

Symmetries of Toric Varieties

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Theorem 1 (Esser–Ji–Moraga 25, N.)

Let X be a complete simplicial toric variety of dimension n . Suppose that S_k acts faithfully on X . If $n = 1, 2$, or 3 , then $k \leq n + 3$; if $n \geq 4$, then $k \leq n + 2$. If equality holds, then X is isomorphic to one of the varieties listed in the following table;

n	Maximal S_k	X
1	S_4	$\mathbb{P}^1$
2	S_5	$\mathbb{P}^1 \times \mathbb{P}^1$
3	S_6	$\mathbb{P}^3$
4	S_6	$\mathbb{P}^4, \mathbb{P}^2 \times \mathbb{P}^2, \mathbb{P}(1^4, m), \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(m))$ ($m \in 2\mathbb{Z}_{\geq 0}$)
$n \geq 5$	S_{n+2}	$\mathbb{P}^n$

The idea of the proof of Theorem 1 is to use the structure of $\text{Aut}(X)$.

Theorem 2 (Cox 95)

Let X be a complete simplicial toric variety, and let R be the Cox ring of X . We denote by $\text{Aut}_g(R)$ the group of automorphisms of R preserving the grading. Let $\text{Aut}^0(X)$ be the identity component of $\text{Aut}(X)$. Then

$$\text{Aut}^0(X) \cong \text{Aut}_g(R) / \text{Hom}_{\mathbb{Z}}(\text{Cl } X, \mathbb{C}^*).$$

Assume S_k acts faithfully on X (i.e., $S_k \leq \text{Aut}(X)$). There are three cases

- $S_k \cap \text{Aut}^0(X)$ is trivial.
In this case, $k \leq n + 1$.
- $S_k \cap \text{Aut}^0(X) = A_k$.
In this case, $(n, k) = (2, 5)$ or $(4, 6)$ and $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2 \times \mathbb{P}^2$.
- $S_k \cap \text{Aut}^0(X) = S_k$.

The key lemma in this case is the following.

Lemma 3 (Esser–Ji–Moraga 25)

If $S_k \leq \text{Aut}^0(X)$, then

$$n \geq \begin{cases} 3 & \text{if } k = 5, 6 \\ k - 2 & \text{if } k \geq 7. \end{cases}$$

Furthermore, if equality holds, then $X \cong \mathbb{P}^n$.

By Lemma 3,

- $n = 2 \Rightarrow k < 5$.
- $(n, k) = (3, 6) \Rightarrow X \cong \mathbb{P}^3$.
- $n \geq 5, k = n + 2 \Rightarrow X \cong \mathbb{P}^n$.
- $n = 4 \Rightarrow k \leq 6$.

To complete the proof of Theorem 1, it remains to classify the varieties with a faithful S_6 action for the case $n = 4$.

Theorem 4 (N.)

Let X be a complete simplicial toric variety of dimension 4. Then $S_6 \leq \text{Aut}^0(X)$ if and only if X is isomorphic to one of the following;

- $\mathbb{P}^4$,
- $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a)) \rightarrow \mathbb{P}^3$ for some even integer $a \in 2\mathbb{Z}$,
- $\mathbb{P}(1^4, m)$ for some positive even integer $m \in 2\mathbb{Z}_{>0}$.

To prove Theorem 4, we introduce some notations. Let Δ be the fan defining X , and $\Delta(1)$ be the set of rays in Δ . For $\rho_i \in \Delta(1)$, denote D_i to be the torus invariant divisor corresponding to ρ_i . Consider the partition $\Delta(1) = \Delta_1 \sqcup \cdots \sqcup \Delta_s$, where $\rho_i, \rho_j \in \Delta_k$ for some k if and only if $D_i \sim D_j$. Let $d_i := |\Delta_i|$, and without loss of generality, we can assume $d_1 \geq d_2 \geq \cdots \geq d_s$.

Lemma 5 (Esser–Ji–Moraga 25)

- $\sum_{i=1}^s (d_i - 1) \leq 4$.
- If $\sum_{i=1}^s (d_i - 1) = 4$, then $X \cong \mathbb{P}^{d_1-1} \times \cdots \times \mathbb{P}^{d_s-1}$.
- If $S_6 \leq \text{Aut}^0(X)$, then $d_1 \geq 4$.

By Lemma 5 and some properties of toric varieties, there are only four cases for s and $(d_1, \dots, d_s)$.

- $s = 1, d_1 = 5 \Rightarrow X \cong \mathbb{P}^4$.
- $s = 2, (d_1, d_2) = (4, 2) \Rightarrow X \cong \mathbb{P}^3 \times \mathbb{P}^1$.
- $s = 2, (d_1, d_2) = (4, 1) \Rightarrow X \cong \mathbb{P}(1^4, m)$ for some $m \in \mathbb{Z}_{>1}$.
- $s = 3, (d_1, d_2, d_3) = (4, 1, 1) \Rightarrow X \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a)) \rightarrow \mathbb{P}^3$ for some $a \in \mathbb{Z}_{\neq 0}$.

Therefore, it is enough to show the following lemma.

Lemma 6 (N.)

- Let $X = \mathbb{P}(1^4, m)$ for $m \in \mathbb{Z}_{>1}$. Then $S_6 \leq \text{Aut}^0(X)$ if and only if m is an even number.
- Let $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a)) \rightarrow \mathbb{P}^3$ for $a \in \mathbb{Z}$. Then $S_6 \leq \text{Aut}^0(X)$ if and only if a is an even integer.

For $X = \mathbb{P}(1^4, m)$, $S_6 \leq \text{Aut}^0(X)$ if and only if $S_6 \leq (\text{GL}_4(\mathbb{C}) \times \mathbb{C}^*) / \mathbb{C}^*$. Thus, the assertion follows from the representation theory of S_6 .

For $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a))$, X is the blow-up of $\mathbb{P}(1^4, a)$ along its unique singular point (for $a \geq 2$). Hence, the assertion follows from Blanchard's lemma.

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■ David A. Cox. The homogeneous coordinate ring of a toric variety. *Journal of Algebraic Geometry* 4, no. 1, 17–50, 1995.