

GENERALIZED ISOMETRIC SPHERES OF ELEMENTS OF $PU(1, n; \mathbb{C})$

SHIGEYASU KAMIYA

神谷 茂保 (岡山理大 工)

Let G be a discrete subgroup of $PU(1, n; \mathbb{C})$. For a boundary point y of the Siegel domain, we define the generalized isometric sphere $I_y(f)$ of an element f of $PU(1, n; \mathbb{C})$. By using the generalized isometric spheres of elements of G , we construct a fundamental domain $P_y(G)$ for G , which is regarded as a generalization of the Ford domain. And we show that the Dirichlet polyhedron $D(w)$ for G with center w converges to $P_y(G)$ as $w \rightarrow y$.

1. First let us recall some definitions and notation. Let $\mathbb{C}$ be the field of complex numbers. Let $V = V^{1,n}(\mathbb{C})$ denote the vector space $\mathbb{C}^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\tilde{\Phi}(z^*, w^*) = -(\overline{z_0^*} w_1^* + \overline{z_1^*} w_0^*) + \sum_{j=2}^n \overline{z_j^*} w_j^*$$

for $z^* = (z_0^*, z_1^*, z_2^*, \dots, z_n^*)$, $w^* = (w_0^*, w_1^*, w_2^*, \dots, w_n^*)$ in V . An automorphism g of V , that is a linear bijection such that $\tilde{\Phi}(g(z^*), g(w^*)) = \tilde{\Phi}(z^*, w^*)$ for z^*, w^* in V , will be called a unitary transformation. We denote the group of all unitary transformations by $U(1, n; \mathbb{C})$. Set $PU(1, n; \mathbb{C}) = U(1, n; \mathbb{C})/(\text{center})$. Let $V_0 = \{w^* \in V \mid \tilde{\Phi}(w^*, w^*) = 0\}$ and $V_- = \{w^* \in V \mid \tilde{\Phi}(w^*, w^*) < 0\}$. It is clear that V_0 and V_- are invariant under $U(1, n; \mathbb{C})$. Set $V^* = V_- \cup V_0 - \{0\}$. Let $\pi: V^* \rightarrow \pi(V^*)$ be the projection map defined by $\pi(w_0^*, w_1^*, w_2^*, \dots, w_n^*) = (w_1, w_2, \dots, w_n)$, where $w_j = w_j^*/w_0^*$ for $j = 1, 2, \dots, n$. We write ∞ for $\pi(0, 1, 0, \dots, 0)$. We may identify $\pi(V_-)$ with the Siegel domain

$$H^n = \{w = (w_1, w_2, \dots, w_n) \in \mathbb{C}^n \mid \operatorname{Re}(w_1) > \frac{1}{2} \sum_{j=2}^n |w_j|^2\}.$$

An element g in $PU(1, n; \mathbb{C})$ acts on the Siegel domain H^n and its boundary ∂H^n . In H^n , we can introduce the hyperbolic metric d (see [3] and [6]). An element of $PU(1, n; \mathbb{C})$ is an isometry of H^n with respect to d . Denote $H^n \cup \partial H^n$ by $\overline{H^n}$. The H -coordinates of a point $(w_1, w_2, \dots, w_n) \in \overline{H^n} - \{\infty\}$ are defined by $(k, t, w')_H \in (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R} \times \mathbb{C}^{n-1}$ such that $k = \operatorname{Re}(w_1) - \frac{1}{2} \sum_{j=2}^n |w_j|^2$, $t = \operatorname{Im}(w_1)$ and $w' = (w_2, \dots, w_n)$. The Cygan metric $\rho(p, q)$ for $p = (k_1, t_1, w')_H$ and $q = (k_2, t_2, w'')_H$ is given by

$$\rho(p, q) = \left| \left\{ \frac{1}{2} \|W' - w'\|^2 + |k_2 - k_1| \right\} + i \{t_1 - t_2 + \operatorname{Im}(\overline{w'} W')\} \right|^{\frac{1}{2}},$$

where $\overline{w'} W' = \sum_{j=2}^n \overline{w_j} W_j$.

Let $f = (a_{ij})_{1 \leq i, j \leq n+1} \in PU(1, n; \mathbf{C})$ with $f(\infty) \neq \infty$. We define the *isometric sphere* $I(f)$ of f by

$$I(f) = \{w = (w_1, w_2, \dots, w_n) \in \overline{H^n} \mid |\tilde{\Phi}(W, Q)| = |\tilde{\Phi}(W, f^{-1}(Q))|\},$$

where $Q = (0, 1, 0, \dots, 0)$, $W = (1, w_1, w_2, \dots, w_n)$ in V^* (see [5]). It follows that the isometric sphere $I(f)$ is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \sqrt{1/|a_{12}|}$, that is,

$$I(f) = \left\{ w = (k, t, w')_H \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C}^{n-1} \mid \rho(w, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

Fix $y \in \partial H^n$ such that $f(y) \neq y$. Let γ be an element of $PU(1, n; \mathbf{C})$ with $\gamma(y) = \infty$. We define the generalized isometric sphere $I_y(f)$ at y of f as

$$I_y(f) = \gamma^{-1}(I_{\gamma f \gamma^{-1}}) = \{z \in \overline{H^n} \mid \rho(\gamma(z), \gamma f^{-1} \gamma^{-1}(\infty)) = R_{\gamma f \gamma^{-1}}\}$$

(see [1]). Note that if $y = \infty$, then $I_\infty(f)$ is the usual isometric sphere $I(f)$. The definition above does not depend on the choice of the element γ such that $\gamma(y) = \infty$.

Unless otherwise stated, we shall always take $f, g, \dots$ to be elements of $PU(1, n; \mathbf{C})$ fixing neither y nor ∞ . Set

$$\alpha_y(f, z) = \frac{R_f \rho(y, z)}{\rho(z, f^{-1}(y)) \rho(y, f(\infty))}.$$

We can write $I_y(f)$ as

$$I_y(f) = \{z \in \overline{H^n} \mid \alpha_y(f, z) = 1\}.$$

Put

$$Ext \ I_y(f) = \{z \in \overline{H^n} \mid \alpha_y(f, z) < 1\},$$

$$Int \ I_y(f) = \{z \in \overline{H^n} \mid \alpha_y(f, z) > 1\},$$

respectively.

Just as in the case of isometric spheres, we have

Proposition 1.1.

- (1) $I_{f(y)}(f) = f(I_y(f)) = I_y(f^{-1})$;
- (2) $f(Ext \ I_y(f)) \subset Int \ I_y(f^{-1})$;
- (3) $f(Int \ I_y(f)) \subset Ext \ I_y(f^{-1})$.

Next we consider the location of fixed points of elements.

Proposition 1.2. *Let f be an element of $PU(1, n; \mathbf{C})$ with fixed point x . If f is elliptic or parabolic, then x lies on the isometric sphere $I(f^{-1})$ of f^{-1} . If f is loxodromic, then $I(f^{-1})$ does not contain x .*

Replacing isometric spheres by generalized isometric spheres leads to the same conclusion as in Proposition 1.2.

Proposition 1.3. *Let f be an element of $PU(1, n; \mathbb{C})$ with fixed point x . If f is elliptic or parabolic, then x lies on $I_y(f)$. If f is loxodromic, then $I_y(f)$ does not contain x .*

2. Let z_1, z_2 be two different points in H^n . Let $E(z_1, z_2)$ be the bisector of $\{z_1, z_2\}$, that is,

$$E(z_1, z_2) = \{w \in H^n \mid d(z_1, w) = d(z_2, w)\}$$

(see [5] for details). Let G be a discrete subgroup of $PU(1, n; \mathbb{C})$ and let w be any point of H^n that is not fixed by any element of G except the identity. The Dirichlet polyhedron $D(w)$ for G with center w is defined by

$$D(w) = \bigcap_{g \in G - \{id\}} H_g(w),$$

where $H_g(w) = \{z \in H^n \mid d(z, w) < d(z, g(w))\}$. We observe that

- (1) $D(w)$ is not necessarily convex,
- (2) $D(w)$ is star-shaped about w ,
- (3) $D(w)$ is locally finite

(see [2], [4], [11] and [12]).

Let $\Omega(G)$ be the ordinary set of G . Assume that $\infty \in \Omega(G)$ and its stability subgroup $G_\infty = \{\text{identity}\}$. Then there is a positive constant M such that $\rho(0, g(\infty)) \leq M$ for any element g of G . The same argument as in [4] leads to the following results.

- (1) *The radii of isometric spheres are bounded above.*
- (2) *The number of isometric spheres with radii exceeding a given positive quantity is finite.*
- (3) *Given any infinite sequence of distinct isometric spheres of elements of G , the radii being $R_{g_1}, R_{g_2}, \dots$, then $\lim_{m \rightarrow \infty} R_{g_m} = 0$.*

We show that the generalized isometric sphere $I_y(f)$ is closely related to the bisector $E(z, f^{-1}(z))$.

Proposition 2.1. *If $z \in H^n$ converges to $y \in \partial H^n$, then $E(z, f^{-1}(z))$ converges to $I_y(f)$.*

By using generalized isometric spheres, we can construct a fundamental domain.

Theorem 2.2. *Let G be a discrete subgroup of $PU(1, n; \mathbb{C})$. Let ∞ be a point of $\Omega(G)$ and let $G_\infty = \{\text{identity}\}$. Suppose that y is a point of $\Omega(G) \cap \partial H^n$ and that G_y consists only of the identity. Then*

$$P_y(G) = \bigcap_{f \in G - \{id\}} \text{Ext } I_y(f)$$

is a fundamental domain for G .

By Proposition 2.1 and Theorem 2.2, we obtain

Theorem 2.3. *Let G be a discrete subgroup of $PU(1, n; \mathbb{C})$. Let $z \in H^n$ and let $y \in \partial H^n \cap \Omega(G)$. Then $D(z) \rightarrow P_y(G)$ as $z \rightarrow y$.*

From the manner of constructing $P_y(G)$, we have

Corollary 2.4. *The fundamental domain $P_y(G)$ is locally finite.*

References

1. B. N. Apanasov, Discrete Groups in Space and Uniformization Problems, Kluwer Acad. Press, 1991.
2. A.F. Beardon, The Geometry of Discrete Groups, Springer-Verlag, New York, 1983.
3. S. S. Chen and L. Greenberg, Hyperbolic spaces, in "Contributions to Analysis," Academic Press, New York (1974), 49-87.
4. L. R. Ford, Automorphic Functions (Second Edition), Chelsea, New York, 1951.
5. W. M. Goldman, Complex hyperbolic geometry, Oxford University Press, 1999.
6. S. Kamiya, Notes on elements of $U(1, n; \mathbb{C})$, Hiroshima Math. J. 21 (1991), 23-45.
7. S. Kamiya, On H-balls and canonical regions of loxodromic elements in complex hyperbolic space, Math. Proc. Camb. Phil. Soc. 113 (1993), 573-582.
8. S. Kamiya, Parabolic elements of $U(1, n; \mathbb{C})$, Rev. Romaine Math. Pures et Appl. 40 (1995), 55-64.
9. S. Kamiya, On discrete subgroups of $PU(1, 2; \mathbb{C})$ with Heisenberg translations, (to appear)
10. S. Kamiya, Generalized isometric spheres and fundamental domains for discrete subgroups of $PU(1, n; \mathbb{C})$, (to appear)
11. J. Lehner, Discontinuous groups and automorphic functions, Mathematical Surveys, No. 8, Amer. Math. Soc., 1964.
12. M. B. Phillips, Dirichlet polyhedra for cyclic groups in complex hyperbolic space, Proc. Amer. Math. Soc., 115 (1992), 221-228.

Okayama University of Science
 1-1 Ridai-cho, Okayama 700-0005 JAPAN
 e-mail:kamiya@mech.ous.ac.jp