

ON THE ZEROS OF EISENSTEIN SERIES ASSOCIATED WITH $\Gamma_0^*(2)$, $\Gamma_0^*(3)$, AND SOME GROUPS

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1. INTRODUCTION

Let $k \geq 4$ be an even integer, for $z \in \mathbb{H} := \{z \in \mathbb{C}; \operatorname{Im}(z) > 0\}$, let

$$(1) \quad E_k(z) := \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-k}$$

be the *Eisenstein series* associated with $\mathrm{SL}_2(\mathbb{Z})$. Then,

$$\mathbb{F} := \left\{ |z| \geq 1, -\frac{1}{2} \leq \operatorname{Re}(z) \leq 0 \right\} \cup \left\{ |z| > 1, 0 \leq \operatorname{Re}(z) < \frac{1}{2} \right\}$$

is a *fundamental domain* of $\mathrm{SL}_2(\mathbb{Z})$.

In [RSD], F. K. C. Rankin and H. P. F. Swinnerton-Dyer considered the problem of locating the zeros of $E_k(z)$ in $\mathbb{F}$. They proved that for $k = 12n + s$ ($s = 4, 6, 8, 10, 0$, and 14), then n zeros are in $A := \{z \in \mathbb{C}; |z| = 1, \pi/2 < \operatorname{Arg}(z) < 2\pi/3\}$. They also said in the last part of the paper, "This method can equally well be applied to Eisenstein series associated with subgroup of the modular group." However, it seems unclear how widely this claim holds.

Here, we consider the same problem for Fricke groups $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$ (See [K], [Q]), which are commensurable groups of $\mathrm{SL}_2(\mathbb{Z})$. For a fixed prime p , we define the following;

$$(2) \quad \Gamma_0^*(p) := \Gamma_0(p) \cup \Gamma_0(p) W_p,$$

where

$$(3) \quad \Gamma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}); c \equiv 0 \pmod{p} \right\}, \quad W_p := \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}.$$

Let $k \geq 4$ be an even integer, for $z \in \mathbb{H}$, let

$$(4) \quad E_{k,p}^*(z) := \frac{1}{p^{k/2} + 1} (p^{k/2} E_k(pz) + E_k(z))$$

be the Eisenstein series associated with $\Gamma_0^*(p)$. Then the next regions

$$\begin{aligned} \mathbb{F}^*(2) &:= \left\{ |z| \geq 1/\sqrt{2}, -\frac{1}{2} \leq \operatorname{Re}(z) \leq 0 \right\} \cup \left\{ |z| > 1/\sqrt{2}, 0 \leq \operatorname{Re}(z) < \frac{1}{2} \right\}, \\ \mathbb{F}^*(3) &:= \left\{ |z| \geq 1/\sqrt{3}, -\frac{1}{2} \leq \operatorname{Re}(z) \leq 0 \right\} \cup \left\{ |z| > 1/\sqrt{3}, 0 \leq \operatorname{Re}(z) < \frac{1}{2} \right\} \end{aligned}$$

are fundamental domains of $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$, respectively.

Define $A_2^* := \{z \in \mathbb{C}; |z| = 1/\sqrt{2}, \pi/2 < \operatorname{Arg}(z) < 3\pi/4\}$, and $A_3^* := \{z \in \mathbb{C}; |z| = 1/\sqrt{3}, \pi/2 < \operatorname{Arg}(z) < 5\pi/6\}$. Then we have $\overline{A_2^*} = A_2^* \cup \{i/\sqrt{2}, e^{3\pi/4}/\sqrt{2}\}$, and $\overline{A_3^*} = A_3^* \cup \{i/\sqrt{3}, e^{5\pi/6}/\sqrt{3}\}$.

In this paper, we will apply the method of F. K. C. Rankin and H. P. F. Swinnerton-Dyer (RSD Method) to the Eisenstein series associated with $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$. We will prove the next theorems.

Theorem 1. Let $k \geq 4$ be an even integer. $E_{k,2}^*(z)$ has all zeros on $\overline{A_2^*}$.

Theorem 2. Let $k \geq 4$ be an even integer. $E_{k,3}^*(z)$ has all zeros on $\overline{A_3^*}$.

2. $\Gamma_0^*(2)$ (PROOF OF THEOREM1)

2.1. **Preliminaries.** We give the next definition;

$$(5) \quad F_{k,2}^*(\theta) := e^{ik\theta/2} E_{k,2}^* \left(e^{i\theta}/\sqrt{2} \right).$$

Before proving Theorem1, we consider an expansion of $F_{k,2}^*(\theta)$.

By the definition of $E_k(z)$, $E_{k,2}^*(z)$ (cf. (1),(4)), we have

$$\begin{aligned} & 2(2^{k/2} + 1)e^{ik\theta/2} E_{k,2}^* \left(e^{i\theta}/\sqrt{2} \right) \\ &= 2^{k/2} \sum_{(c,d)=1} (ce^{-i\theta/2} + \sqrt{2}de^{i\theta/2})^{-k} + 2^{k/2} \sum_{(c,d)=1} (ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2})^{-k}. \end{aligned}$$

Now, we consider the case if c is even. We have

$$\begin{aligned} 2^{k/2} \sum_{\substack{(c,d)=1 \\ c:\text{even}}} (ce^{-i\theta/2} + \sqrt{2}de^{i\theta/2})^{-k} &= 2^{k/2} \sum_{\substack{(c,d)=1 \\ d:\text{odd}}} (2c'e^{-i\theta/2} + \sqrt{2}de^{i\theta/2})^{-k} \quad (c = 2c') \\ &= \sum_{\substack{(c,d)=1 \\ d:\text{odd}}} (\sqrt{2}c'e^{-i\theta/2} + de^{i\theta/2})^{-k} = \sum_{\substack{(c,d)=1 \\ c:\text{odd}}} (ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2})^{-k}. \end{aligned}$$

Thus we can write as follows;

$$(6) \quad F_{k,2}^*(\theta) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c:\text{odd}}} (ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2})^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c:\text{odd}}} (ce^{-i\theta/2} + \sqrt{2}de^{i\theta/2})^{-k}.$$

Hence we use this expression as a definition.

In the last part of this section, we compare the two series in this expression. Note that for any pair (c, d) , $(ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2})^{-k}$ and $(ce^{-i\theta/2} + \sqrt{2}de^{i\theta/2})^{-k}$ are conjugates of each other. The next lemma follows.

Lemma 2.1. $F_{k,2}^*(\theta)$ is real, for $\forall \theta \in \mathbb{R}$.

2.2. **Application of the RSD Method.** We will apply the method of F. K. C. Rankin and H. P. F. Swinnerton-Dyer (RSD Method) to the Eisenstein series associated with $\Gamma_0^*(2)$. We note that $N := c^2 + d^2$.

Firstly, we consider the case $N = 1$. Because c is odd, there are two cases, $(c, d) = (1, 0)$ and $(c, d) = (-1, 0)$. Then

$$(7) \quad F_{k,2}^*(\theta) = 2 \cos(k\theta/2) + R_2^*,$$

where R_2^* is the summation of the rest terms.

Let $v_k(c, d, \theta) := |ce^{i\theta/2} + \sqrt{2}de^{-i\theta/2}|^{-k}$, then $v_k(c, d, \theta) = 1/(c^2 + 2d^2 + 2\sqrt{2}cd \cos \theta)^{k/2}$, and $v_k(c, d, \theta) = v_k(-c, -d, \theta)$.

Now we will consider the next three cases, namely $N = 2, 5$, and $N \geq 10$. Note that $\theta \in [\pi/2, 3\pi/4]$. When $N = 2$, $v_k(1, 1, \theta) \leq 1$, $v_k(1, -1, \theta) \leq (1/3)^{k/2}$. When $N = 5$, $v_k(1, 2, \theta) \leq (1/5)^{k/2}$, $v_k(1, -2, \theta) \leq (1/3)^k$. When $N \geq 10$, $|ce^{i\theta/2} \pm \sqrt{2}de^{-i\theta/2}|^2 \geq (c^2 + d^2)/3 = N/3$, and the rest of the question is about the number of terms with $c^2 + d^2 = N$. Because c is odd, $|c| = 1, 3, \dots, 2N' - 1 \leq N^{1/2}$, so the number of $|c|$ is not more than $(N^{1/2} + 1)/2$. Thus the number of terms with $c^2 + d^2 = N$ is not more than $2(N^{1/2} + 1) \leq 3N^{1/2}$, for $N \geq 5$. Then we get the upper bound $\frac{162}{k-3} \left(\frac{1}{3}\right)^{k/2}$.

Thus

$$(8) \quad |R_2^*| \leq 2 + 2 \left(\frac{1}{3}\right)^{k/2} + 2 \left(\frac{1}{5}\right)^{k/2} + 2 \left(\frac{1}{3}\right)^k + \frac{162}{k-3} \left(\frac{1}{3}\right)^{k/2}.$$

Recalling "RSD Method", we want to show that $|R_2^*| < 2$. But the right-hand side is greater than 2. The point is the case $(c, d) = \pm(1, 1)$. We will consider the expansion of the method.

2.3. Expansion of the RSD Method (1). In the previous subsection, the point was the case $(c, d) = \pm(1, 1)$. Notice that " $v_k(1, 1, \theta) < 1 \Leftrightarrow \theta < 3\pi/4$ ". So we can easily expect that we get a good bound for $\theta \in [\pi/2, 3\pi/4 - x]$ for small $x > 0$. But if $k = 8n$, we need $|R_2^*| < 2$ for $\theta = 3\pi/4$ in this method. We will consider the case when $k = 8n, \theta = 3\pi/4$ in the next section.

Let $k = 8n + s$ ($n = m(k)$, $s = 4, 6, 0$, and 10). If $k < 8$, then $n < 1$. Consequently, $F_{k,2}^*(\theta)$ has at least 0 zeros, which does not make sense. So we may assume that $k \geq 8$.

The first problem is how small x should be. We consider each of the cases $s = 4, 6, 0$, and 10 .

When $s = 4$, $(2n+1)\pi \leq k\theta/2 \leq (3n+1)\pi + \pi/2$. So the last integer point (i.e. ± 1) is $k\theta/2 = (3n+1)\pi$, then $\theta = 3\pi/4 - \pi/k$. Similarly, when $s = 6$, and 10 , we have $\theta = 3\pi/4 - \pi/2k$, $3\pi/4 - 3\pi/2k$, respectively. When $s = 0$, the second to the last integer point is $\theta = 3\pi/4 - \pi/k$.

Thus we need $x \leq \pi/2k$.

Lemma 2.2. Let $k \geq 8$. For $\forall \theta \in [\pi/2, 3\pi/4 - x]$ ($x = \pi/2k$), $|R_2^*| < 2$.

Before proving the above lemma, we need the following preliminaries.

Proposition 2.1.

- (1) If $0 \leq x \leq \pi/2$, then $\sin x \geq 1 - \cos x$.
- (2) If $0 \leq x \leq \pi/16$, then $1 - \cos x \geq \frac{31}{64}x^2$.

Proof of Lemma 2.2. Let $k \geq 8$ and $x = \pi/2k$, then $0 \leq x \leq \pi/16$.

$$\begin{aligned} |e^{i\theta/2} + \sqrt{2}e^{-i\theta/2}|^2 &\geq 1 + \frac{31}{16}x^2. \quad (\text{Prop.2.1}) \\ |e^{i\theta/2} + \sqrt{2}e^{-i\theta/2}|^k &\geq 1 + \frac{k}{2} \frac{31}{16}x^2 \geq 1 + \frac{31}{4}x^2. \quad (k \geq 8) \\ v_k(1, 1, \theta) &\leq 1 - \frac{(31/4)}{1 + (31/4)x^2}x^2 \leq 1 - \frac{31 \times 256}{31\pi^2 + 1024}x^2. \end{aligned}$$

Thus

$$2v_k(1, 1, \theta) \leq 2 - \frac{31 \times 512}{31\pi^2 + 1024} \left(\frac{\pi}{2k}\right)^2 \leq 2 - \frac{265}{9} \frac{1}{k^2}.$$

In inequality(15), replace 2 with the bound $2 - \frac{265}{9} \frac{1}{k^2}$. Then

$$|R_2^*| \leq 2 - \frac{265}{9} \frac{1}{k^2} + 35 \left(\frac{1}{3}\right)^{k/2} \quad (k \geq 8).$$

Finally, we can show that $35 \left(\frac{1}{3}\right)^{k/2} < \frac{265}{9} \frac{1}{k^2}$. So, the proof is complete. $\square$

2.4. Expansion of the RSD Method (2). For the case " $k = 8n, \theta = 3\pi/4$ ", we need the next lemma.

Lemma 2.3. Let k be an integer such that $k = 8n$ for $\exists n \in \mathbb{N}$. If n is even, then $F_{k,2}^*(3\pi/4) > 0$. On the other hand if n is odd, then $F_{k,2}^*(3\pi/4) < 0$.

Before proving this lemma, recall that $E_k(z)$ is the modular form of weight k for $\text{SL}_2(\mathbb{Z})$ for $k \geq 4$: even. Then

$$(9) \quad E_k(z+1) = E_k(z), \quad E_k(-1/z) = z^k E_k(z).$$

Proof of Lemma 2.3. Let $k = 8n$ ($n \geq 1$). By the definition of $E_{k,2}^*(z)$, $F_{k,2}^*(z)$ (cf. (4),(12)), we have

$$F_{k,2}^*(3\pi/4) = \frac{e^{i3(k/8)\pi}}{2^{k/2} + 1} \left(2^{k/2} E_k(-1+i) + E_k\left(\frac{-1+i}{2}\right) \right).$$

By using the equations (9), $E_k(-1+i) = E_k(i)$, $E_k((-1+i)/2) = 2^{k/2} E_k(i)$. Then

$$F_{k,2}^*(3\pi/4) = 2e^{i(k/8)\pi} \frac{2^{k/2}}{2^{k/2} + 1} F_k(\pi/2).$$

The next question is: "Which one holds; $F_k(\pi/2) < 0$ or $F_k(\pi/2) > 0$?"

In [RSD], they showed $F_k(\theta) := e^{ik\theta/2} E_k(\theta) = 2 \cos(k\theta/2) + R_1$. Then they proved $|R_1| < 2$ for $k \geq 12$. Moreover, for $k = 8$, $|R_1|$ is not more than $1.29658... < 2$. It is monotonically decreasing in k . Thus we can show

$$(10) \quad |R_1| < 2 \quad \text{for } \forall k \geq 8.$$

When $k = 8n$,

$$F_{8n,2}^*(3\pi/4) = 2e^{in\pi} \frac{2^{4n}}{2^{4n} + 1} F_{8n}(\pi/2),$$

where $\frac{2^{4n}}{2^{4n} + 1} > 0$, $F_{8n}(\pi/2) = 2\cos(2n\pi) + R_1 > 0$. So the $\text{sign}(\pm)$ of $F_{k,2}^*(3\pi/4)$ is that of $e^{in\pi}$. Thus the proof is complete. $\square$

2.5. Valence formula for $\Gamma_0^*(2)$. In order to decide the locating of all zeros of $E_{k,2}^*(z)$, we need the valence formula for $\Gamma_0^*(2)$:

Proposition 2.2. *Let f be a modular function of weight k for $\Gamma_0^*(2)$, which is not identically zero. We have*

$$(11) \quad v_\infty(f) + \frac{1}{2}v_{i/\sqrt{2}}(f) + \frac{1}{4}v_{\rho_2}(f) + \sum_{\substack{p \in \Gamma_0^*(2) \setminus \mathbb{H} \\ p \neq i/\sqrt{2}, \rho_2}} v_p(f) = \frac{k}{8},$$

where $v_p(f)$ is the order of f at p , and $\rho := e^{2\pi/3}$, and $\rho_2 := e^{3\pi/4}/\sqrt{2}$.

Remark 1. *Let $k \geq 4$ be an even integer. We have*

$$\begin{aligned} v_{i/\sqrt{2}}(E_{k,2}^*) &= s_k \quad (s_k = 0, 1 \text{ such that } 2s_k \equiv k \pmod{4}), \\ v_{\rho_2}(E_{k,2}^*) &= t_k \quad (t_k = 0, 1, 2, 3 \text{ such that } -2t_k \equiv k \pmod{8}). \end{aligned}$$

3. $\Gamma_0^*(3)$ (PROOF OF THEOREM2)

3.1. Preliminaries. We give the next definition;

$$(12) \quad F_{k,3}^*(\theta) := e^{ik\theta/2} E_{k,3}^*(e^{i\theta}/\sqrt{3}).$$

By the definition of $E_k(z)$, $E_{k,3}^*(z)$ (cf. (1),(4)), we have

$$\begin{aligned} &2(3^{k/2} + 1)e^{ik\theta/2} E_{k,3}^*(e^{i\theta}/\sqrt{3}) \\ &= 3^{k/2} \sum_{(c,d)=1} (ce^{-i\theta/2} + \sqrt{3}de^{i\theta/2})^{-k} + 3^{k/2} \sum_{(c,d)=1} (ce^{i\theta/2} + \sqrt{3}de^{-i\theta/2})^{-k}. \end{aligned}$$

We consider the case if 3 is divisible by c . Then we can write as follows;

$$(13) \quad F_{k,3}^*(\theta) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 3 \nmid c}} (ce^{i\theta/2} + \sqrt{3}de^{-i\theta/2})^{-k} + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 3 \nmid c}} (ce^{-i\theta/2} + \sqrt{3}de^{i\theta/2})^{-k}.$$

The next lemma follows.

Lemma 3.1. $F_{k,3}^*(\theta)$ is real, for $\forall \theta \in \mathbb{R}$.

3.2. Application of the RSD Method. We note that $N := c^2 + d^2$, and consider the case $N = 1$. Then we can write;

$$(14) \quad F_{k,3}^*(\theta) = 2\cos(k\theta/2) + R_3^*. \quad (\exists R_3^* \in \mathbb{R})$$

Let $v_k(c, d, \theta) := |ce^{i\theta/2} + \sqrt{3}de^{-i\theta/2}|^{-k}$. Now we will consider the next cases, namely $N = 2, 5, 10, 13, 17$, and $N \geq 25$. Considering $\theta \in [\pi/2, 5\pi/6]$, we calculate $v_k(c, d, \theta)$ for $N = 2, 5, 10, 13, 17$. Furthermore, for $N \geq 25$, we get the upper bound $\frac{352\sqrt{6}}{k-3} \left(\frac{1}{2}\right)^k$. Thus

$$(15) \quad |R_3^*| \leq 4 + 176 \left(\frac{1}{2}\right)^k$$

Now, we want to show that $|R_3^*| < 2$. But the right-hand side is much greater than 2. The points are the cases $(c, d) = \pm(1, 1), \pm(2, 1)$.

3.3. Expansion of the RSD Method (1). In this subsection, we will prove following lemma.

Lemma 3.2. Let $k \geq 8$. For $\forall \theta \in [\pi/2, 5\pi/6 - x]$ ($x = \pi/3k$), $|R_3^*| < 2$.

Before proving the above lemma, we need the following preliminaries.

Proposition 3.1.

- (1) For $k \geq 8$, $(\frac{3}{2})^{2/k} \leq 1 + (2 \log \frac{3}{2}) \frac{1}{k} + \frac{1}{2} (2 \log \frac{3}{2})^2 (\frac{3}{2})^{2/k} \frac{1}{k^2}$.
- (2) For $k \geq 8$, $3 + 2\sqrt{3} \cos(\frac{5\pi}{6} - \frac{\pi}{3k}) \geq \frac{2\pi}{\sqrt{3}} \frac{1}{k}$.
- (3) For $k \geq 8$, and let $x = \pi/3k$, then $4 + 2\sqrt{3} \cos(\frac{5\pi}{6} - x) \geq (\frac{3}{2})^{2/k} \left(1 + \frac{256 \times 7 \times 13}{3 \times 127 \times k} x^2\right)$.

Proposition 3.2.

- (1) For $k \geq 8$, $3^{2/k} \leq 1 + (2 \log 3) \frac{1}{k} + \frac{1}{2} (2 \log 3)^2 3^{2/k} \frac{1}{k^2}$.
- (2) For $k \geq 8$, $6 + 4\sqrt{3} \cos(\frac{5\pi}{6} - \frac{\pi}{3k}) \geq \frac{2\pi}{\sqrt{3}} \frac{1}{k}$.
- (3) For $k \geq 8$, and let $x = \pi/3k$, then $7 + 4\sqrt{3} \cos(\frac{5\pi}{6} - x) \geq 3^{2/k} \left(1 + \frac{256 \times 7 \times 13}{3 \times 127 \times k} x^2\right)$.

Proof of Lemma 3.2. Let $k \geq 8$ and $x = \pi/3k$, then $0 \leq x \leq \pi/24$.

By Proposition 3.1

$$|e^{i\theta/2} + \sqrt{3}e^{-i\theta/2}|^2 \geq \left(\frac{3}{2}\right)^{2/k} \left(1 + \frac{256 \times 7 \times 13}{3 \times 127 \times k} x^2\right). \quad (\text{Prop.3.1(3)})$$

$$v_k(1, 1, \theta) \leq \frac{2}{3} - \frac{107}{8} x^2.$$

Similarly, by Proposition 3.2

$$|2e^{i\theta/2} + \sqrt{3}e^{-i\theta/2}|^2 \geq 3^{2/k} \left(1 + \frac{256 \times 7 \times 13}{3 \times 127 \times k} x^2\right). \quad (\text{Prop.3.1(3)})$$

$$v_k(2, 1, \theta) \leq \frac{1}{3} - \frac{107}{16} x^2.$$

In inequality(15), replace 4 with these bounds. Then

$$|R_3^*| \leq 2 - \frac{107\pi^2}{24} \frac{1}{k^2} + 176 \left(\frac{1}{2}\right)^k.$$

We can show that $176 \left(\frac{1}{2}\right)^k < \frac{107\pi^2}{24} \frac{1}{k^2}$. □

3.4. Expansion of the RSD Method (2). For the case “ $k = 12n, \theta = 5\pi/6$ ”, we need the next lemma.

Lemma 3.3. Let k be the integer such that $k = 12n$ for $\exists n \in \mathbb{N}$. If n is even, then $F_{k,3}^*(5\pi/6) > 0$. On the other hand, if n is odd, then $F_{k,3}^*(5\pi/6) < 0$.

Proof. Let $k = 12n$ ($n \geq 1$). By the definition of $E_{k,3}^*(z), F_{k,3}^*(z)$ (cf. (4),(12)), we have

$$F_{k,3}^*(5\pi/6) = \frac{e^{i5(k/12)\pi}}{3^{k/2} + 1} \left(3^{k/2} E_k \left(\frac{-3 + \sqrt{3}i}{2} \right) + E_k \left(\frac{-\sqrt{3} + i}{2\sqrt{3}} \right) \right).$$

By using the equations (9), for $k = 12n$,

$$F_{12n,3}^*(5\pi/6) = 2e^{in\pi} \frac{3^{6n}}{3^{6n} + 1} F_{12n}(2\pi/3),$$

where $\frac{3^{6n}}{3^{6n} + 1} > 0$, $F_{12n}(2\pi/3) = 2 \cos(4n\pi) + R_1 > 0$ (cf. (10)). So the sign($\pm$) of $F_{k,3}^*(5\pi/6)$ is that of $e^{in\pi}$. Thus the proof is complete. □

3.5. Valence formula for $\Gamma_0^*(3)$.

Proposition 3.3. *Let f be a modular function of weight k for $\Gamma_0^*(3)$, which is not identically zero. We have*

$$(16) \quad v_\infty(f) + \frac{1}{2}v_{i/\sqrt{3}}(f) + \frac{1}{6}v_{\rho_3}(f) + \sum_{\substack{p \in \Gamma_0^*(3) \setminus \mathbb{H} \\ p \neq i/\sqrt{3}, \rho_3}} v_p(f) = \frac{k}{6},$$

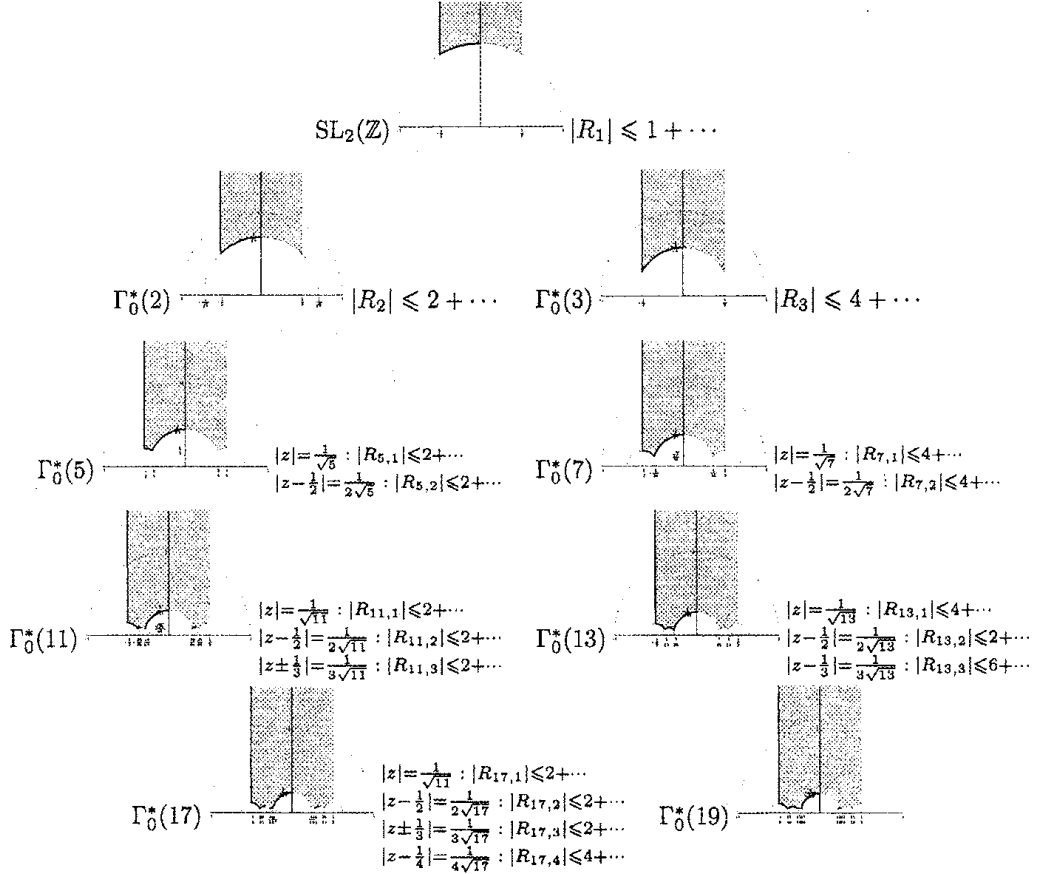
where $\rho_3 := e^{5\pi/6}/\sqrt{3}$.

Remark 2. *Let $k \geq 4$ be an even integer. We have*

$$\begin{aligned} v_{i/\sqrt{3}}(E_{k,3}^*) &= s_k \quad (s_k = 0, 1 \text{ such that } 2s_k \equiv k \pmod{4}), \\ v_{\rho_3}(E_{k,3}^*) &= t_k \quad (t_k = 0, 1, 2, 3, 4, 5 \text{ such that } -2t_k \equiv k \pmod{12}). \end{aligned}$$

4. ON SOME OTHER GROUPS

4.1. **Frick group $\Gamma_0^*(p)$.** For $\text{SL}_2(\mathbb{Z})$ and $\Gamma_0^*(p)$ for prime p such that $2 \leq p \leq 19$, their fundamental domains and bounds for Eisenstein series are following:

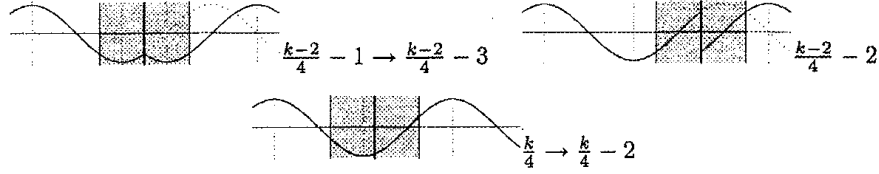


4.1.1. $\Gamma_0^*(5)$. We can write:

$$\begin{aligned} |z| = \frac{1}{\sqrt{5}} : e^{\frac{ik\theta_1}{2}} E_{k,5}^* \left(\frac{e^{i\theta_1}}{\sqrt{5}} \right) &= 2 \cos \left(\frac{k\theta_1}{2} \right) + R_{5,1}, \\ \left| z - \frac{1}{2} \right| = \frac{1}{2\sqrt{5}} : e^{\frac{ik\theta_2}{2}} E_{k,5}^* \left(\frac{e^{i\theta_2}}{2\sqrt{5}} - \frac{1}{2} \right) &= 2 \cos \left(\frac{k\theta_2}{2} \right) + R_{5,2}. \end{aligned}$$

We have $\theta_2 = \theta_1 - \pi/2$, and $e^{\frac{ik\theta_2}{2}} = (-i)^{k/2} e^{\frac{ik\theta_1}{2}}$.

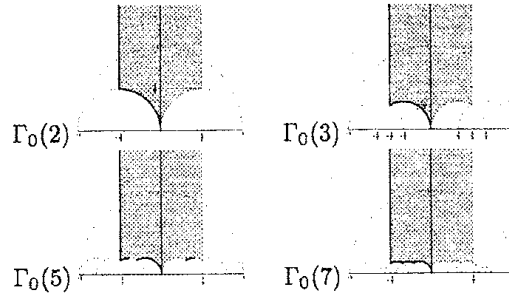
We consider the bound for $\theta_1 \in [\pi/2, \pi - \alpha - \pi/k]$ and $\theta_2 \in [\pi/2 - \alpha - k/2, \pi/2]$, where $\tan \alpha = 1/2$. If $4 \nmid p$, then we want to show $\frac{k-2}{4} - 1$ zeros are on the arcs. Also, if $4 \mid p$, then we want $\frac{k}{4}$ zeros to be there.



We proved $E_{k,5}^*$ has almost all zeros on the arcs $|z| = 1/\sqrt{5}$ and $|z - 1/2| = 1/2\sqrt{5}$ for $k \geq 12$, except for at most 2 points.

Similarly, we proved $E_{k,11}^*$ has almost all zeros on the arcs $|z| = 1/\sqrt{11}$, $|z - 1/2| = 1/2\sqrt{11}$, and $|z - 1/3| = 1/3\sqrt{11}$ for $k \geq 16$, except for at most 4 points.

4.2. Congruence subgroup $\Gamma_0(p)$. For $\Gamma_0^*(p)$ for prime p , their fundamental domains are following:



$\Gamma_0^*(p)$ has two cusps ∞ and 0 . Thus we define Eisenstein Series for $\Gamma_0^*(p)$ as follows:

$$E_{k,p}^\infty(z) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p|c}} (cz+d)^{-k}, \quad \text{for cusp } \infty.$$

$$E_{k,p}^0(z) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p \nmid c}} (cz+d)^{-k}, \quad \text{for cusp } 0.$$

4.2.1. $\Gamma_0(2)$. Considering that $\begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix} : z \mapsto -1 - \frac{1}{2z}$, we have

$$E_{k,2}^\infty\left(\frac{e^{i\theta}}{2} - \frac{1}{2}\right) = \left(-\frac{1}{2} - \frac{i}{2} \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)^k E_{k,2}^0\left(-\frac{1}{2} - \frac{i}{2} \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)$$

Furthermore, we have

$$e^{\frac{ik\theta}{2}} E_{k,2}^\infty\left(\frac{e^{i\theta}}{2} - \frac{1}{2}\right) = 2 \cos(k\theta/2) + R : \text{real.}$$

However, for $\theta \in [\pi/2, 0]$, we have $|R| \leq \infty$. This problem is much more difficult than $\Gamma_0^*(p)$. We proved

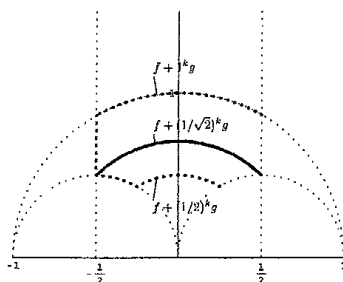
$$|R| < 2 \quad \text{for } \theta \in [\pi/2, \pi/6] \text{ and } k \geq 8.$$

We also proved

$$|R| < 2 \quad \text{for } \theta \in [\pi/2, \pi/12] \text{ and } k \geq 16,$$

$$|R| < 2 \quad \text{for } \theta \in [\pi/2, \pi/18] \text{ and } k \geq 20.$$

Finally, let $f = E_{k,2}^\infty(z)$, and $g = E_{k,2}^0(z)$. The locations of zeros of some linear combinations of f and g are following:



It is interesting question to consider the location of zeros of $f + r^k g$.

Remark 3. Getz[G] considered a similar problem for the zeros of extremal modular forms of $SL_2(\mathbb{Z})$. It seems that similar results do not hold for extremal modular forms of $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$. We plan to look into this in the near future.

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