

Haraux-Weissler型方程式の正值球対称解について

On the Positive Radial Solutions to the Haraux-Weissler Equation

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1. Introduction

The aim of this talk is to investigate the structure of positive radial solutions to

$$(1.1) \quad \Delta u + \frac{1}{2}x \cdot \nabla u + \lambda u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^n,$$

where $p > 1$, $n \geq 3$ and $\lambda \geq 0$. Since we are interested in radial solutions (i.e., $u = u(r)$ with $r = |x|$), we will study the following initial value problem

$$(IVP) \quad \begin{cases} u_{rr} + \frac{n-1}{r}u_r + \frac{r}{2}u_r + \lambda u + |u|^{p-1}u = 0, & r > 0, \\ u(0) = \alpha > 0. \end{cases}$$

Equation (1.1) comes from the study of a semilinear heat equation of the form

$$(1.2) \quad f_t - \Delta f - |f|^{p-1}f = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n.$$

When we discuss the following function, which is called a *self-similar solution*,

$$f(t, x) := t^{-1/(p-1)}u(x/\sqrt{t}),$$

it can be seen that f satisfies (1.2) if and only if u satisfies (1.1) with $\lambda = 1/(p-1)$.

In Section 3, it will be shown that (IVP) has a unique solution $u(r) \in C^2([0, \infty))$ with $u_r(0) = 0$, which is denoted by $u(r; \alpha)$. Moreover, if we define $z := \inf \{r > 0; u(r; \alpha) = 0\}$, then $u(r; \alpha)$ is decreasing in $[0, z)$. By the decreasing property of $u(r; \alpha)$, we can classify solutions of (IVP) in the following manner:

- (i) $u(r; \alpha)$ is a *crossing solution* if $0 < z < +\infty$,
- (ii) $u(r; \alpha)$ is a *decaying solution* if $z = +\infty$, i.e. $u(r; \alpha) > 0$ in $[0, \infty)$.

These terminologies are used by Yanagida and Yotsutani [YY1].

Many authors have studied (IVP). Weissler [W1] has proved that, if $\lambda \geq n/2$, then $u(r; \alpha)$ is a crossing solution for every $\alpha > 0$. For $0 < \lambda < n/2$, the critical exponent $p = (n+2)/(n-2)$ is important. Set $L := \lim_{r \rightarrow \infty} r^{2\lambda} u(r; \alpha)$. In the supercritical case $p \geq (n+2)/(n-2)$, Atkinson and Peletier [AP] and Peletier, Terman and Weissler [PTW] have proved that, if $0 < \lambda \leq \max\{1, n/4\}$, then $u(r; \alpha)$ is a decaying solution with $0 < L < +\infty$ for every $\alpha > 0$. Especially in the critical case $p = (n+2)/(n-2)$, Escobedo and Kavian [EK] have got the following result; if $\max\{1, n/4\} < \lambda < n/2$, then there exists a decaying solution with $L = 0$, i.e.,

$$u(r; \alpha) = C \exp(-r^2/4) r^{2\lambda-n} [1 + O(r^{-2})] \text{ as } r \rightarrow \infty,$$

where C is a positive constant. In the subcritical case $1 < p < (n+2)/(n-2)$, Weissler [W1] has proved that, if $\lambda > 0$, then $u(r; \alpha)$ is a crossing solution for sufficiently large α . Moreover, Haraux and Weissler [HW] have given an interesting result. Put

$$\alpha_* := \inf \{ \alpha > 0 ; u(r; \alpha) \text{ is a crossing solution} \}.$$

If $\lambda > 1/2(p-1)$ and $\lambda < n/2$, then $0 < \alpha_* < +\infty$ and $u(r; \alpha_*)$ is a decaying solution with $L = 0$. Moreover, $u(r; \alpha)$ is a decaying solution with $0 < L < +\infty$ for sufficiently small α .

Although we have picked up a part of known results, it seems that there are no works about the structure of solutions to (IVP) with $\lambda = 0$, and that the complete information for the structure of solutions to (IVP) with $\lambda > 0$ has not known. In this paper, we will show the structure of *positive* radial solutions to (IVP) with $\lambda = 0$, using the classification theorem by Yanagida and Yotsutani (see Section 4). Moreover, we will apply the same argument to (IVP) with $\lambda = 1$, and give more detailed information than the result in [HW].

2. Main Results

Our problem is to decide whether each $u(r; \alpha)$ is a crossing solution or a decaying solution when initial value α moves from 0 to $+\infty$. In the case $\lambda = 0$, we obtain the following result.

Theorem 1. Let $\lambda = 0$.

- (i) If $p \geq (n+2)/(n-2)$, then $u(r; \alpha)$ is a decaying solution for every $\alpha > 0$.
- (ii) If $1 < p < (n+2)/(n-2)$, then there exists a unique positive number α_0 such that $u(r; \alpha)$ is a decaying solution for every $\alpha \in (0, \alpha_0]$ and a crossing solution for every $\alpha \in (\alpha_0, \infty)$. Moreover, $u(r; \alpha_0)$ is the most rapidly decaying solution among decaying solutions such that

$$(2.1) \quad u(r; \alpha_0) = O(r^{-n} \exp(-r^2/4)) \text{ as } r \rightarrow \infty.$$

In [YY1], Yanagida and Yotsutani have studied the structure of positive radial solutions to the Lane-Emden equation

$$(2.2) \quad \Delta u + u^p = 0, \quad x \in \mathbb{R}^n.$$

A fundamental difference to the structure of positive radial solutions between (1.1) with $\lambda = 0$ and (2.2) appears in the subcritical case $1 < p < (n+2)/(n-2)$ because every positive radial solution to (2.2) is a crossing solution.

In the case $\lambda = 1$, we can show a similar result to the case $\lambda = 0$.

Theorem 2. Let $\lambda = 1$.

- (i) If $p \geq (n+2)/(n-2)$, then $u(r; \alpha)$ is a decaying solution for every $\alpha > 0$.
- (ii) If $1 < p < (n+2)/(n-2)$, then there exists a unique positive number α_1 such that $u(r; \alpha)$ is a decaying solution for every $\alpha \in (0, \alpha_1]$ and a crossing solution for every $\alpha \in (\alpha_1, \infty)$. Moreover, $u(r; \alpha_1)$ is the most rapidly decaying solution among decaying solutions such that

$$(2.3) \quad u(r; \alpha_1) = O(r^{2-n} \exp(-r^2/4)) \text{ as } r \rightarrow \infty.$$

Theorem 2 gives us more detailed structure of solutions to (IVP) with $\lambda = 1$ than the result established by Haraux and Weissler [HW] .

3. Preliminary Results

In this section, we will give some fundamental properties of solutions to (IVP).

Proposition 3.1. The following two conditions are equivalent:

(i) $u(r; \alpha) \in C([0, \infty)) \cap C^2((0, \infty))$ satisfies (IVP).

(ii) $u(r; \alpha) \in C([0, \infty))$ satisfies

$$(3.1) \quad u(r; \alpha) = \alpha - \int_0^r dt \int_0^t (s/t)^{n-1} \exp\{(s^2 - t^2)/4\} (\lambda u + |u|^{p-1} u) ds.$$

Moreover, in both cases, the following properties holds;

(a) $u(r; \alpha)$ is decreasing in $[0, z)$, where $z := \inf \{r > 0 ; u(r; \alpha) = 0\}$. (If $u(r; \alpha) > 0$ in $[0, \infty)$, then we put $z = \infty$.)

(b) $u(r; \alpha) \in C^2([0, \infty))$ and $u_r(0; \alpha) = 0$.

(c) $|u(r; \alpha)| \leq C(1+r)^{-2\lambda}$ and $|u_r(r; \alpha)| \leq C(1+r)^{-2\lambda-1}$ for all $r \geq 0$, where C depends boundedly on α .

Proof. We first show that (i) implies (ii). For this purpose, we begin with the proof of (a).

First we note that the equation of (IVP) is equivalent to

$$(3.2) \quad \{r^{n-1} \exp(r^2/4) u_r\}_r + r^{n-1} \exp(r^2/4) (\lambda u + |u|^{p-1} u) = 0.$$

Integrating (3.2) over $[\theta, r]$ leads to

$$(3.3) \quad r^{n-1} \exp(r^2/4) u_r(r; \alpha) - \theta^{n-1} \exp(\theta^2/4) u_r(\theta; \alpha) = - \int_\theta^r s^{n-1} \exp(s^2/4) (\lambda u + |u|^{p-1} u) ds.$$

Since $s^{n-1} \exp(s^2/4) (\lambda u + |u|^{p-1} u) \in L^1(0, r)$, there exists $\lim_{\theta \rightarrow 0} \theta^{n-1} u_r(\theta; \alpha)$. Now we will prove $\lim_{r \rightarrow 0} r^{n-1} u_r(r; \alpha) = 0$ by contradiction. Suppose that

$$(3.4) \quad \lim_{r \rightarrow 0} r^{n-1} u_r(r; \alpha) =: \eta > 0.$$

(We can also derive a contradiction in the case $\eta < 0$.) Let ε be any sufficiently small positive number. From (3.4), we can take sufficiently small $\delta(\varepsilon) > 0$ such that

$$(3.5) \quad r^{1-n}(\eta - \varepsilon) < u_r(r; \alpha) < r^{1-n}(\eta + \varepsilon)$$

for $r \in (0, \delta(\varepsilon))$. Integrating (3.5) from r to δ , we get

$$u(\delta; \alpha) - \frac{\eta + \varepsilon}{n-2}(r^{2-n} - \delta^{2-n}) < u(r; \alpha) < u(\delta; \alpha) - \frac{\eta - \varepsilon}{n-2}(r^{2-n} - \delta^{2-n});$$

which implies $\lim_{r \rightarrow 0} u(r; \alpha) = -\infty$. Since this is absurd, we get $\lim_{\theta \rightarrow 0} \theta^{n-1} u_r(\theta; \alpha) = 0$.

Therefore, letting $\theta \rightarrow 0$ in (3.3), we obtain

$$(3.6) \quad u_r(r; \alpha) = - \int_0^r (s/r)^{n-1} \exp\left\{(s^2 - r^2)/4\right\} (\lambda u + |u|^{p-1} u) ds.$$

Thus as far as $u(r; \alpha)$ is positive, $u_r(r; \alpha)$ is negative; so that $u(r; \alpha)$ is decreasing in $[0, \infty)$.

Moreover, Integrating (3.6) over $[0, r]$ and using $u(0) = \alpha$, we get (3.1). Thus we have shown that (i) implies (ii). Conversely, it is readily seen that (ii) implies (i). Concerning the proofs of (b) and (c), see [W2] and [HW], respectively. Q.E.D.

Proposition 3.2. There exists a unique solution $u(r; \alpha) \in C^2([0, \infty))$ of (IVP).

Proof. By Proposition 3.1, it is sufficient to show the uniqueness and existence of solutions for (3.1). The uniqueness is easily proved by Gronwall's inequality. The existence is obtained as follows. For $0 \leq r \leq \delta$ with a suitably small $\delta > 0$, we use the successive approximation method to obtain the local existence. For $r > \delta$, we introduce

$$E(r) = \frac{1}{2} u_r(r; \alpha)^2 + \frac{\lambda}{2} u(r; \alpha)^2 + \frac{1}{p+1} |u(r; \alpha)|^{p+1}.$$

Differentiating $E(r)$, we obtain

$$E'(r) = - \left\{ \frac{n-1}{r} + \frac{r}{2} \right\} u_r^2 \leq 0.$$

Thus, since $u(r; \alpha)$ and $u_r(r; \alpha)$ can never blow up, the global existence of $u(r; \alpha)$ for every $r > 0$ can be proved in the standard manner. Q.E.D.

4. The Classification Theorem by Yanagida and Yotsutani

In this section, for the purpose to prove Theorems 1 and 2, we will explain *the classification theorem by Yanagida and Yotsutani* (see [YY2] or [Y]) for the following initial value problem

$$(4.1) \quad \begin{cases} (g(r)u_r)_r + g(r)K(r)(u^+)^p = 0, & r > 0, \\ u(0) = \alpha > 0, \end{cases}$$

where $u^+ = \max\{u, 0\}$. We suppose that $g(r)$ and $K(r)$ satisfy

$$(g) \quad \begin{cases} g(r) \in C^1([0, \infty)); \\ g(r) > 0 \text{ in } (0, \infty); \\ 1/g(r) \notin L^1(0, 1); \\ 1/g(r) \in L^1(1, \infty), \end{cases}$$

and

$$(K) \quad \begin{cases} K(r) \in C(0, \infty); \\ K(r) \geq 0 \text{ and } K(r) \neq 0 \text{ in } (0, \infty); \\ h(r)K(r) \in L^1(0, 1); \\ h(r)\{h(r)/g(r)\}^p K(r) \in L^1(1, \infty), \end{cases}$$

where

$$h(r) := g(r) \int_r^\infty \{1/g(s)\} ds.$$

Moreover, define the following functions

$$\begin{aligned} G(r) &:= \frac{2}{p+1} g(r) h(r) K(r) - \int_0^r g(s) K(s) ds, \\ H(r) &:= \frac{2}{p+1} h(r)^2 \left\{ \frac{h(r)}{g(r)} \right\}^p K(r) - \int_r^\infty h(s) \left\{ \frac{h(s)}{g(s)} \right\}^p K(s) ds, \end{aligned}$$

and set

$$r_G := \inf \{r \in (0, \infty); G(r) < 0\}, \quad r_H := \sup \{r \in (0, \infty); H(r) < 0\}.$$

Remark 4.1. We can show that (4.1) has a unique solution $u(r; \alpha)$ for each $\alpha > 0$ under the first, second and third conditions in (K).

Now we will state their result.

Theorem 4.1. ([YY2]) Suppose that $G(r) \neq 0$ in $[0, \infty)$. Let $u(r; \alpha)$ be the solution of (4.1).

- (a) If $r_G = \infty$ (i.e., $G(r) \geq 0$ in $(0, \infty)$), then $u(r; \alpha)$ is a crossing solution for every $\alpha > 0$.
- (b) If $r_G < \infty$ and $r_H = 0$ (i.e., $H(r) \geq 0$ in $(0, \infty)$), then $u(r; \alpha)$ is a decaying solution with $\lim_{r \rightarrow \infty} \{g(r)/h(r)\}u(r; \alpha) = \infty$ for every $\alpha > 0$.
- (c) If $0 < r_H \leq r_G < \infty$, then there exists a unique positive number α_f such that $u(r; \alpha)$ is a crossing solution for every $\alpha \in (\alpha_f, \infty)$, and a decaying solution with $\lim_{r \rightarrow \infty} \{g(r)/h(r)\}u(r; \alpha) = \infty$ for every $\alpha \in (0, \alpha_f)$. Moreover, if $\alpha = \alpha_f$, then $u(r; \alpha)$ is a decaying solution with $0 < \lim_{r \rightarrow \infty} \{g(r)/h(r)\}u(r; \alpha) < \infty$, which means that $u(r; \alpha_f)$ is the most rapidly decaying solution among decaying solutions.

Remark 4.2. If $G(r) \equiv 0$ in $[0, \infty)$, then for every $\alpha > 0$, $u(r; \alpha)$ is a decaying solution with $0 < \lim_{r \rightarrow \infty} \{g(r)/h(r)\}u(r; \alpha) < \infty$.

5. Proof of Theorem 1

In this section, we will study the following initial value problem

$$(5.1) \quad \begin{cases} u'' + \frac{n-1}{r}u' + \frac{r}{2}u + (u^+)^p = 0, & r > 0, \\ u(0) = \alpha > 0, \end{cases}$$

where $u^+ = \max\{u, 0\}$. The equation of (5.1) is equivalent to

$$\left\{ r^{n-1} \exp(r^2/4) u_r \right\}_r + r^{n-1} \exp(r^2/4) (u^+)^p = 0.$$

If we put $g(r) := r^{n-1} \exp(r^2/4)$ and $K(r) := 1$ in (4.1), then it is easily seen that $g(r)$ and $K(r)$ satisfy (g) and (K), respectively. Moreover, we obtain

$$\begin{aligned} G(r) &= 2(p+1)^{-1} r^{2n-2} \exp(r^2/2) \int_r^\infty s^{1-n} \exp(-s^2/4) ds - \int_0^r s^{n-1} \exp(s^2/4) ds, \\ H(r) &= 2(p+1)^{-1} r^{2n-2} \exp(r^2/2) \left\{ \int_r^\infty s^{1-n} \exp(-s^2/4) ds \right\}^{p+2} \\ &\quad - \int_r^\infty s^{n-1} \exp(s^2/4) \left\{ \int_s^\infty t^{1-n} \exp(-t^2/4) dt \right\}^{p+1} ds. \end{aligned}$$

After some calculations,

$$(5.2) \quad G'(r) = 2(p+1)^{-1} r^{n-1} \exp(r^2/4) \{ \Phi(r) - (p+3)/2 \} = \left\{ \int_r^\infty s^{1-n} \exp(-s^2/4) ds \right\}^{-p-1} H'(r),$$

where

$$(5.3) \quad \Phi(r) := \{2(n-1) + r^2\} r^{n-2} \exp(r^2/4) \int_r^\infty s^{1-n} \exp(-s^2/4) ds.$$

In order to apply Theorem 4.1, we must know the location of r_G and r_H . For this purpose, we will investigate the profiles of $G(r)$ and $H(r)$. In view of (5.2), it is important to study $\Phi(r)$. First we obtain the following lemma.

Lemma 5.1.

- (i) $\lim_{r \rightarrow 0} \Phi(r) = 2(n-1)/(n-2)$.
- (ii) $\Phi(r) = 2 - 4r^{-2} + o(r^{-2})$ as $r \rightarrow \infty$.
- (iii) There exists a unique number $r_0 \in (0, \sqrt{6(n-1)})$ such that $\Phi(r)$ is decreasing in $[0, r_0]$ and increasing in (r_0, ∞) . Moreover, $\Phi(r_0) < 2$.

Proof. (i) By l'Hospital's theorem,

$$\begin{aligned} \lim_{r \rightarrow 0} \Phi(r) &= \lim_{r \rightarrow 0} \{2(n-1) + r^2\} r^{n-2} \exp(r^2/4) \int_r^\infty s^{1-n} \exp(-s^2/4) ds \\ &= \lim_{r \rightarrow 0} \frac{\left\{ \int_r^\infty s^{1-n} \exp(-s^2/4) ds \right\}_r}{\left\{ \left[\{2(n-1) + r^2\} r^{n-2} \right]^{-1} \right\}_r} \\ &= \lim_{r \rightarrow 0} \frac{4(n-1)^2 + 4(n-1)r^2 + r^4}{2(n-1)(n-2) + nr^2} = \frac{2(n-1)}{n-2}. \end{aligned}$$

(ii) Integrating by parts, we obtain

$$\begin{aligned} (5.4) \quad & \int_r^\infty s^{1-n} \exp(-s^2/4) ds \\ &= 2r^{-n} \exp(-r^2/4) - 2n \int_r^\infty s^{-1-n} \exp(-s^2/4) ds \\ &= 2r^{-n} \exp(-r^2/4) - 4nr^{-n-2} \exp(-r^2/4) + 4n(n+2) \int_r^\infty s^{-3-n} \exp(-s^2/4) ds. \end{aligned}$$

Thus we get

$$\Phi(r) = 2 - 4r^{-2} - 8n(n-1)r^{-4} + 4n(n+2) \{2(n-1) + r^2\} r^{n-2} \exp(r^2/4) \int_r^\infty s^{-3-n} \exp(-s^2/4) ds,$$

which implies (ii).

(iii) From (ii), $\Phi(r)$ is increasing for sufficiently large r and converges to 2. Moreover, since $2(n-1)/(n-2) > 2$, $\Phi(r)$ must have a local minimum at some $r_0 \in (0, \infty)$, and it is smaller than 2. We will show that there are no other critical points of $\Phi(r)$. By direct calculations,

$$(5.5) \quad \Phi'(r) = -2(n-1)r^{-1} - r$$

$$(5.6) \quad \Phi''(r) = -2(n-1)(n-3)r^{-2} - 2n - r^2/2 + \{2(n-1)(n-2) + (2n-1)r^2 + r^4/2\}r^{n-3}\exp(r^2/4)\int_r^\infty s^{1-n}\exp(-s^2/4)ds.$$

Suppose that there exists a positive number $\tilde{r}$ such that $\Phi'(\tilde{r}) = 0$. It follows from (5.5) that

$$(5.7) \quad \tilde{r}^{n-2}\exp(\tilde{r}^2/4)\int_{\tilde{r}}^\infty s^{1-n}\exp(-s^2/4)ds = \frac{2\tilde{r}^2 + 4(n-1)}{\tilde{r}^4 + 2(2n-1)\tilde{r}^2 + 4(n-1)(n-2)}.$$

Combining (5.6) and (5.7) leads to

$$(5.8) \quad \Phi''(\tilde{r}) = \frac{-4(\tilde{r} + \sqrt{6(n-1)})(\tilde{r} - \sqrt{6(n-1)})}{\tilde{r}^4 + 2(2n-1)\tilde{r}^2 + 4(n-1)(n-2)}.$$

From (5.8), $\Phi''(\tilde{r}) > 0$ if $\tilde{r} \in (0, \sqrt{6(n-1)})$ and $\Phi''(\tilde{r}) < 0$ if $\tilde{r} \in (\sqrt{6(n-1)}, \infty)$. Therefore, if $\Phi(r)$ has a critical point, then it must be a local minimum in $(0, \sqrt{6(n-1)})$ and a local maximum in $(\sqrt{6(n-1)}, \infty)$. This result says that there exist at most one local minimum and one local maximum since a local maximum cannot exist in $(0, \sqrt{6(n-1)})$ and a local minimum cannot exist in $(\sqrt{6(n-1)}, \infty)$. We have already known that $\Phi(r)$ has a local minimum, and now we will show that $\Phi(r)$ cannot have a local maximum. In fact, suppose that there exists a local maximum. Then $\Phi(r)$ decreases for large r . But it is impossible, because (ii) of this lemma means that $\Phi(r)$ increasingly converges to 2. Thus we finish the proof of (iii). (See Fig.1.)

Q.E.D.

From Lemma 5.1, since $2 < (p+3)/2 < 2(n-1)/(n-2)$ if $1 < p < (n+2)/(n-2)$, there exists a unique number $r_* \in (0, \infty)$ such that $\Phi(r) > (p+3)/2$ in $(0, r_*)$, $\Phi(r_*) = (p+3)/2$ and $\Phi(r) < (p+3)/2$ in (r_*, ∞) (see Fig.2). Moreover, since $(p+3)/2 \geq 2(n-1)/(n-2)$ if $p \geq (n+2)/(n-2)$, $\Phi(r) \leq (p+3)/2$ in $[0, \infty)$. Therefore, in view of the expressions of (5.2), we get the following lemma.

Lemma 5.2.

- (i) If $p \geq (n+2)/(n-2)$, then $G(r)$ and $H(r)$ are decreasing in $[0, \infty)$.
- (ii) If $1 < p < (n+2)/(n-2)$, then there exists a unique number $r_* \in (0, \infty)$ such that $G(r)$ and $H(r)$ are increasing in $[0, r_*)$ and decreasing in (r_*, ∞) .

The behaviors of $G(r)$ and $H(r)$ near $r = 0$ and $r = \infty$ are shown by the following result.

Lemma 5.3.

- (i) $\lim_{r \rightarrow \infty} G(r) = -\infty$.
- (ii) $\lim_{r \rightarrow 0} G(r) = 0$.
- (iii) $\liminf_{r \rightarrow \infty} H(r) \geq 0$.
- (iv) If $1 < p < (n+2)/(n-2)$, then $\limsup_{r \rightarrow 0} H(r) < 0$.

Remark 5.1. If $p \geq (n+2)/(n-2)$, then $H(r) \geq 0$ and $H(r) \neq 0$ in $[0, \infty)$ from Lemma 5.2 (i) and Lemma 5.3 (iii).

Proof. (i) By Lemma 5.1, $\{\Phi(r) - (p+3)/2\}$ is finitely negative for sufficiently large r and does not decay to zero as $r \rightarrow \infty$. Moreover, since $\lim_{r \rightarrow \infty} r^{n-1} \exp(r^2/4) = +\infty$, we obtain $\lim_{r \rightarrow \infty} G(r) = -\infty$. Therefore, we get (i).

(ii) Since $\lim_{r \rightarrow 0} \int_0^r s^{n-1} \exp(s^2/4) ds = 0$, it is sufficient to show

$$\lim_{r \rightarrow 0} r^{2n-2} \exp(r^2/2) \int_r^\infty s^{1-n} \exp(-s^2/4) ds = 0.$$

In fact, by l'Hospital's theorem,

$$\lim_{r \rightarrow 0} \frac{\left\{ \int_r^\infty s^{1-n} \exp(-s^2/4) ds \right\}_r}{(r^{2-2n})_r} = \lim_{r \rightarrow 0} \frac{r^{1-n} \exp(-r^2/4)}{(2n-2)r^{1-2n}} = 0.$$

$$\begin{aligned} \text{(iii)} \quad H(r) &> - \int_r^\infty s^{n-1} \exp(s^2/4) \left\{ \int_s^\infty t^{1-n} \exp(-t^2/4) dt \right\}^{p+1} ds \\ &> -(n-2)^{-p-1} \int_r^\infty s^{n-1+(2-n)(p+1)} \exp(-ps^2/4) ds. \end{aligned}$$

Therefore, we get

$$\liminf_{r \rightarrow \infty} H(r) \geq -(n-2)^{-p-1} \lim_{r \rightarrow \infty} \int_r^\infty s^{n-1+(2-n)(p+1)} \exp(-ps^2/4) ds = 0.$$

(iv) Let $p \in (1, (n+2)/(n-2))$. Assume ε be any sufficiently small positive number with $\varepsilon < \{(n+2) - (n-2)p\} / (n-2)(p+1)$ and fix ρ such that $\exp\{-(p+1)\rho^2/4\} > 1 - \varepsilon$. Then for $0 < r < \rho$,

$$\begin{aligned} (5.9) \quad H(r) &< \frac{2}{p+1} r^{2n-2} \exp\left(\frac{r^2}{2}\right) \left\{ \int_r^\infty s^{1-n} \exp\left(-\frac{s^2}{4}\right) ds \right\}^{p+2} \\ &\quad - \int_r^\rho s^{n-1} \exp\left(\frac{s^2}{4}\right) \left\{ \int_s^\rho t^{1-n} \exp\left(-\frac{t^2}{4}\right) dt \right\}^{p+1} ds \\ &< \frac{2}{p+1} r^{2n-2} \exp\left(\frac{r^2}{4}\right) \exp\left\{-\frac{(p+1)r^2}{4}\right\} \frac{1}{(n-2)^{p+2}} r^{(2-n)(p+2)} \\ &\quad - \int_r^\rho s^{n-1} \exp\left(\frac{s^2}{4}\right) \exp\left\{-\frac{(p+1)\rho^2}{4}\right\} \frac{1}{(n-2)^{p+1}} s^{(2-n)(p+1)} \left\{1 - \left(\frac{s}{\rho}\right)^{n-2}\right\}^{p+1} ds. \end{aligned}$$

First we consider the case $2 < p < 5$ for $n = 3$ and $1 < p < (n+2)/(n-2)$ for $n \geq 4$. Since $p+1 < 6$ and $2 - (n-2)p < 0$, we obtain

$$\begin{aligned} H(r) &< \frac{2}{(p+1)(n-2)^{p+2}} r^{2-(n-2)p} \exp\left(\frac{r^2}{4}\right) \\ &\quad - \frac{1}{(n-2)^{p+1}} \exp\left(\frac{r^2}{4}\right) \exp\left\{-\frac{(p+1)\rho^2}{4}\right\} \int_r^\rho s^{1-(n-2)p} \left\{1 - \left(\frac{s}{\rho}\right)^{n-2}\right\}^6 ds \\ &< \frac{2}{(p+1)(n-2)^{p+2}} r^{2-(n-2)p} \exp\left(\frac{r^2}{4}\right) \\ &\quad + \frac{1}{\{2 - (n-2)p\}(n-2)^{p+1}} r^{2-(n-2)p} \exp\left(\frac{r^2}{4}\right) (1 - \varepsilon) + o(r^{2-(n-2)p}) \\ &= -\frac{(n+2) - (n-2)p - \varepsilon(n-2)(p+1)}{(p+1)\{(n-2)p-2\}(n-2)^{p+2}} r^{2-(n-2)p} \exp\left(\frac{r^2}{4}\right) + o(r^{2-(n-2)p}); \end{aligned}$$

so that

$$\lim_{r \rightarrow 0} H(r) = -\infty.$$

In the case $p = 2$ for $n = 3$, it follows from the last inequality of (5.9) that

$$\begin{aligned} H(r) &< 2 \exp(-r^2/2) / 3 - \exp(r^2/4) \exp(-3\rho^2/4) \int_r^\rho s^{-1} \{1 - (s/\rho)\}^3 ds \\ &< 2/3 - (1 - \varepsilon)(\log \rho - \log r + O(1)) \\ &= (1 - \varepsilon) \log r + O(1) \end{aligned}$$

Then we arrive at the same result as before. It remains to discuss the case $1 < p < 2$ for $n = 3$.

Since $p + 1 < 3$, we get

$$\begin{aligned} H(r) &< \frac{2}{p+1} r^{2-p} \exp\left(\frac{r^2}{4}\right) \exp\left\{-\frac{(p+1)r^2}{4}\right\} - \exp\left(\frac{r^2}{4}\right) \exp\left\{-\frac{(p+1)\rho^2}{4}\right\} \int_r^\rho s^{1-p} \left(1 - \frac{s}{\rho}\right)^3 ds \\ &< \frac{2}{p+1} r^{2-p} \exp\left(\frac{r^2}{4}\right) - \exp\left(\frac{r^2}{4}\right) (1-\varepsilon) \int_r^\rho \left\{s^{1-p} - 3\frac{s^{2-p}}{\rho} + 3\frac{s^{3-p}}{\rho^2} - \frac{s^{4-p}}{\rho^3}\right\} ds \\ &= \left[\left\{\frac{2}{p+1} + \frac{1-\varepsilon}{2-p}\right\} r^{2-p} + o(r^{2-p})\right] \exp\left(\frac{r^2}{4}\right) - \frac{6(1-\varepsilon)}{(2-p)(3-p)(4-p)(5-p)} \exp\left(\frac{r^2}{4}\right) \rho^{2-p} \end{aligned}$$

from (5.9). Thus we obtain

$$\limsup_{r \rightarrow 0} H(r) \leq -\frac{6(1-\varepsilon)}{(2-p)(3-p)(4-p)(5-p)} \rho^{2-p} < 0.$$

Q.E.D.

Proof of Theorem 1. From Lemmas 5.2 and 5.3, we can draw the graphs of $G(r)$ and $H(r)$. Then we obtain $r_G = 0 (< \infty)$ and $r_H = 0$ in the case $p \geq (n+2)/(n-2)$ (see Fig.3) and $0 < r_H < r_G < \infty$ in the case $1 < p < (n+2)/(n-2)$ (see Fig.4). So we can apply Theorem 4.1 to show Theorem 1.

We will show (2.1). From Theorem 4.1, there exists a positive finite number β such that

$$\lim_{r \rightarrow \infty} \left\{ \int_r^\infty s^{1-n} \exp(-s^2/4) ds \right\}^{-1} u(r; \alpha_0) = \beta.$$

Moreover, by using the fact that $\left\{ \int_r^\infty s^{1-n} \exp(-s^2/4) ds \right\}^{-1} u(r; \alpha_0)$ is increasing in $[0, \infty)$, it follows from (5.4) that

$$\begin{aligned} u(r; \alpha_0) &< \beta \int_r^\infty s^{1-n} \exp(-s^2/4) ds \\ &= 2\beta \left\{ r^{-n} \exp(-r^2/4) - 2nr^{-n-2} \exp(-r^2/4) + 2n(n+2) \int_r^\infty s^{-3-n} \exp(-s^2/4) ds \right\}. \end{aligned}$$

This implies (2.1).

Q.E.D.

6. Proof of Theorem 2

In this section, we will study (IVP) with $\lambda = 1$. Put

$$u(r) := v(r) \varphi(r),$$

then the equation of (IVP) is rewritten as

$$v_{rr} + \left(2\frac{\varphi_r}{\varphi} + \frac{n-1}{r} + \frac{r}{2}\right)v_r + |\varphi|^{p-1}|v|^{p-1}v + \left\{\frac{\varphi_{rr}}{\varphi} + \left(\frac{n-1}{r} + \frac{r}{2}\right)\frac{\varphi_r}{\varphi} + \lambda\right\}v = 0.$$

Therefore, if we take $\varphi(r)$ which satisfies the following initial value problem

$$(6.1) \quad \begin{cases} \varphi_{rr} + \left(\frac{n-1}{r} + \frac{r}{2}\right)\varphi_r + \lambda\varphi = 0, & r > 0, \\ \varphi(0) = 1, \quad \varphi_r(0) = 0, \end{cases}$$

then $v(r)$ must satisfy

$$\begin{cases} v_{rr} + \left(2\frac{\varphi_r}{\varphi} + \frac{n-1}{r} + \frac{r}{2}\right)v_r + |\varphi|^{p-1}|v|^{p-1}v = 0, & r > 0, \\ v(0) = \alpha > 0. \end{cases}$$

In the special case $\lambda = 1$, it is possible to express the $C^2[0, \infty)$ -solution of (6.1) by

$$\varphi(r) = (n-2)r^{2-n} \exp(-r^2/4) \int_0^r s^{n-3} \exp(s^2/4) ds.$$

Note that $\varphi(r) > 0$ in $[0, \infty)$. In order to know the structure of solutions to (IVP) with $\lambda = 1$, we have only to verify whether $v(r; \alpha)$ has a zero or not. In this section, we will mainly study

$$(6.2) \quad \begin{cases} v_{rr} + \left(2\frac{\varphi_r}{\varphi} + \frac{n-1}{r} + \frac{r}{2}\right)v_r + \varphi^{p-1}(v^+)^p = 0, & r > 0, \\ v(0) = \alpha > 0. \end{cases}$$

The equation of (6.2) is equivalent to

$$\{r^{n-1} \exp(r^2/4) \varphi^2 v_r\}_r + r^{n-1} \exp(r^2/4) \varphi^2 \cdot \varphi^{p-1}(v^+)^p = 0;$$

to which Theorem 4.1 is applicable. In fact, we obtain following proposition.

Proposition 6.1. Put $g(r) := r^{n-1} \exp(r^2/4) \varphi^2$ and $K(r) := \varphi^{p-1}$. Then $g(r)$ and $K(r)$ satisfy (g) and (K) , respectively.

Proof. We can readily see that $g(r)$ and $K(r)$ satisfy $(g)_1$, $(g)_2$, $(K)_1$ and $(K)_2$, where $(g)_i$ and $(K)_i$ mean the i -th condition of (g) and (K) , respectively. Moreover,

$(g)_3$ Since $1/g(r) = r^{1-n} + o(r^{1-n})$ as $r \rightarrow 0$, we get $1/g(r) \notin L^1(0,1)$.

$(g)_4$ Integrating by parts, we obtain

$$\begin{aligned} \int_0^r s^{n-3} \exp(s^2/4) ds &= 2r^{n-4} \exp(r^2/4) - 4(n-4)r^{n-6} \exp(r^2/4) \\ &\quad + 4(n-4)(n-6) \int_1^r s^{n-7} \exp(s^2/4) ds + \int_0^1 s^{n-3} \exp(s^2/4) ds + (4n-18)e^{1/4}; \end{aligned}$$

so that

$$(6.3) \quad \varphi(r) = 2(n-2)r^{-2} - 4(n-2)(n-4)r^{-4} + o(r^{-4}) \quad \text{as } r \rightarrow \infty.$$

From (6.3), since

$$1/g(r) = r^{5-n} \exp(-r^2/4)(1+o(1))/4(n-2)^2 \quad \text{as } r \rightarrow \infty,$$

we have $1/g(r) \in L^1(1, \infty)$.

(K)₃ Note that

$$\begin{aligned} h(r) &= g(r) \int_r^\infty \{1/g(s)\} ds \\ &= r^{3-n} \exp(-r^2/4) \left\{ \int_0^r s^{n-3} \exp(s^2/4) ds \right\}^2 \left[\int_r^\infty s^{n-3} \exp(s^2/4) \left\{ \int_0^s t^{n-3} \exp(t^2/4) dt \right\}^{-2} ds \right] \\ &= r^{3-n} \exp(-r^2/4) \left\{ \int_0^r s^{n-3} \exp(s^2/4) ds \right\}^2 \int_r^\infty (1/T^2) dT \\ &= r^{3-n} \exp(-r^2/4) \left\{ \int_0^r s^{n-3} \exp(s^2/4) ds \right\} = r\varphi(r)/(n-2), \end{aligned}$$

where $\tau := \int_0^r t^{n-3} \exp(t^2/4) dt$. So we readily obtain

$$h(r)K(r) = r\varphi(r)^p/(n-2) \in L^1(0,1).$$

Condition (K)₄ is readily seen by

$$h(r)\{h(r)/g(r)\}^p K(r) = r^{1+(2-n)p} \exp(-pr^2/4)/(n-2)^{p+1} \in L^1(1, \infty). \quad \text{Q.E.D.}$$

Now we obtain

$$\begin{aligned} G(r) &= (n-2)^{p+1} \left[\frac{2}{p+1} r^{4-n+(2-n)p} \exp\left\{-\frac{(p+1)r^2}{4}\right\} \left\{ \int_0^r s^{n-3} \exp\left(\frac{s^2}{4}\right) ds \right\}^{p+2} \right. \\ &\quad \left. - \int_0^r s^{1+(2-n)p} \exp\left(-\frac{ps^2}{4}\right) \left\{ \int_0^s t^{n-3} \exp\left(\frac{t^2}{4}\right) dt \right\}^{p+1} ds \right], \\ H(r) &= \frac{1}{(n-2)^{p+1}} \left[\frac{2}{p+1} r^{4-n+(2-n)p} \exp\left\{-\frac{(p+1)r^2}{4}\right\} \int_0^r s^{n-3} \exp\left(\frac{s^2}{4}\right) ds \right. \\ &\quad \left. - \int_r^\infty s^{1+(2-n)p} \exp\left(-\frac{ps^2}{4}\right) ds \right]. \end{aligned}$$

Differentiating $G(r)$ and $H(r)$, we get

$$(6.4) \quad H'(r) = \frac{2}{(p+1)(n-2)^{p+1}} r^{1+(2-n)p} \exp\left(-\frac{pr^2}{4}\right) \left\{ \Psi(r) - \frac{p+3}{2} \right\} \equiv \left\{ \int_r^\infty \frac{1}{g(s)} ds \right\}^{p+1} G'(r),$$

where

$$(6.5) \quad \Psi(r) := (p+3) - \frac{1}{n-2} \varphi(r) \left[\{(n-2)p+n-4\} + \frac{p+1}{2} r^2 \right]$$

by recalling the expression of $\varphi(r) = (n-2)r^{2-n} \exp(-r^2/4) \int_0^r s^{n-3} \exp(s^2/4) ds$.

In order to prove Theorem 2, we will use the same argument as in Section 5. First, we will investigate the profile of $\Psi(r)$.

Lemma 6.1.

- (i) $\lim_{r \rightarrow 0} \Psi(r) = 2(n-1)/(n-2)$.
- (ii) $\Psi(r) = 2 - 4pr^{-2} + o(r^{-2})$ as $r \rightarrow \infty$.
- (iii) There exists a unique number $r_1 \in \left(\sqrt{2(p+2)\{(n-2)p+n-4\} / \{p(p+1)\}}, \infty \right)$ such that $\Psi(r)$ is decreasing in $[0, r_1)$ and increasing in (r_1, ∞) . Moreover, $\Psi(r_1) < 2$.

Proof. (i) Since $\lim_{r \rightarrow 0} \varphi(r) = 1$ and $\lim_{r \rightarrow 0} r^2 \varphi(r) = 0$, the conclusion easily follows.

(ii) Using (6.3) for sufficiently large r , we obtain

$$\begin{aligned} \Psi(r) &= (p+3) - \{2r^{-2} - 4(n-4)r^{-4} + o(r^{-4})\} \left[\{(n-2)p+n-4\} + \frac{p+1}{2} r^2 \right] \\ &= 2 - 4pr^{-2} + o(r^{-2}). \end{aligned}$$

(iii) Since $\Psi(r)$ increasingly converges to 2 from (ii) and $2(n-1)/(n-2) > 2$, $\Psi(r)$ must have a local minimum at some $r_1 \in (0, \infty)$ and $\Psi(r_1) < 2$. We will show that there are no other critical points of $\Psi(r)$. Direct calculations yield

$$\begin{aligned} (6.6) \quad \Psi'(r) &= -\{(n-2)p+n-4\}r^{-1} - (p+1)r/2 \\ &\quad + [(n-2)\{(n-2)p+n-4\} + \{(n-3)p+n-4\}r^2 + (p+1)r^4/4] \\ &\quad \times r^{1-n} \exp(-r^2/4) \int_0^r s^{n-3} \exp(s^2/4) ds, \end{aligned}$$

$$\begin{aligned} (6.7) \quad \Psi''(r) &= (n-1)\{(n-2)p+n-4\}r^{-2} + \{(2n-7)p+2n-9\}/2 + (p+1)r^2/4 \\ &\quad + [(1-n)(n-2)\{(n-2)p+n-4\} + \{(-3n^2+16n-22)p-3n^2+20n-32\}r^2/2 \\ &\quad + \{(-3n+11)p-3n+13\}r^4/4 - (p+1)r^6/8]r^{-n} \exp(-r^2/4) \int_0^r s^{n-3} \exp(s^2/4) ds. \end{aligned}$$

Suppose that there exists a positive number $\hat{r}$ such that $\Psi'(\hat{r}) = 0$. Then by (6.6), we have

$$(6.8) \quad \hat{r}^{-n} \exp(-\hat{r}^2/4) \int_0^{\hat{r}} s^{n-3} \exp(s^2/4) ds \\ = \frac{\{(n-2)p+n-4\} + (p+1)\hat{r}^2/2}{(n-2)\{(n-2)p+n-4\}\hat{r}^2 + \{(n-3)p+n-4\}\hat{r}^4 + (p+1)\hat{r}^6/4}.$$

When $n = 3$, the right hand side of (6.8) is non-positive for some $\hat{r}$. But the left hand side of (6.8) is positive for every $\hat{r}$. Therefore, for $n = 3$, we observe that $\Psi(r)$ cannot have any critical points for r satisfying

$$(p-1)r^2 - r^4 + (p+1)r^6/4 \leq 0.$$

Combining (6.7) and (6.8) leads to

$$(6.9) \quad \Psi''(\hat{r}) = \frac{-2(p+2)\{(n-2)p+n-4\} + p(p+1)\hat{r}^2}{(n-2)\{(n-2)p+n-4\}\hat{r}^2 + \{(n-3)p+n-4\}\hat{r}^4 + (p+1)\hat{r}^6/4}.$$

Let $r_p := \sqrt{2(p+2)\{(n-2)p+n-4\}/\{p(p+1)\}}$. From (6.9), $\Psi''(\hat{r}) < 0$ for $\hat{r} \in (0, r_p)$ and $\Psi''(\hat{r}) > 0$ for $\hat{r} \in (r_p, \infty)$. Therefore, if $\Psi(r)$ has a critical point, then it must be a local maximum in $(0, r_p)$ and a local minimum in (r_p, ∞) . This result says that there exists at most one local maximum and one local minimum since a local minimum cannot exist in $(0, r_p)$ and a local maximum cannot exist in (r_p, ∞) . Moreover, we will evaluate the critical value for $\Psi(r)$.

Combining (6.5) and (6.8), we get

$$\Psi(\hat{r}) = \frac{(p+1)\hat{r}^4/2 - \{p^2 - (2n-7)p - 2n + 8\}\hat{r}^2 + 2(n-1)\{(n-2)p+n-4\}}{(p+1)\hat{r}^4/4 + \{(n-3)p+n-4\}\hat{r}^2 + (n-2)\{(n-2)p+n-4\}}.$$

Define

$$\psi(r) := \frac{(p+1)r^4/2 - \{p^2 - (2n-7)p - 2n + 8\}r^2 + 2(n-1)\{(n-2)p+n-4\}}{(p+1)r^4/4 + \{(n-3)p+n-4\}r^2 + (n-2)\{(n-2)p+n-4\}} \quad \text{in } [0, \infty).$$

Then $\psi(r)$ satisfies $\psi(0) = 2(n-1)/(n-2)$, $\lim_{r \rightarrow \infty} \psi(r) = 2$ and

$$(6.10) \quad \psi'(r) \\ = \frac{p(p+1)^2 r [r^4 - 4\{(n-2)p+n-4\}r^2 / p(p+1) - 4(p+2)\{(n-2)p+n-4\}^2 / p(p+1)^2] / 2}{[(p+1)r^4/4 + \{(n-3)p+n-4\}r^2 + (n-2)\{(n-2)p+n-4\}]^2} \\ = \frac{p(p+1)^2 r [r^2 + 2\{(n-2)p+n-4\} / (p+1)] (r+r_p)(r-r_p) / 2}{[(p+1)r^4/4 + \{(n-3)p+n-4\}r^2 + (n-2)\{(n-2)p+n-4\}]^2}.$$

Since $2\{(n-2)p+n-4\} > 0$ for $n \geq 3$, it follows from (6.10) that $\psi(r)$ is decreasing in $(0, r_p)$ and increasing in (r_p, ∞) . Therefore, $\Psi(r)$ has at most one local maximum in $(0, r_p)$, and it is smaller than $2(n-1)/(n-2)$. But this is impossible from (i) of Lemma 6.1. Therefore, $\Psi(r)$ does not have any local maximum. Thus we can finish the proof of (iv). Q.E.D.

Correspondingly to Lemma 5.2, we obtain the following lemma.

Lemma 6.2.

- (i) If $p \geq (n+2)/(n-2)$, then $G(r)$ and $H(r)$ are decreasing in $[0, \infty)$.
- (ii) If $1 < p < (n+2)/(n-2)$, then there exists a unique number $r_{..} \in (0, \infty)$ such that $G(r)$ and $H(r)$ are increasing in $[0, r_{..})$ and decreasing in $(r_{..}, \infty)$.

The behaviors of $G(r)$ and $H(r)$ near $r = 0$ and $r = \infty$ are given as follows.

Lemma 6.3.

- (i) $\lim_{r \rightarrow \infty} G(r) = -\infty$.
- (ii) $\lim_{r \rightarrow 0} G(r) = 0$.
- (iii) $\liminf_{r \rightarrow \infty} H(r) \geq 0$.
- (iv) If $1 < p < (n+2)/(n-2)$, then $\limsup_{r \rightarrow 0} H(r) < 0$.

Remark 6.1. If $p \geq (n+2)/(n-2)$, then $H(r) \geq 0$ and $H(r) \neq 0$ in $[0, \infty)$ from Lemma 6.2 (i) and Lemma 6.3 (iii).

Proof. (i) Note that (6.4) can be rewritten as

$$G'(r) = \frac{2}{p+1} (r^2 \varphi(r))^{p+1} r^{n-2p-3} \exp\left(\frac{r^2}{4}\right) \left\{ \Psi(r) - \frac{p+3}{2} \right\}.$$

By Lemma 6.1, $\{\Psi(r) - (p+3)/2\}$ is finitely negative for sufficiently large r and does not

converge to zero as $r \rightarrow \infty$. Moreover, since $\lim_{r \rightarrow \infty} r^2 \varphi(r) = 2$ from (6.3) and $\lim_{r \rightarrow \infty} r^{n-2p-3} \exp(r^2/4) = \infty$, we get (i).

(ii) Since $\lim_{r \rightarrow 0} \int_0^r s^{1+(2-n)p} \exp(-ps^2/4) \left\{ \int_0^s t^{n-3} \exp(t^2/4) dt \right\}^{p+1} ds = 0$, it is sufficient to prove

$$\lim_{r \rightarrow 0} r^{4-n+(2-n)p} \exp\{-(p+1)r^2/4\} \left\{ \int_0^r s^{n-3} \exp(s^2/4) ds \right\}^{p+2} = 0;$$

which comes from the identity

$$r^{4-n+(2-n)p} \exp\{-(p+1)r^2/4\} \left\{ \int_0^r s^{n-3} \exp(s^2/4) ds \right\}^{p+2} = r^n \exp(r^2/4) \varphi(r)^{p+2} / (n-2)^{p+2}.$$

(iii) The assertion is readily seen from the following inequality

$$H(r) > -(n-2)^{-p-1} \int_r^\infty s^{1+(2-n)p} \exp(-ps^2/4) ds.$$

(iv) Let $p \in (1, (n+2)/(n-2))$. Assume ε be any sufficiently small positive number with $\varepsilon < \{(n+2) - (n-2)p\} / (n-2)(p+1)$ and fix ρ such that $\exp\{-(p+1)\rho^2/4\} > 1 - \varepsilon$. Then for $0 < r < \rho$,

$$\begin{aligned} (6.11) \quad H(r) &< \frac{1}{(n-2)^{p+1}} \left[\frac{2}{p+1} r^{4-n+(2-n)p} \exp\left\{-\frac{(p+1)r^2}{4}\right\} \int_0^r s^{n-3} \exp\left(\frac{s^2}{4}\right) ds \right. \\ &\quad \left. - \int_r^\rho s^{1+(2-n)p} \exp\left(-\frac{ps^2}{4}\right) ds \right] \\ &< \frac{1}{(n-2)^{p+1}} \left[\frac{2}{(p+1)(n-2)} r^{2+(2-n)p} \exp\left(-\frac{pr^2}{4}\right) - \exp\left(-\frac{p\rho^2}{4}\right) \int_r^\rho s^{1+(2-n)p} ds \right]. \end{aligned}$$

First considering the case $2 < p < 5$ for $n = 3$ and $1 < p < (n+2)/(n-2)$ for $n \geq 4$, we obtain

$$H(r) < -\frac{(n+2) - (n-2)p - \varepsilon(n-2)(p+1)}{(p+1)\{(n-2)p-2\}(n-2)^{p+2}} r^{2-(n-2)p} \exp\left(\frac{r^2}{4}\right) + o(r^{2-(n-2)p});$$

so that

$$\lim_{r \rightarrow 0} H(r) = -\infty.$$

In the case $p = 2$ for $n = 3$, observing that

$$\begin{aligned} H(r) &< 2 \exp(-r^2/2) / 3 - \exp(-\rho^2/2)(\log \rho - \log r) \\ &< (1-\varepsilon) \cdot \log r + O(1) \end{aligned}$$

from (6.11), we arrive at the same result as before. Moreover, in the case $1 < p < 2$ for $n = 3$, we get

$$H(r) < \frac{1}{(n-2)^{p+1}} \left\{ \frac{2}{p+1} r^{2-p} \exp(-pr^2/4) - \frac{1}{2-p} \exp(-p\rho^2/4)(\rho^{2-p} - r^{2-p}) \right\}$$

from (6.11). Thus we obtain

$$\limsup_{r \rightarrow 0} H(r) \leq -\frac{1}{(2-p)(n-2)^{p+1}} \exp\left(-\frac{p\rho^2}{4}\right) \rho^{2-p} < 0$$

since $2-p > 0$.

Q.E.D.

In the same way as the proof of Theorem 1, we obtain the following theorem.

Theorem 6.1. The structure of positive solutions to (6.2) is as follows.

- (i) If $p \geq (n+2)/(n-2)$, then $v(r; \alpha)$ is a decaying solution for every $\alpha > 0$.
- (ii) If $1 < p < (n+2)/(n-2)$, then there exists a unique positive number α_1 such that $v(r; \alpha)$ is a decaying solution for every $\alpha \in (0, \alpha_1]$ and a crossing solution for every $\alpha \in (\alpha_1, \infty)$. Moreover, $v(r; \alpha_1)$ is the most rapidly decaying solution among decaying solutions and there exists a positive finite number γ such that

$$\lim_{r \rightarrow \infty} \left\{ (n-2)^2 \int_0^r s^{n-3} \exp(s^2/4) ds \right\} v(r, \alpha_1) = \gamma.$$

Proof of Theorem 2.

The structure of positive solutions to (IVP) with $\lambda = 1$ is readily obtained by Theorem 6.1. We will show (2.3). Using the fact that $\left\{ (n-2)^2 \int_0^r s^{n-3} \exp(s^2/4) ds \right\} v(r, \alpha_1)$ is increasing in $[0, \infty)$, we get

$$v(r, \alpha_1) < \gamma \left\{ (n-2)^2 \int_0^r s^{n-3} \exp(s^2/4) ds \right\}^{-1}.$$

Therefore, we have

$$\begin{aligned} u(r; \alpha_1) &= v(r; \alpha_1) \varphi(r) \\ &< \gamma \left\{ (n-2)^2 \int_0^r s^{n-3} \exp(s^2/4) ds \right\}^{-1} \cdot (n-2) r^{2-n} \exp(-r^2/4) \left\{ \int_0^r s^{n-3} \exp(s^2/4) ds \right\} \\ &= (n-2)^{-1} \gamma r^{2-n} \exp(-r^2/4). \end{aligned}$$

This implies (2.3).

Q.E.D.

7. Appendix

After this talk, I have obtained the following result on the structure of solutions to (IVP).

Theorem 7.1. Suppose that $0 \leq \lambda \leq (n-2)/2$. If $1 < p < (n+2)/(n-2)$, then there exists a unique positive number α_λ such that $u(r; \alpha)$ is a decaying solution for every $\alpha \in (0, \alpha_\lambda]$ and a crossing solution for every $\alpha \in (\alpha_\lambda, \infty)$. Moreover, $u(r; \alpha_\lambda)$ is the most rapidly decaying solution among decaying solutions.

References

- [AP] F.V. Atkinson and L.A. Peletier, Sur les solutions radiales de l'équation $\Delta u + (x \cdot \nabla u)/2 + \lambda u/2 + |u|^{p-1}u = 0$, C. R. Acad. Sci. Paris Ser. I, 302 (1986), 99-101.
- [EK] M. Escobedo and O. Kavian, Variational problems related to self-similar solutions of the heat equation, Nonlinear Anal., 11 (1987), 1103-1133.
- [HW] A. Haraux and F.B. Weissler, Nonuniqueness for a semilinear initial value problem, Indiana Univ. Math. J., 31 (1982), 167-189.
- [PTW] L.A. Peletier, D. Terman and F.B. Weissler, On the equation $\Delta u + (x \cdot \nabla u)/2 + f(u) = 0$, Arch. Rational Mech. Anal., 94 (1986), 83-99.
- [W1] F.B. Weissler, Asymptotic analysis of an ODE and non-uniqueness for a semilinear PDE, Arch. Rational Mech. Anal., 91 (1986), 231-245.
- [W2] F.B. Weissler, Rapidly decaying solutions of an ODE with application to semilinear elliptic on parabolic PDEs., Arch. Rational Mech. Anal., 91 (1986), 247-266.
- [YY1] E. Yanagida and S. Yotsutani, Classification of the structure of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbb{R}^n$, Arch. Rational Mech. Anal., 124 (1993), 239-259.
- [YY2] E. Yanagida and S. Yotsutani, A unified approach to the structure of radial solutions to semilinear elliptic problems, in preparation.
- [Y] S. Yotsutani, Pohozaev identity and its applications, Kyoto University Sûrikaiseikikenkyûsho Kôkyûroku, 834 (1993), 80-90.

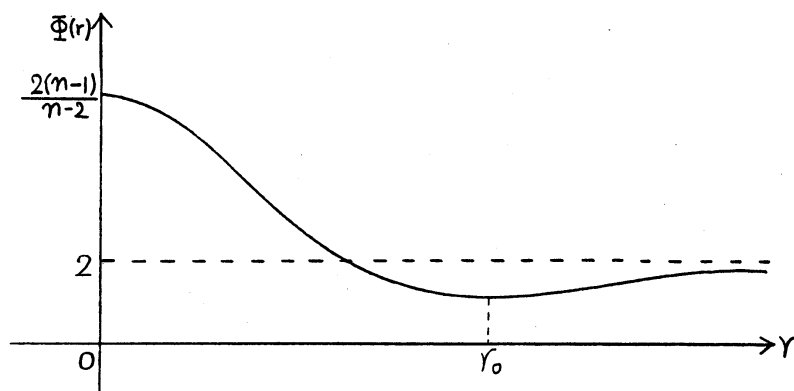


Fig.1

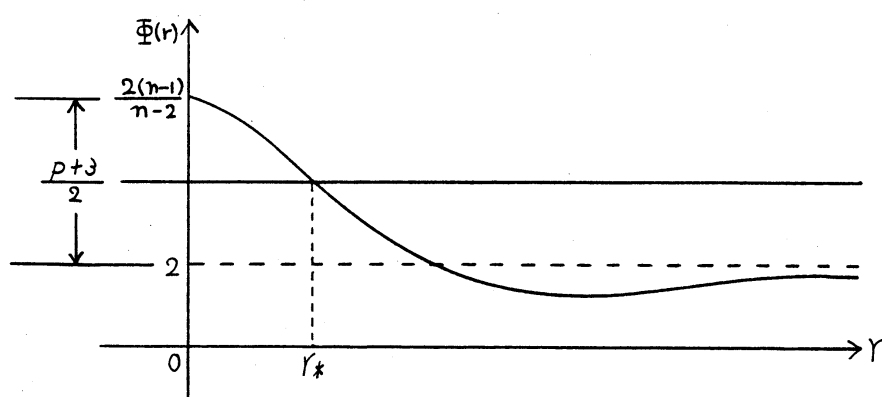


Fig.2

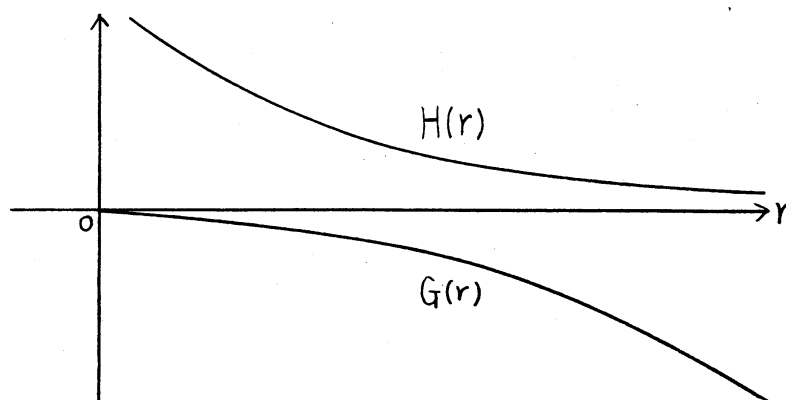


Fig.3

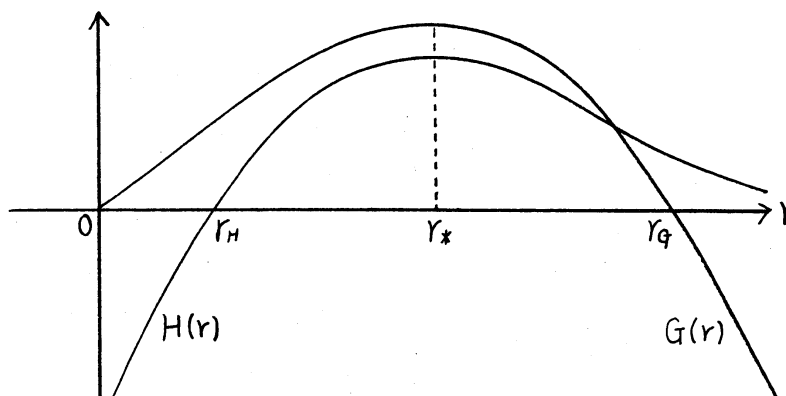


Fig.4