

# Abelian coverings of links

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Recently, the Smith Conjecture was proved by W.P. Thurston, H.H. Bass, P. Shalen, W. Meeks - S.T. Yau, and C.McA. Gordon - R.A. Litherland. In fact, they proved the following:

Branched Covering Theorem. Let  $\Sigma$  be a homotopy 3-sphere and let  $K$  be a knot in  $\Sigma$ . Then, if the  $n$ -fold branched cyclic covering space of  $\Sigma$  branched along  $K$  is a homotopy 3-sphere for some  $n \geq 2$ ,  $K$  is a trivial knot.

In the Topology Symposium in Sapporo 1979, Professor K. Murasugi showed that Branched Covering Theorem implies the following:

Theorem M. Let  $S^3$  be the 3-sphere. Then if an abelian covering  $M$  of a link in  $S^3$  is simply connected,  $M$  is  $S^3$ .

Furthermore he conjectured the following:

Conjecture M. The only link in  $S^3$ , other than the trivial knot, which has a homotopy sphere as an abelian covering is



In this paper, we will prove this conjecture.

I would like to express my sincere gratitude to the members of Kobe Topology Seminar for their helpful suggestions and conversations.

# § 1. Reducing the conjecture to another conjecture.

In this section we review Murasugi's proof of Theorem M and reduce Conjecture M to another conjecture.

First, we give a simple proof to a theorem of Murasugi and Mayberry [5] which gives a necessary condition for an abelian covering of a link to be a homology sphere and which is a key lemma to prove Theorem M. To do this, we use a method of [6].

Let  $L = K_1 \cup \dots \cup K_\mu$  be an oriented link of  $\mu$ -components in an oriented homology 3-sphere  $S$ , and let  $X = S - L$ . By Alexander duality the first integral homology group  $H_1(X)$  is the free abelian group on  $\mu$ -generators  $t_1, \dots, t_\mu$ , where  $t_i$  is the meridian of  $K_i$ . Let  $\tilde{X}_a$  be the universal abelian covering space of  $X$ ; that is, the covering space of  $X$  corresponding to the kernel of the Hurewicz homomorphism  $\gamma: \pi_1(X) \rightarrow H_1(X)$ . Let  $A$  be a finite abelian group and  $\psi: H_1(X) \rightarrow A$  be an epimorphism. Let  $\tilde{X}_\psi$  be the covering space of  $X$  corresponding to  $\text{Ker}(\psi \circ \gamma)$ , and  $M_\psi$  be the branched covering space of  $S$  obtained by the completion of  $\tilde{X}_\psi$ . We use the symbol  $q$  (resp.  $j$ ) to denote the natural projection  $\tilde{X}_a \rightarrow \tilde{X}_\psi$  (resp. the inclusion  $\tilde{X}_\psi \hookrightarrow M_\psi$ ). Let  $R(\psi)$  be the factor module  $H_1(\tilde{X}_\psi) / (j_* q_* H_1(\tilde{X}_a))$ . Then we have:

Proposition 1.  $|R(\psi)| = (\prod_{i=1}^{\mu} n_i) / |A|$ ,

where  $| |$  denotes the order of a group and  $n_i$  is the order of the element  $\psi(t_i)$  of  $A$ .

Proof. From the definition of  $\tilde{X}_\psi$ , the following sequence is exact:  $1 \rightarrow \pi_1(\tilde{X}_\psi) \rightarrow \pi_1(X) \xrightarrow{\psi \circ \gamma} A \rightarrow 1$ .

Factoring this sequence by  $q_* \pi_1(\tilde{X}_a)$ , we obtain the following exact

sequence:  $1 \rightarrow \pi_1(\tilde{X}_\psi)/q_*\pi_1(\tilde{X}_a) \rightarrow H_1(X) \xrightarrow{\psi} A \rightarrow 1$ .

Since  $\pi_1(\tilde{X}_\psi)/q_*\pi_1(\tilde{X}_a)$  is abelian,  $H_1(\tilde{X}_\psi)/q_*H_1(\tilde{X}_a) \cong \text{Ker } \psi$ .

Consider the following diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \rightarrow & q_*H_1(\tilde{X}_a) & \rightarrow & H_1(\tilde{X}_\psi) & \xrightarrow{\eta} & \text{Ker } \psi \rightarrow 0 \\ & & \downarrow & & \downarrow j_* & & \downarrow [j_*] \\ 0 & \rightarrow & (j \circ q)_*H_1(\tilde{X}_a) & \rightarrow & H_1(M_\psi) & \rightarrow & R(\psi) \rightarrow 0 \end{array}$$

where  $\eta$  is a natural map and  $[j_*]$  is the homomorphism induced by  $j_*$ . Since  $M_\psi$  is the completion of  $\tilde{X}_\psi$ ,  $j_*$  is onto and  $\text{Ker } j_*$  is equal to the branch relation  $B \subset H_1(\tilde{X}_\psi)$ . Hence, from the kernel and cokernel exact sequence (see [3]),  $\text{Ker}[j_*] = \eta(B)$ . From the definition of the branch relation  $B$ , we can see

$$\eta(B) = \langle t_i^{n_i} \ (1 \leq i \leq \mu) \rangle \subset \text{Ker } \psi \subset H_1(X),$$

where  $\langle \dots \rangle$  denotes the subgroup generated by the elements in  $\langle \dots \rangle$ . Hence  $R(\psi) \cong \text{Ker } \psi / \langle t_i^{n_i} \ (1 \leq i \leq \mu) \rangle$ . Factoring the exact sequence  $0 \rightarrow \text{Ker } \psi \rightarrow H_1(X) \xrightarrow{\psi} A \rightarrow 0$  by  $\langle t_i^{n_i} \ (1 \leq i \leq \mu) \rangle$ , we obtain the following exact sequence:

$$0 \rightarrow R(\psi) \rightarrow \bigoplus_{i=1}^{\mu} \langle t_i \mid t_i^{n_i} = 1 \rangle \rightarrow A \rightarrow 0.$$

Hence  $|R(\psi)| \cdot |A| = \prod_{i=1}^{\mu} n_i$ . This completes the proof.

Example. Let  $M_n(L)$  be the  $n$ -fold branched cyclic covering of  $L$ ; that is, the branched covering space corresponding to  $\text{Ker}(p_n \circ \gamma)$ , where  $p_n$  is the homomorphism  $H_1(X) \rightarrow \langle t \mid t^n = 1 \rangle$  defined by the equality  $p_n(t_i) = t \ (1 \leq i \leq \mu)$ . Then  $|R(p_n)| = n^\mu/n = n^{\mu-1}$  (compare [2] or [6]). In particular,  $M_n(L)$  can not be a homology sphere unless  $\mu = 1$ .

Corollary. (Theorem 11.1 of [5])

If  $M_\psi$  is a homology sphere, then  $A = \bigoplus_{i=1}^\mu \langle \psi(t_i) \rangle$ . That is,  $\psi$  is of the following form:

$$\psi: H_1(X) \rightarrow \bigoplus_{i=1}^\mu \langle t_i \mid t_i^{n_i} = 1 \rangle, \quad \psi(t_i) = t_i \quad (1 \leq i \leq \mu).$$

Now, let us review Murasugi's argument.

Theorem M. Let  $M_\psi$  be an abelian covering of a link  $L$  in the 3-sphere  $S^3$ . Then, if  $M_\psi$  is simply connected,  $M_\psi$  is  $S^3$ .

Proof by Murasugi. From the corollary of Proposition 1,  $\psi$  is of the following form:

$$\psi: H_1(X) \rightarrow \bigoplus_{i=1}^\mu \langle t_i \mid t_i^{n_i} = 1 \rangle, \quad \psi(t_i) = t_i \quad (1 \leq i \leq \mu).$$

Let  $\psi_k$  ( $0 \leq k \leq \mu$ ) be the homomorphism  $H_1(X) \rightarrow \bigoplus_{i=1}^k \langle t_i \mid t_i^{n_i} = 1 \rangle$  defined by the equality  $\psi_k(t_i) = \begin{cases} t_i & (1 \leq i \leq k) \\ 1 & (k+1 \leq i \leq \mu) \end{cases}$ .

Then we obtain the following sequence of branched coverings:

$$M_\psi = M_{\psi_\mu} \rightarrow M_{\psi_{\mu-1}} \rightarrow \dots \rightarrow M_{\psi_1} \rightarrow M_{\psi_0} = S^3.$$

Note that  $M_{\psi_i}$  is the  $n_i$ -fold branched cyclic covering space of  $M_{\psi_{i-1}}$  branched along  $\tilde{K}_i$ , the lift of  $K_i$  in  $M_{\psi_{i-1}}$ . Since  $M_\psi$  is simply connected,  $M_{\psi_i}$  ( $0 \leq i \leq \mu$ ) is simply connected. From the example of Proposition 1,  $\tilde{K}_i \subset M_{\psi_{i-1}}$  is connected. Hence, by Branched Covering Theorem,  $\tilde{K}_i \subset M_{\psi_{i-1}}$  is a trivial knot. Since the bottom of the sequence is  $S^3$ , every  $M_{\psi_i}$  ( $1 \leq i \leq \mu$ ) is  $S^3$ .

From the above argument, Conjecture M is equivalent to the following:

Conjecture  $\tilde{M}$ . If a link  $L = K_1 \cup \dots \cup K_\mu$  ( $\mu \geq 2$ ) in  $S^3$  has the following property  $(\star)$ , then  $L \subset S^3$  is .

$(\star)$  There exist integers  $n_1, \dots, n_\mu$  such that

$$(1) \quad n_i \geq 2 \quad (1 \leq i \leq \mu),$$

$$(2) \quad \tilde{K}_i \subset M_{\psi_{i-1}} \quad (1 \leq i \leq \mu) \text{ is a trivial knot, where } \psi_i$$

is the homomorphism defined in the proof of Theorem M.

## § 2. Proof of the conjecture.

Lemma 1. Let  $L = K_1 \cup K_2$  be a 2-components link in  $S^3$  with  $K_1$  a trivial knot. Let  $\tilde{K}_2$  be the lift of  $K_2$  in  $M_n(K_1)$ , the  $n$ -fold branched cyclic covering of  $K_1 \subset S^3$ . Let  $\tilde{\Delta}(t)$  be the reduced Alexander polynomial of  $\tilde{K}_2$ . Then, if  $\lambda \equiv |lk(K_1, K_2)| \neq 1$  and  $n \geq 2$ ,  $\tilde{\Delta}(t) \neq 1$ .

Proof. Let  $n = p^r m$ ,  $\text{g.c.d.}(p, m) = 1$ ,  $p$  a prime,  $r > 0$ , and  $\lambda \neq 0$ . Then, by Theorem 1 of Murasugi [4],

$$\rho_\lambda(t) \tilde{\Delta}(t) \equiv \left[ \prod_{j=0}^{m-1} \Delta(\eta^j, t) \right]^{p^r} \pmod{p},$$

where  $\Delta(t_1, t_2)$  is the Alexander polynomial of  $L$ ,  $\rho_\lambda(t) = 1 + t + \dots + t^{\lambda-1}$  and  $\eta$  is a primitive  $m$ -th root of 1. In particular,  $\Delta(1, t)^{p^r}$  divides  $\rho_\lambda(t) \tilde{\Delta}(t) \pmod{p}$ . Since  $\rho_\lambda(t)$  divides  $\Delta(1, t)$  by the Torres's condition [10],  $\rho_\lambda(t)^{p^r-1}$  divides  $\tilde{\Delta}(t) \pmod{p}$ . Hence, if  $\lambda > 1$ ,  $\tilde{\Delta}(t) \neq 1$ . If  $\lambda = 0$ ,  $\tilde{K}_2$  is a  $n$ -components link; so,  $(t-1)^{n-1}$  divides  $\tilde{\Delta}(t)$  and  $\tilde{\Delta}(t) \neq 1$ .

Proposition 2. If  $L = K_1 \cup \dots \cup K_\mu \subset S^3$  has Property  $(\star)$ , then  $\mu = 2$  and  $|lk(K_1, K_2)| = 1$ .

Proof. Suppose that  $\mu \geq 3$ . Let  $\tilde{K}_i$  ( $i = 2, 3$ ) be the lift of  $K_i$  in  $M_{\psi_1}$ , the  $n_1$ -fold branched cyclic covering of  $K_1 \subset S^3$ . Then  $|\text{lk}(\tilde{K}_2, \tilde{K}_3)| = n_1 |\text{lk}(K_2, K_3)| \neq 1$ . Hence the lift  $\tilde{K}_3$  of  $K_3$  in  $M_{\psi_2}$ , the  $n_2$ -fold branched cyclic covering of  $\tilde{K}_2 \subset M_{\psi_1}$ , can not be a trivial knot from Lemma 1; this is a contradiction. Hence  $\mu = 2$  and  $|\text{lk}(K_1, K_2)| = 1$  by Lemma 1.

Hence we may consider only 2-components links. Using Branched Covering Theorem, we obtain the following:

Proposition 3. Let  $L = K_1 \cup K_2$  be a 2-components link in  $S^3$  with  $K_1$  a trivial knot. Suppose  $L$  has Property  $(\star)$ ; that is, there is an integer  $n_1 \geq 2$  such that the lift  $\tilde{K}_2$  of  $K_2$  in  $M_{n_1}(K_1)$ , the  $n_1$ -fold branched cyclic covering of  $K_1 \subset S^3$ , is a trivial knot. Then, for any integer  $n \geq 1$ , the lift  $\tilde{K}_2$  of  $K_2$  in  $M_n(K_1)$  is a trivial knot.

Proof. Let  $n_2 \geq 1$  be an integer and let  $\psi$  be the homomorphism  $H_1(S^3 - L) \rightarrow \bigoplus_{i=1}^2 \langle t_i \mid t_i^{n_i} = 1 \rangle$  defined by the equality  $\psi(t_i) = t_i$  ( $i = 1, 2$ ). Then  $M_{\psi}$  is the  $n_2$ -fold branched cyclic covering space of  $M_{n_1}(K_1)$  branched along  $\tilde{K}_2$ , the lift of  $K_2$ . Since  $\tilde{K}_2 \subset M_{n_1}(K_1)$  is a trivial knot in  $S^3$ ,  $M_{\psi}$  is  $S^3$ . Let  $\tilde{K}_1$  be the lift of  $K_1$  in  $M_{n_2}(K_2)$ . Then  $M_{\psi}$  is the  $n_1$ -fold branched cyclic covering space of  $M_{n_2}(K_2)$  branched along  $\tilde{K}_1$ . Hence, by the proof of Theorem M,  $M_{n_2}(K_2)$  is  $S^3$  and  $\tilde{K}_1$  is a trivial knot. Repeat the above argument by exchanging the roles of  $K_1$  and  $K_2$ , and we obtain the desired result.

Now, the proof of Conjecture  $\tilde{M}$  is completed by the following proposition:

Proposition 4. Let  $L = K_1 \cup K_2$  be a link in  $S^3$  with  $K_1$  a trivial knot. Let  $\tilde{K}_2$  be the lift of  $K_2$  in  $M_2(K_1)$ , the 2-fold branched cyclic covering of  $K_1 \subset S^3$ . Then, if  $\tilde{K}_2$  is a trivial knot,  $L \subset S^3$  is .

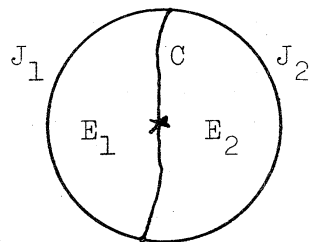
Proof. Let  $T$  be the involution of  $M_2(K_1)$  generating the covering transformation group. Then the fixed-point set of  $T$  is  $\tilde{K}_1$ , the lift of  $K_1$ , and  $T(\tilde{K}_2) = \tilde{K}_2$ . Let  $D$  be a disk in  $M_2(K_1)$  bounding  $\tilde{K}_2$ . Using this disk, we will construct a disk  $\underline{D}$  in  $S^3$  such that  $\partial \underline{D} = K_2$  and  $\underline{D}$  intersects  $K_1$  transversely in a single point. Let  $S(D)$  be the closure of  $\overset{\circ}{D} \cap T(\overset{\circ}{D})$  in  $M_2(K_1)$ , where  $\overset{\circ}{D}$  denotes the interior of  $D$ . Then we have

Lemma 2. We can deform  $D$ , without moving  $\partial D$ , so that  $\overset{\circ}{D}$  is transverse to both  $T(\overset{\circ}{D})$  and  $\tilde{K}_1$ , and  $S(D)$  is a proper 1-dim. submanifold of  $D$ .

Proof. See Lemma 1 of [1]. (Lemma 1 of [1] does not require that the deformation fixes  $\partial D$ . But it is not so hard to accomplish the deformation without moving  $\partial D$ .)

Hence we may assume that  $S(D)$  consists of simple closed curves and arcs with end points on  $\partial D$ . If  $S(D)$  contains simple closed curves, we can eliminate them by the cut and paste method (see [1], [8]). So we may assume that  $S(D)$  consists only of proper arcs. Since  $|\text{lk}(\tilde{K}_1, \tilde{K}_2)| = |\text{lk}(K_1, K_2)| = 1$  by Proposition 2,  $D \cap \tilde{K}_1 \neq \emptyset$ . Hence there is a connected component  $C$  of  $S(D)$  such that  $C \cap \tilde{K}_1$

$\neq \emptyset$ . Then  $D$  is a union of two disks  $E_1$  and  $E_2$  with  $E_1 \cap E_2 = C$ . Let  $J_i = \partial E_i - C$  for each  $i=1,2$ . Suppose there is a component  $C'$  other than  $C$  such that  $C' \cap \tilde{K}_1$



$\neq \emptyset$ . Without loss of generality we may assume

that  $C' \subset E_1$ . Then  $T(\partial C') \subset T(J_1) = J_2$ . On the other hand, since  $C' \cap \tilde{K}_1 \neq \emptyset$ ,  $T(\partial C') = \partial C' \subset J_1$ . This is a contradiction.

Hence any component of  $S(D)$ , other than  $C$ , does not intersect  $S(D)$ . From this, we can see that  $T(E_1) \cap E_1 = C$ . Let  $\underline{D} = p(E_1)$ , where  $p$  is the covering projection  $M_2(K_1) \rightarrow S^3$ . Then, from the above argument,  $\underline{D}$  is a disk with  $\partial \underline{D} = K_1$  and  $\underline{D} \cap K_1 = \text{one point}$ . This completes the proof of Proposition 4.

Thus we have proved Conjecture M.

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