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JØRGENSEN'S INEQUALITY FOR COMPLEX HYPERBOLIC 2- SPACE

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1 Introduction

Jørgensen's inequality gives a necessary condition for non-elementary two generator group of isometries of hyperbolic space to be discrete. We give analogues of Jørgensen's inequality for non-elementary groups of isometries of complex hyperbolic 2-space generated by two elements, one of which is either loxodromic or boundary elliptic.

This is a joint work with Jiang Yueping (Hunan University) and John. R. Parker (University of Durham).

2 The classical Jørgensen's inequality

We discuss the original inequality of Jørgensen and reformulate in a way that we can generalize. Jørgensen takes two elements A and B in SL(2,C) and says that if

$$|tr^2(A)-4|+|tr(ABA^{-1}B^{-1})-2|<1,$$

then the group $<A,B>$ generated by A and B is either elementary or not discrete. In this paper we will only be concerned with the cases where A is loxodromic or elliptic. We may reformulate Jørgensen's inequality in terms of cross ratios of fixed points. Jørgensen's inequality is equivalent to

Theorem 1. *Let A and B be elements of* SL(2,C) *so that A is either loxodromic or elliptic with fixed points μ and ν in $\hat{\mathbf{C}}$. Let $M=|tr^2(A)-4|^{\frac{1}{2}}$. If either*

$$M^2(|[B(\mu),\nu,\mu,B(\nu)]|+1)<1 \quad or \quad M^2(|[B(\mu),\mu,\nu,B(\nu)]|+1)<1,$$

then the group $<A,B>$ is either elementary or not discrete.

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3 Preliminaries

Let $\mathbf{C}^{2,1}$ be a complex vector space of dimension 3 with the Hermitian form of signature (2,1). We choose the Hermitian form on $\mathbf{C}^{2,1}$ to be given by the matrix J

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Thus $< z, w >= w^* J z = z_1 \overline{w_3} + z_2 \overline{w_2} + z_3 \overline{w_1}$.

We define the Siegel domain model of complex 2-space, $\mathbf{H}^2_{\mathbf{C}}$ as follows: We identify points of $\mathbf{H}^2_{\mathbf{C}}$ with their horospherical coordinates, $z = (\zeta, v, u) \in \mathbf{C} \times \mathbf{R} \times \mathbf{R}_+ = \mathbf{H}^2_{\mathbf{C}}$. Similarly, points in $\partial\mathbf{H}^2_{\mathbf{C}} = \mathbf{C} \times \mathbf{R} \cup \{\infty\}$ are either $z = (\zeta, v, 0) \in \mathbf{C} \times \mathbf{R} \times \{0\}$ or a point at infnity, which is denoted by ∞. Define the map $\phi : \overline{\mathbf{H}^2_{\mathbf{C}}} \to \mathbf{PC}^{2,1}$ by

$$\begin{aligned} \phi &: (\zeta, v, u) \longmapsto [(-|\zeta|^2 - u + iv)/2, \zeta, 1]^t, \\ \phi &: \infty \longmapsto [1, 0, 0]^t. \end{aligned}$$

The map ϕ is a homeomorphism from $\mathbf{H}^2_{\mathbf{C}}$ to the set of points z in $\mathbf{PC}^{2,1}$ with $< z, z >< 0$. Also ϕ is a homeomorphism from $\partial\mathbf{H}^2_{\mathbf{C}}$ to the set of points z in $\mathbf{PC}^{2,1}$ with $< z, z >= 0$. Let L be a complex line intersecting $\mathbf{H}^2_{\mathbf{C}}$. Then $\phi(L)$ is a 2-dimensional subspace of $\mathbf{C}^{2,1}$. The orthogonal complement of this space is a one (complex) dimensional subspace of $\mathbf{C}^{2,1}$ spanned by a vector p with $< p, p >> 0$. Without loss of generality, we take $< p, p >= 1$ and call p the polar vector corresponding to the complex line L.

The Bergman metric on $\mathbf{H}^2_{\mathbf{C}}$ is defined by the following formula for the distance ρ between points z and w of $\mathbf{H}^2_{\mathbf{C}}$:

$$\cosh\left(\frac{\rho(z, w)}{2}\right) = \frac{< \phi(z), \phi(w) >< \phi(w), \phi(z) >}{< \phi(z), \phi(z) >< \phi(w), \phi(w) >}.$$

The holomorphic isometry group of $\mathbf{H}^2_{\mathbf{C}}$ with respect to the Bergman metric is the projective unitary group $PU(2,1)$ and acts on $\mathbf{PC}^{2,1}$ by matrix multiplication. A matrix $g \in \mathbf{GL}(3, \mathbf{C})$ is in $PU(2,1)$ if and only if it preserves the Hermitian form given by J. For four distinct points z_1, z_2, w_1, w_2 of $\overline{\mathbf{H}^2_{\mathbf{C}}}$ the cross-ratio is defined as

$$|[z_1, z_2, w_1, w_2]| = \left| \frac{< \phi(w_1), \phi(z_1) >< \phi(w_2), \phi(z_2) >}{< \phi(w_2), \phi(z_1) >< \phi(w_1), \phi(z_2) >} \right|.$$

In order to represent the holomorphic isometries of $\mathbf{H}^2_{\mathbf{C}}$, we work with the special unitary group $SU(2,1)$ throughout this paper.

4 Subgroups with loxodromic generators

We give our results about the subgroups with loxodromic elements. Let A be a loxodromic element with fixed points μ and ν in $\partial\mathbf{H}^2_{\mathbb{C}}$. Suppose that A has a complex dilation factor $\lambda(A)$. We define a conjugation invariant factor M by

$$M = |\lambda(A) - 1| + |\lambda(A)^{-1} - 1|.$$

Theorem 2. *Let A be a loxodromic element of $SU(2,1)$ fixing μ and ν, and let B be any element of $SU(2,1)$. If either*

$$M(|[B(\mu), \nu, \mu, B(\nu)]|^{1/2} + 1) < 1 \quad \text{or} \quad M(|[B(\mu), \mu, \nu, B(\nu)]|^{1/2} + 1) < 1,$$

then the group $< A, B >$ is either elementary or not discrete.

Theorem 3. *Let A be a loxodromic element of $SU(2,1)$ fixing μ and ν, and let B be any element of $SU(2,1)$. If $M \leq \sqrt{2} - 1$ and*

$$|[B(\mu), \nu, \mu, B(\nu)]| + |[B(\mu), \mu, \nu, B(\nu)]| < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},$$

then the group $< A, B >$ is either elementary or not discrete.

We can show that neither theorem is a consequence of the other one.

5 Subgroups with boundary elliptic elements

Let A be a boundary elliptic element of $SU(2,1)$. Then A fixes a complex line in $\mathbf{H}^2_{\mathbb{C}}$. We denote this complex line by L_A and its polar vector by p_A. The fixed complex line of BAB^{-1} is $B(L_A)$, which has the polar vector $B(p_A)$. Normalizing p_A and $B(p_A)$ so that $< p_A, p_A > = < B(p_A), B(p_A) > = 1$, we have three cases:

(1) If $| < p_A, B(p_A) > | < 1$, then L_A and $B(L_A)$ intersect at a point in $\mathbf{H}^2_{\mathbb{C}}$. Moreover, $| < p_A, B(p_A) > | = \cos\psi$, where ψ is the angle of intersection between L_A and $B(L_A)$. In particular, if $| < p_A, B(p_A) > | = 0$, then L_A and $B(L_A)$ intersect orthogonally.

(2) If $| < p_A, B(p_A) > | = 1$, then either $B(L_A) = L_A$ or L_A and $B(L_A)$ are asymptotic at at a point in $\partial\mathbf{H}^2_{\mathbb{C}}$.

(3) If $| < p_A, B(p_A) > | > 1$, then L_A and $B(L_A)$ are ultraparallel, that is, they are disjoint and have a common orthogonal complex geodesic. Moreover, $| < p_A, B(p_A) > | = \cosh\frac{\rho}{2}$, where ρ is the distance between L_A and $B(L_A)$.

For a boundary elliptic element $A \in SU(2,1)$ we define the order of A as

$$ord(A) = \inf\{\mathrm{m} > 0 : \mathrm{A}^{\mathrm{m}} = \mathrm{I}\}.$$

Theorem 4. *Let A be a boundary elliptic element of $SU(2,1)$ which rotates through an angle $\theta = 2\pi/n$ about a complex line L_A. Let B be any element of $SU(2,1)$ so that $B(L_A) \neq L_A$. If one of the following three conditions (1), (2) and (3) is satisfied, then the group $< A, B >$ is not discrete.*

(1) L_A and $B(L_A)$ intersect at an angle $\psi \neq \pi/2$ and $ord(A) = n \geq 6$.

(2) L_A and $B(L_A)$ are asymptotic and $ord(A) = n \geq 7$.

(3) L_A and $B(L_A)$ are ultraparallel and

$$|\cosh\frac{\rho}{2}\sin\frac{\theta}{2}| < \frac{1}{2},$$

where ρ is the distance between L_A and $B(L_A)$.

If L_A and $B(L_A)$ intersect orthogonally and

$$|tr(B)\sin\frac{\theta}{2}| < \frac{1}{2},$$

then the group $< A, B >$ is either elementary or not discrete.

Theorem 5. *Let A be a boundary elliptic element fixing the complex line L_A spanned by μ and ν in $\partial\mathbf{H}^2_{\mathbf{C}}$. Suppose that B is any element of $SU(2,1)$ for which L_A and $B(L_A)$ do not intersect orthogonally. If either*

$$M(|[B(\mu), \nu, \mu, B(\nu)]|^{1/2} + 1) < 1 \quad \text{or} \quad M(|[B(\mu), \mu, \nu, B(\nu)]|^{1/2} + 1) < 1,$$

then the group $< A, B >$ is either elementary or not discrete.

Theorem 6. *Let A be a boundary elliptic element fixing the complex line L_A spanned by μ and ν in $\partial\mathbf{H}^2_{\mathbf{C}}$. Suppose that B is any element of $SU(2,1)$ for which L_A and $B(L_A)$ do not intersect orthogonally. If $M \leq \sqrt{2} - 1$ and*

$$|[B(\mu), \nu, \mu, B(\nu)]| + |[B(\mu), \mu, \nu, B(\nu)]| < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},$$

then the group $< A, B >$ is either elementary or not discrete.

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