

# Fractional part sums and divisor functions II

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## § 1. Introduction

This is a continuation of Part I [1] and deal with the negative power case. That is, we consider ( $t > 0$ )

$$\sum_{n \leq x} n^{-t} \sigma_a(n), \quad \sum_{n \leq x} E_{-t}^a(n), \quad \int_1^x E_{-t}^a(u) du,$$

where

$$\sigma_a(n) = \sum_{d|n} d^a,$$

$E_{-t}^a(x)$  : the error term associated with  $\sum_{n \leq x} n^{-t} \sigma_a(n)$ .

Our theorem generalize and refine and in some cases correct MacLeod's theorem [3].

Let  $B_k(x)$  denote the  $k$ -th Bernoulli polynomial,  $[x]$  the integral part of  $x$ ,  $\bar{B}_k(x) = B_k(x - [x])$  the  $k$ -th periodic Bernoulli polynomial, and define the basic function  $G_{a,k}(x)$  by

$$G_{a,k}(x) = \sum_{n \leq x} n^a \bar{B}_k\left(\frac{x}{n}\right),$$

for  $a \in \mathbb{R}$ ,  $k \in \mathbb{N}$ .

Our aim amounts to writing the error terms associated with above three sums explicitly in terms of the function  $G_{a,k}(x)$ , and get informations on the behaviour of the error terms and infer new results on  $G_{a,k}(x)$ .

## § 2. Statement of results

Theorem 1 If  $a > 0$  and  $t \leq a$ ,  $a-t \in \mathbb{Z}$ , then we have

$$\sum_{n \leq x} n^{-t} \sigma_a(n) = \frac{\zeta(a+1)}{a-t+1} x^{a-t+1} + \sum_{r=1}^{a-t+1} \frac{(-1)^r}{r} \binom{a-t}{r-1} x^{a-t+1-r} G_{r-a-1,r}(x)$$

$$+ \begin{cases} \zeta(t-a)\zeta(t) - x^{-t} G_{a,1}(x), & t \geq 2, \\ -\frac{x^{1-t}}{a(a-t+1)} + \begin{cases} \frac{\zeta(t-a)}{1-t} x^{1-t} + \zeta(t-a)\zeta(t), & t \neq 1, \\ \zeta(t-a)\log x + \gamma\zeta(t-a), & t = 1, \end{cases} \\ + x^{1-t} \sum_{\substack{r \\ \max\{a-1,0\} < r \\ \leq a-t+1}} \frac{(-1)^r}{r} \binom{a-t}{r-1} E_{r-a-1,r} - x^{-t} G_{a,1}(x), & 0 < t < 2, \end{cases}$$

$$+ O(x^{-t+a/2}),$$

where the constants  $E_{r-a-1,r}$  are given in Lemm11[1],  $\zeta(s)$  denotes the Riemann zeta-function, and  $\gamma$  Euler constant.

For the case  $t > a$  we state it in Theorem 3, since it needs a different argument.

For the second sum, under the same assumptions as Th.1, we distinguish 8 cases according to the values of  $a, t$ . Here we state the 1st and 5th cases.

Let  $\sum_{n \leq x} n^{-t} \sigma_a(n) = g_{-t}^a(x) + E_{-t}^a(x)$ , where we define  $E_{-t}^a(x)$  by

$$g_{-t}^a(x) = \begin{cases} \frac{\zeta(a+1)}{a} x^a + \zeta(t-a) \log x, & t=1, a \leq 2, \\ \frac{\zeta(a+1)}{a-t+1} x^{a-t+1}, & \text{otherwise.} \end{cases}$$

### Theorem 2

$$\sum_{n \leq x} E_{-a}^a(n) = \begin{cases} \frac{\zeta(1-a)}{(1-a)(2-a)} x^{2-a} + \frac{\zeta(a+1)-\zeta(a)}{2} x + (1/2 - \bar{B}_1(x)) \frac{\zeta(1-a)}{1-a} x^{1-a}, & 0 \leq a=t < 1, \\ \frac{\zeta(2)-\log 2\pi-\gamma}{2} x - \log x/4 - (1/2 - \bar{B}_1(x)) G_{-1,1}(x), & a=t=1, \\ \frac{\zeta(1-a)}{(1-a)(2-a)} x^{2-a} + \frac{\zeta(a+1)-\zeta(a)}{2} x + \frac{\zeta(a-1)+\zeta(a+1)}{12} \\ - 1/2(1/2 - \bar{B}_1(x)) \zeta(a) - (1/2 - \bar{B}_1(x)) G_{-a,1}(x), & 1 < a=t < 2, \\ \frac{\zeta(3)-\zeta(2)}{2} x + \log x/12 + \gamma/12 + \zeta(3)/12 - \zeta'(-1) \\ - 1/2(1/2 - \bar{B}_1(x)) \zeta(2) - (1/2 - \bar{B}_1(x)) G_{-2,1}(x), & a=t=2, \\ - G_{a+1,2}(x)/2x^a - G_{1-a,2}(x)/2 + O(x^{-a/2 + 1/2}). \end{cases}$$

The 5th case is

$$\sum_{n \leq x} E_{-1}^1(n) = \zeta(3)x^2/4 - (-1/6 + \gamma/12 - \zeta'(-1) + \zeta(3)\bar{B}_2(x)/2 + (1/2 - \bar{B}_1(x))G_{-2,1}(x))x \\ - \frac{x}{2} G_{-1,2}(x) - \frac{1}{2x} G_{3,2}(x) + O(x^{1/2}).$$

The corresponding MacLeods Th.8(vi) should be corrected as this.

Applying the best known result of Peng-Recknagel[4], we have

$$\begin{aligned} \sum_{n \leq x} E_{-1}^1(n) &= \frac{\zeta(2) - \log 2\pi - \gamma}{2} x - \frac{1}{2} G_{0,2}(x) + O(x^{1/4}) \\ &= \frac{\zeta(2) - \log 2\pi - \gamma}{2} x + O(x^{2/7 + \varepsilon}), \quad \forall \varepsilon > 0. \end{aligned}$$

For the formula for the last integral, it is from Segal's formula[5], and Theorem 4-(c) of MacLeod[3].

In the end, we state

Theorem 3 We restrict ourselves to the case  $0 \leq a \leq 2$ ,  $t=a+1$ .

$$\begin{aligned} \sum_{n \leq x} n^{-a-1} \sigma_a(n) &= \begin{cases} \frac{1}{2} \log^2 x + 2\gamma \log x + \gamma^2 + 2\gamma_1, & a=0, \\ \zeta(a+1) \log x + \zeta(a+1) + \gamma \zeta(a+1), & a > 0, \end{cases} \\ &\quad - x^{-1} G_{-a,1}(x) - x^{-a-1} G_{a,1}(x) \\ &\quad + \begin{cases} 0, & a=0, \\ -\frac{\zeta(1-a)}{a} x^{-a}, & 0 < a < 2, \\ 0, & a=2, \end{cases} \\ &\quad + O(x^{-1-a/2}), \end{aligned}$$

where  $\gamma$  is Euler constant,  $\gamma_1 = \lim_{x \rightarrow \infty} (\sum_{n \leq x} n^{-1} \log n - (\log^2 x)/2)$  the 1st generalized Euler constant.

In particular

$$\sum_{n \leq x} \frac{d(n)}{n} = \frac{1}{2} \log^2 x + 2\gamma \log x + \gamma^2 + 2\gamma_1 - \frac{2}{x} G_{0,1}(x) + O(x^{-1})$$

holds.

It is known that solving the Dirichlet's divisor problem is equivalent to verifying

$$G_{0,1}(x) = O(x^{1/4 + \varepsilon}) \quad , \quad \forall \varepsilon > 0 \quad .$$

Using the best known estimate due to Kolesnik[2], we have

$$\sum_{n \leq x} \frac{d(n)}{n} = \frac{1}{2} \log^2 x + 2\gamma \log x + \gamma^2 + 2\gamma_1 + O(x^{-73/108} \log^{\varepsilon} x) \quad , \quad \forall \varepsilon > 0 \quad .$$

#### References

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