

On orientations of fixed point sets of spin structure preserving involutions on manifolds.

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1 Introduction

Let X be an oriented connected closed smooth manifold of dimension n with $n \geq 4$, and F an embedded closed submanifold of codimension 2 with $[F]_2 = 0 \in H_{n-2}(X; \mathbf{Z}_2)$, where $[F]_2$ denote the homology class represented by F in X with coefficients in $\mathbf{Z}_2$. Then we have a double branched covering map $\tilde{X} \rightarrow X$ branched along F . In [2] and [3], we have obtained the following result;

Theorem 1.1 *Suppose that $H_1(X; \mathbf{Z}_2) = 0$. Then $\tilde{X}$ admits a spin structure if and only if F admits an orientation such that $[F]^\sharp \in H^2(X; \mathbf{Z})$ is twice a cohomology class of which reduction modulo 2 coincides with the second Stiefel-Whitney class. Here, $[F]^\sharp$ denote the Poincaré-dual of $[F]$.*

The assumption that $H_1(X; \mathbf{Z}_2) = 0$ is essential. As a generalization of the above theorem, we first obtain;

Theorem 1.2 *Let H be a connected closed surface smoothly embedded in X . Suppose that $H_1(X; \mathbf{Z}_2) = 0$, that $n = 4$, and that $[H]_m = 0 \in H_2(X; \mathbf{Z}_m)$, where $[H]_m$ denote the homology class represented by the oriented H . Then $\tilde{X}$ is spin if and only if $[F]^\sharp \in H^2(X; \mathbf{Z})$ is m times a cohomology class of which reduction modulo 2 coincides with the second Stiefel-Whitney class.*

Although Theorem 1.2 is the case for $n = 4$, it should hold for all positive integer n .

Next we have obtained the following ([3]) as an another generalization of Theorem 1.1;

Theorem 1.3 *$\tilde{X}$ admits a spin structure that is preserved by the covering transformation map $T : \tilde{X} \rightarrow \tilde{X}$ if and only if $[F]^\sharp \in H^2(X; \mathbf{Z})$ is twice a cohomology class of which reduction modulo 2 coincides with the second Stiefel-Whitney class.*

Suppose that F admits an orientation such that $[F]^\sharp = 2w \in H^2(X; \mathbf{Z})$ with $(w)_2 = w_2(X)$, where $(w)_2 \in H^2(X; \mathbf{Z}_2)$ denotes the reduction modulo 2. Then we have

$$H_1(X; \mathbf{Z}) \cong \oplus_{i=1}^n \mathbf{Z}_2 \oplus_{i=1}^{N_0} \mathbf{Z} \langle p_i \rangle \oplus_{i=1}^{N_1} \mathbf{Z}_{2^{r_i}} \langle q_i \rangle \oplus_{i=1}^{N_2} \mathbf{Z}_{k_i},$$

where $r_i \geq 2$ and k_i odd. Therefore we obtain

$$H_1(X; \mathbf{Z}_2) \cong \oplus_{i=1}^n \mathbf{Z}_2 \oplus_{i=1}^{N_0} \mathbf{Z}_2 \langle p_i \rangle \oplus_{i=1}^{N_1} \mathbf{Z}_2 \langle q_i \rangle$$

Then the 1-st homology group $H_1(X - F; \mathbf{Z}_2)$ is isomorphic to $\mathbf{Z}_2 \langle \mu_1, \dots, \mu_s \rangle \oplus H_1(X; \mathbf{Z}_2)$, where μ_i is homology class represented by a meridian circle to F .

We choose a homomorphism $v : H_1(X - F; \mathbf{Z}_2) \rightarrow \mathbf{Z}_2$ so that $v(\mu_i) = 1 \in \mathbf{Z}_2$ holds for all $1 \leq i \leq s$. Let $\Omega \subset X$ be an oriented closed $n - 2$ -submanifold of X such that $[\Omega]^\sharp = \omega \in H^2(X; \mathbf{Z})$. Let $L_i \subset X$ be an oriented loop such that $[L]_i = l_i \in H_1(X)$. Then fix an embedding $f_i : S^1 \times D^{n-1} \rightarrow X$ so that

$$\begin{cases} f_i(S^1 \times 0) = L_i \\ f_i(S^1 \times D^{n-1}) \cap (F \cup \Omega) = \emptyset. \end{cases}$$

Since $2l_i = 0$, we can choose an emmbedded surface G such that $\partial G = f_i(S^1 \times \{a, b\})$, where $a, b \in \partial D^{n-1}$. Then by setting $X' = \overline{X - f_i(S^1 \times D^{n-1})}$, we have that $(G_i, \partial G_i) \subset (X', \partial X')$. Then define $v \in H^1(X - F; \mathbf{Z}_2)$ as

$$v(l_i) = [\Omega] \cdot [G_i, \partial G_i] - \frac{1}{2}([F] \cdot [G_i, \partial G_i]).$$

Here, $\cdot : H_{n-2}(X'; \mathbf{Z}) \times H_2(X', \partial X'; \mathbf{Z}) \rightarrow \mathbf{Z}$ denote the intersection pairing. Then the coveing trnsformation map $T : \tilde{X} \rightarrow \tilde{X}$ of the double branched covering $\tilde{X} \rightarrow X$ determined by v is a spin structure preserving.

Remark 1.1 *Ono gives semi-orientation of fixed point set for each spin structure on $\tilde{X}$ that is preserved by $T : \tilde{X} \rightarrow \tilde{X}$ as follows([6]): let $SO(\tilde{X}) \rightarrow \tilde{X}$ denote the orthonormal frame bundle of $\tilde{X}$ together with a spin structure $Spin(\tilde{X}) \rightarrow SO(\tilde{X})$ that is presered by T . Then the differential $dT : SO(\tilde{X}) \rightarrow SO(\tilde{X})$ has a lift $\tilde{dT} : Spin(\tilde{X}) \rightarrow Spin(\tilde{X})$. Since the restriction $\tilde{dT}|_{\tilde{F}}$ is a bundle automorphism, it is a section of the adjoint bundle $Ad(Spin(\tilde{X})) \rightarrow \tilde{X}$. Because $Ad(Spin(\tilde{X}))$ is a subbundle of the Clliford algebra bundle $Cl(\tilde{X}) \rightarrow \tilde{X}$, and $Cl(\tilde{X}) \rightarrow \tilde{X}$ is isomorphic to the exterior bundle $\wedge^* T\tilde{X}$, $\tilde{dT}$ is a section of $\wedge^* T\tilde{X}$. Moreover we can see that $\wedge^* T\tilde{X}$ is a section of a exterior bundle of a normal bundle ν of $\tilde{F}$ in $\tilde{X}$. Thus $\tilde{dT}$ determines an orientation of ν , and given $\tilde{X}$ determines an orientation of $\tilde{F}$. In [4], we have shown that the homology class $[F]^\sharp \in H^2(X; \mathbf{Z})$ represented by this orientaion on $F \approx \tilde{F}$ is twice a characteristic cohomology class.*

Example 1.1 *Let $A \approx S^1 \times (0, 1)$ be an annulus embedded in $\mathbf{R}^2$ and $t : A \rightarrow A$ the involution given by $t(x, y) = (-x, y)$. Let the tangent bundle TA be trivialized so that its framing be bounding one. Then the differential $dt : TA|_{S^1 \times 0} \approx S^1 \times \mathbf{R}^2 \rightarrow TA|_{S^1 \times 0}$ of t has the following form;*

$$dt : \left(e^{\theta i}, \begin{pmatrix} a \\ b \end{pmatrix} \right) \rightarrow \left(e^{-\theta i}, -R(-2\theta) \begin{pmatrix} a \\ b \end{pmatrix} \right),$$

where $R(\theta)$ denote the matrix $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. Then the lift $\tilde{dt} : Spin(A) \approx A \times Spin(2) \rightarrow Spin(A)$ of $dt : SO(A) \rightarrow SO(A)$ to the spin structure $Spin(A) \rightarrow SO(A)$ with respect to the given framing has the form;

$$\left(e^{\theta i}, \xi \right) \rightarrow \left(e^{-\theta i}, (\cos \theta - \sin \theta \frac{\partial}{\partial x} \frac{\partial}{\partial y}) \xi \right).$$

Therefore at the north pole $N = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0) \in A$ (resp. south pole $S = (\begin{pmatrix} -1 \\ 0 \end{pmatrix}, 0) \in A$), the given spin structure determines the orientation $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ (resp. $-\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$).

The above argument shows that the hyperelliptic involution $t : S^2 \rightarrow S^2$ together with the unique spin structure S^2 gives the orientation of the branched locus $N \cup S$ of the quotient space $S^2 \approx S^2/\langle t \rangle$ so that $[N \cup S]^\sharp = 0 \in \mathbf{Z} \cong H^2(S^2; \mathbf{Z})$, which is twice a characteristic cohomology class.

Next we consider the case for annulus. Set $T = S^1 \times S^1$, $l = S^1 \times *$ and $m = * \times S^1$. Then the cohomology classes represented by l and m generate the cohomology group $H^1(T; \mathbf{Z}_2) \cong (\mathbf{Z}_2)^2$. If we give a spin structure on T that restricts to Lie group spin structures on l and m , then the induced orientation of the branched locus F in the quotient space $S^2 \approx T/\langle t \rangle$ by the hyperelliptic involution $t : T \rightarrow T$ satisfies $[F]^\sharp = \pm 4 \in \mathbf{Z} \cong H^2(S^2; \mathbf{Z})$. If we consider a spin structure that restricts to Lie group spin structures on l and to bounding spin structure on m , then the induced orientation satisfies $[F]^\sharp = \pm 0 \in \mathbf{Z}$. These are orientations which are twice a characteristic cohomology class.

By considering the Gysin exact sequence for the double cover $\tilde{X} - \tilde{F} \rightarrow X - F$, we obtain the following([5]);

Theorem 1.4 $\tilde{X}$ is spin if and only there exists a class $w \in H^1(X - F; \mathbf{Z}_2)$ such that $v \cup w = w_2(X - F)$ and that $\langle v, \mu \rangle = 1 \in \mathbf{Z}_2$, where $v \in H^1(X - F; \mathbf{Z}_2)$ determines the double cover $\tilde{X} - \tilde{F} \rightarrow X - F$, and $\mu \in H_1(X - F; \mathbf{Z}_2)$ is a homology class represented by a meridian to F .

Theorem 1.4 implies another way to state Theorem 1.3([5]);

Theorem 1.5 $\tilde{X}$ admits a spin structure that is preserved by the covering transformation map $T : \tilde{X} \rightarrow \tilde{X}$ if and only if $v \cup v = w_2(X - F)$.

As a corollary, we have([5]);

Corollary 1.1 Let $\tilde{Y} \rightarrow Y$ be an unbranched double cover determined by $\rho \in H^1(Y; \mathbf{Z}_2)$. Then $\tilde{Y}$ admits a spin structure that is preserved by the covering transformation map $t : \tilde{Y} \rightarrow \tilde{Y}$ of odd type if and only if $\rho \cup \rho = w_2(Y)$.

Example 1.2

Let $\tau : S^n \rightarrow S^n$ denote the antipodal map with odd n and $q : S^n \rightarrow RP^n$ its quotient map. Then τ is a spin structure preserving map and q is the double covering map that corresponds to $\rho = 1 \in \mathbf{Z}_2 \cong H^1(RP^n; \mathbf{Z}_2)$. Recall that τ is odd type with respect to the unique spin structure on S^n if and only if $n \equiv 3 \pmod{4}$ ([1]). Note that $n \equiv 3 \pmod{4}$ if and only if $\rho \cup \rho = w_2(RP^n)$ because $w_2(RP^n) = \frac{n(n+1)}{2} \in \mathbf{Z}_2 \cong H^2(RP^n; \mathbf{Z}_2)$.

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