.3 (2004), 327–347.