

ON UNIVALENCE CRITERIA FOR MEROMORPHIC FUNCTIONS

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ABSTRACT. We propose a way of deduction of various univalence criteria for meromorphic functions on the outside of the unit circle in terms of the range of their derivatives. This is a summary of the forthcoming joint paper [15] of S. Ponnusamy and the author.

1. INTRODUCTION

Let $\mathcal{A}$ denote the set of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized so that $f(0) = 0$ and $f'(0) = 1$. The set $\mathcal{S}$ of univalent functions in $\mathcal{A}$ has been intensively studied by many authors. Let Σ denote the set of univalent functions F in the domain $\Delta = \{\zeta : |\zeta| > 1\}$ of the form

$$(1.1) \quad F(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}.$$

Note that the function $1/f(1/\zeta)$ belongs to Σ for each $f \in \mathcal{S}$. The converse is, however, not true in general. More precisely, for $F \in \Sigma$, the function $f(z) = 1/F(1/z)$ belongs to $\mathcal{S}$ if and only if F omits 0, namely, $F(\zeta) \neq 0$ for $\zeta \in \Delta$.

In parallel with the analytic case, we consider the set $\mathcal{M}$ of meromorphic functions in Δ with the expansion (1.1) around $\zeta = \infty$. For some technical reason, we also consider the sets $\mathcal{A}_n = \{f \in \mathcal{A} : f^{(m)}(0) = 0 \text{ for } m = 2, \dots, n\}$ and $\mathcal{M}_n = \{F \in \mathcal{M} : b_0 = \dots = b_n = 0\}$. Note that $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{M}_{-1} = \mathcal{M}$.

Practically, it is an important problem to determine univalence of a given function in $\mathcal{A}_n$ or in $\mathcal{M}_n$. The best known conditions for univalence are probably those involving pre-Schwarzian or Schwarzian derivatives, which are defined by

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2.$$

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We define quantities for functions $f \in \mathcal{A}$ and $F \in \mathcal{M}$ by

$$\begin{aligned} B(f) &= \sup_{|z|<1} (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right|, \\ B(F) &= \sup_{|\zeta|>1} (|\zeta|^2 - 1) \left| \frac{\zeta F''(\zeta)}{F'(\zeta)} \right|, \\ N(f) &= \sup_{|z|<1} (1 - |z|^2)^2 |S_f(z)|, \\ N(F) &= \sup_{|\zeta|>1} (|\zeta|^2 - 1)^2 |S_F(\zeta)|. \end{aligned}$$

Note that these quantities may take ∞ as their values. For example, if F has a pole at a finite point, then $B(F) = \infty$.

If $f \in \mathcal{A}$ and $F \in \mathcal{M}$ have the relation $f(z) = 1/F(1/z)$, then we can easily see that

$$(1 - |z|^2)^2 S_f(z) = (|\zeta|^2 - 1)^2 S_F(\zeta)$$

holds for $z = 1/\zeta$. In particular, we have $N(f) = N(F)$.

Theorem A (Nehari [14]). *Every $f \in \mathcal{S}$ satisfies $N(f) \leq 6$. Conversely, if $f \in \mathcal{A}$ satisfies $N(f) \leq 2$ then f must be univalent. The constants 6 and 2 are best possible. The same is true for meromorphic F .*

Though $zf'(z)/f(z) = \zeta F'(\zeta)/F(\zeta)$, there is no such a simple relation between $zf''(z)/f'(z)$ and $\zeta F''(\zeta)/F'(\zeta)$, and thus, between $B(f)$ and $B(F)$ for $f(z) = 1/F(1/z)$, $\zeta = 1/z$. Nevertheless, it is rather surprising that the formally same conclusions can be deduced for f and F . Compare Theorem B with Theorem C.

Theorem B. *Every $f \in \mathcal{S}$ satisfies $B(f) \leq 6$. Conversely, if $f \in \mathcal{A}$ satisfies $B(f) \leq 1$ then $f \in \mathcal{S}$. Moreover, if $B(f) \leq k < 1$, then f extends to a k -quasiconformal mapping of the extended plane. The constants 6 and 1 are best possible.*

Here and hereafter, a quasiconformal mapping g is called k -quasiconformal if its Beltrami coefficient $\mu = g_{\bar{z}}/g_z$ satisfies $\|\mu\|_\infty \leq k$.

The sufficiency of univalence and quasiconformal extendibility are due to Becker [6]. The sharpness of the constant 1 is due to Becker and Pommerenke [8]. The sharp inequality $B(f) \leq 6$ follows from a standard argument in the coefficient estimation (see, e.g., [9, Theorem 2.4]).

Theorem C. *Every $F \in \Sigma$ satisfies $B(F) \leq 6$. Conversely, if $F \in \mathcal{M}$ satisfies $B(F) \leq 1$ then $F \in \Sigma$. Moreover, if $B(F) \leq k < 1$, then F extends to a k -quasiconformal mapping of the extended plane. The constants 6 and 1 are best possible.*

The sufficiency of univalence and quasiconformal extendibility are due to Becker [7]. The sharpness of the constant 1 is also due to Becker and Pommerenke [8]. On the other hand, the estimate $B(F) \leq 6$ lies deeper. Avhadiev [3] first showed the sharp inequality $B(F) \leq 6$ by appealing to Goluzin's inequality (see [10, p. 139]).

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Note that many authors use a different norm for the pre-Schwarzian derivative of $f \in \mathcal{A}$, namely,

$$\|T_f\| = \sup_{|z|<1} (1 - |z|^2) |T_f(z)|.$$

By definition, we observe $B(f) \leq \|T_f\|$.

Recall that a plane domain $\Omega \subset \mathbb{C}$ is called *hyperbolic* if $\partial\Omega$ contains at least two points. Let Ω be a hyperbolic plain domain such that $1 \in \Omega$ but $0 \notin \Omega$ and set

$$\Pi(\Omega) = \{F \in \mathcal{M} : F'(\zeta) \in \Omega \text{ for all } \zeta \in \Delta\}.$$

Set also $\Pi_n(\Omega) = \Pi(\Omega) \cap \mathcal{M}_n$ for $n = -1, 0, 2, \dots$. One of our main results in the present paper is an estimate of $B(F)$ for $F \in \Pi(\Omega)$. The proof is given in [15].

Theorem 1. *Let Ω be a domain such that $1 \in \Omega$ but $0 \notin \Omega$. For every $F \in \Pi_n(\Omega)$, $n \geq 0$, the inequality*

$$B(F) \leq C_n W(\Omega)$$

holds, where C_n is the constant given by

$$(1.2) \quad C_n = \sup_{0 < r < 1} \frac{(n+2)(1-r^2)r^n}{1-r^{2n+4}}$$

and $W(\Omega)$ is the circular width of Ω with respect to the origin, namely,

$$W(\Omega) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{p'(z)}{p(z)} \right|$$

for an analytic universal covering projection p of $\mathbb{D}$ onto Ω .

Note that $W(\Omega)$ does not depend on the particular choice of p . For more details on circular width, see [12]. As one sees easily, $C_0 = 2$ and $1 \leq C_n \leq (n+2)/(n+1)$. If we write $F \in \Pi(\Omega)$ in the form $F = F_0 + b_0$, where $F_0 \in \Pi_0(\Omega)$, the relation $B(F) = B(F_0)$ holds. Therefore, the above theorem can be applicable to the whole family $\Pi(\Omega)$. We note that the analytic counterpart of this theorem is known and much simpler to prove (see [11, Theorem 4.1]); $B(f) \leq \|T_f\| \leq W(\Omega)$ holds for $f \in \mathcal{A}$ with $f'(\mathbb{D}) \subset \Omega$.

As is well known, if $f \in \mathcal{A}$ satisfies $\operatorname{Re} f' > 0$ then f is necessarily univalent (cf. [9, Theorem 2.16]). However, the meromorphic counterpart does not hold (see, for instance, the example given in Section 3). The following univalence criterion is due to Aksent'ev [1] (see also [5, Theorem 11]). Later, Krzyż [13] gave quasiconformal extensions for the functions.

Theorem D (Aksent'ev, Krzyż). *Let $0 \leq k \leq 1$. If $F \in \mathcal{M}$ satisfies the inequality*

$$(1.3) \quad |F'(\zeta) - 1| \leq k, \quad |\zeta| > 1,$$

then F is univalent. Furthermore, if $k < 1$, then F extends to a k -quasiconformal mapping of the extended plane. The radii 1 and k are best possible.

Note that the range of F' cannot be enlarged to $\{w : |w - 1| < a\}$, $a > 1$, for univalence [2].

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2. EXAMPLES

The following examples can be found in [12].

Example 1 (sectors). For $S(\beta) = \{w : |\arg w| < \pi\beta/2\}$, $0 < \beta \leq 2$, we have $W(S(\beta)) = 2\beta$.

Example 2 (annuli). For the annulus $A(r, R) = \{w : r < |w| < R\}$, $0 < r < R < \infty$, we have $W(A(r, R)) = (2/\pi) \log(R/r)$.

Example 3 (disks). Let $\mathbb{D}(a, r) = \{w : |w - a| < r\}$ for $0 < r \leq a$. Then

$$W(\mathbb{D}(a, r)) = \frac{2r/a}{1 + \sqrt{1 - (r/a)^2}}.$$

Example 4 (parallel strips). Let $P(a, b) = \{w : a < \operatorname{Re} w < b\}$ for $0 \leq a < b < \infty$. Then

$$W(P(a, b)) = \max_{0 \leq \theta \leq \pi/2} \frac{2t \cos \theta}{1 - t\theta},$$

where t is a number with $0 < t \leq 2/\pi$ determined by

$$\frac{\pi t}{2} = \frac{b - a}{b + a}.$$

Example 5 (truncated wedges). Let $S(\beta, r, R) = \{w : |\arg w| < \pi\beta/2, r < |w| < R\}$, $0 < \beta \leq 2, 0 < r < R < \infty$. Then

$$W(S(\beta, r, R)) = \frac{\log(R/r)}{(1 + t)\mathcal{K}(t)},$$

where

$$\mathcal{K}(t) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - t^2x^2)}}$$

is the complete elliptic integral of the first kind and $0 < t < 1$ is a number such that

$$\frac{\mathcal{K}(\sqrt{1 - t^2})}{\mathcal{K}(t)} = \frac{2\pi\beta}{\log(R/r)}.$$

3. APPLICATIONS

We apply Theorem 1 and Theorem C to the above examples to obtain several results on univalence of meromorphic functions. As samples, we state a few theorems. Note that the univalence criteria in Theorems 2 and 3 were first given by Avhadiev and Aksent'ev [4].

Let $x_2 \approx 0.4198$ denote the unique zero of the equation

$$\sqrt{x} \log((1 + \sqrt{x})/(1 - \sqrt{x})) = 1$$

in $0 < x < 1$.

Theorem 2. Let $0 \leq k \leq 1$. Suppose that a function $F \in \mathcal{M}$ satisfies the condition

$$|\arg F'(\zeta)| \leq \frac{k\pi}{8}, \quad |\zeta| > 1,$$

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then F must be univalent. Furthermore, if $k < 1$, then F extends to a k -quasiconformal mapping of the extended plane. As for univalence, the constant $\pi/8$ cannot be replaced by any smaller number than $(4/\pi) \arctan x_2$.

Note that $(4/\pi) \arctan x_2 \approx 0.506057 \approx 1.28866(\pi/8)$. The number x_2 appears in the following example.

We consider the function $F_n \in \mathcal{M}$ given by

$$\begin{aligned} F_n(\zeta) &= \zeta - 2 \sum_{j=1}^{\infty} \frac{\zeta^{1-nj}}{nj-1} \\ &= \zeta \left({}_2F_1\left(1, -\frac{1}{n}; 1 - \frac{1}{n}; \zeta^{-n}\right) - 1 \right), \quad |\zeta| > 1, \end{aligned}$$

for each integer $n \geq 2$, where ${}_2F_1(a, b; c; x)$ stands for the hypergeometric function. Note that F_n has the n -fold symmetry

$$F_n(e^{2\pi i/n} \zeta) = e^{2\pi i/n} F_n(\zeta)$$

and belongs to the class $\mathcal{M}_{n-2}$. Since the function h_n defined by

$$h_n(x) = {}_2F_1\left(1, -\frac{1}{n}; 1 - \frac{1}{n}; x\right) - 1 \quad (x \in (0, 1))$$

has the properties that h_n is monotone decreasing, $h_n(0) = 1$ and $\lim_{x \rightarrow 1-} h_n(x) = -\infty$, there is the unique point x_n such that $h(x_n) = 0$ in the interval $0 < x < 1$. Hence, the function F_n has the n zeros $e^{2\pi i j/n} x_n^{-1/n}$, $j = 0, 1, \dots, n-1$, in Δ and, in particular, is not univalent in Δ . On the other hand, we have

$$F'_n(\zeta) = 1 + 2 \sum_{j=1}^{\infty} \zeta^{-nj} = p(\zeta^{-n}),$$

where $p(z)$ is the function given by $p(z) = (1+z)/(1-z)$. It is a standard fact that p maps the unit disk onto the right half-plane $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$. Therefore, F'_n maps Δ onto $\mathbb{H}$ in an n -to-1 way and thus $\operatorname{Re} F'_n > 0$ holds.

In the next criterion, F' may take values with negative real part.

Theorem 3. Let $0 \leq k \leq 1$. Suppose that a function $F \in \mathcal{M}$ satisfies the condition

$$|\log |F'(\zeta)|| \leq \frac{k\pi}{8}, \quad |\zeta| > 1,$$

then F must be univalent. Furthermore, if $k < 1$, then F extends to a k -quasiconformal mapping of the extended plane. As for univalence, the constant $\pi/8$ cannot be replaced by any smaller number than $\log((1+x_2)/(1-x_2))$.

Note that $\log((1+x_2)/(1-x_2)) \approx 0.894894 \approx 2.27883(\pi/8)$. In these results, if we assume F to be in $\mathcal{M}_n$ for larger n , then we can make the involved constants better.

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