

(0, 1) 区間上の作用素単調関数と Kwong 行列

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We review results on operator monotone functions on $(0, 1)$ and Kwong matrices. For details, we refer [10].

1 Operator monotone functions on $(0, 1)$

Let f be a real-valued C^1 function on an interval (a, b) . For n distinct real numbers $t_1, \dots, t_n \in (a, b)$ a *Loewner* (or *Pick*) *matrix* $L_f(t_1, \dots, t_n)$ is defined as

$$L_f(t_1, \dots, t_n) = \left[\frac{f(t_i) - f(t_j)}{t_i - t_j} \right].$$

In the case where $(a, b) \subseteq (0, \infty)$, a *Kwong* (or an *anti-Loewner*) *matrix* $K_f(t_1, \dots, t_n)$ is defined by

$$K_f(t_1, \dots, t_n) = \left[\frac{f(t_i) + f(t_j)}{t_i + t_j} \right].$$

In this paper we study positive operator monotone functions on $(0, 1)$ to continue our preceding studies on Loewner and Kwong matrices [3, 4, 7, 8, 11]. we show that the similar results of Loewner/Kwong matrices do not hold in general. For basic facts on operator monotone functions, we refer the reader to [2, 5, 6].

The following is useful for our study.

Lemma 1.1.

$$\begin{aligned} (1) \quad K_f(t_1, \dots, t_n) + L_f(t_1, \dots, t_n) &= 2 \left[\frac{t_i f(t_i) - t_j f(t_j)}{t_i^2 - t_j^2} \right] \\ &= 2 C \circ L_{tf(t)}(t_1, \dots, t_n) \\ &= 2 L_{\sqrt{t}f(\sqrt{t})}(s_1, \dots, s_n), \end{aligned}$$

where C is given as $C = \left[\frac{1}{t_i + t_j} \right]$, $\circ$ stands for the Schur product and $s_i = t_i^2$.

$$\begin{aligned}
 (2) \quad K_f(t_1, \dots, t_n) - L_f(t_1, \dots, t_n) &= 2 \left[\frac{t_i f(t_j) - t_j f(t_i)}{t_i^2 - t_j^2} \right] \\
 &= 2 D \left[\frac{t_i/f(t_i) - t_j/f(t_j)}{t_i^2 - t_j^2} \right] D \\
 &= 2 C \circ \left(DL_{t/f(t)}(t_1, \dots, t_n) D \right) \\
 &= 2 DL_{\sqrt{t}/f(\sqrt{t})}(s_1, \dots, s_n) D,
 \end{aligned}$$

where C and s_i are the same as in (1) and D is given as $D = \text{diag}(f(t_1), \dots, f(t_n))$.

For our study we prepare the representation of positive operator monotone functions on $(0, 1)$.

Theorem 1.2 A positive operator monotone function $f(s)$ on $(0, 1)$ is of the form

$$f(s) = \int_{[0,1]} \frac{s}{s + \zeta - 2s\zeta} dm(\zeta),$$

where m is a positive measure on $[0, 1]$.

For $0 \leq \zeta \leq 1$, put

$$f_\zeta(s) := \frac{s}{(1 - 2\zeta)s + \zeta} = \frac{s}{s + \zeta - 2s\zeta}. \quad (1.1)$$

Theorem 1.3 Let $f_\zeta(s)$ be the function in (1.1). Then $s/f_\zeta(s)$ is operator monotone if and only if $\zeta \leq 1/2$.

Corollary 1.4 Let $f(s)$ be a positive operator monotone function on $(0, 1)$ which is of the form

$$f(s) = \int_{[0,1/2]} f_\zeta(s) dm(\zeta) = \int_{[0,1/2]} \frac{s}{(1 - 2\zeta)s + \zeta} dm(\zeta), \quad (1.2)$$

where m is a positive measure on $[0, 1/2]$. Then $s/f(s)$ is operator monotone on $(0, 1)$.

The following corresponds to Kwong [9].

Theorem 1.5 If $f(s)$ is the operator monotone function in (1.2), then all Kwong matrices associated with f are positive semidefinite.

Theorem 1.6 Let $f_\zeta(s)$ be the function in (1.1). Then all Kwong matrices associated with f_ζ are positive semidefinite if and only if $\zeta \leq 1/2$.

The following is a counterpart to Audenaert [1].

Theorem 1.7 Let $f(s)$ be a positive function on $(0, 1)$. If $\sqrt{s}f(\sqrt{s})$ or $\sqrt{s}/f(\sqrt{s})$ is the operator monotone function in (1.2), then all Kwong matrices associated with f are positive semidefinite.

For $0 \leq \zeta \leq 1$, let us consider the function on $(0, 1)$

$$g_\zeta(s) := \frac{f_\zeta(s^2)}{s} = \frac{s}{(1 - 2\zeta)s^2 + \zeta}. \quad (1.3)$$

We note the following:

Theorem 1.8 Let $g_\zeta(s)$ be the function in (1.3). Then $g_\zeta(s)$ is operator monotone if and only if $1/2 \leq \zeta$, and all Kwong matrices associated with g_ζ are positive semidefinite if and only if $\zeta \leq 1/2$.

Proposition 1.9 Let $f(s)$ be the operator monotone function in (1.2). Then for any positive integer m ,

$$\left[\frac{f(s_i)^m - f(s_j)^m}{s_i^m - s_j^m} \right]$$

are positive semidefinite for all n and $s_1, \dots, s_n$ in $(0, 1)$.

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