

A duality theorem for iterated infinite cyclic coverings

by

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This paper will derive a dual structure immanent in a manifold by establishing a duality analogous to the classical Poincaré duality for iterated infinite cyclic covers of a compact manifold. An application will be done in Section three. Spaces and maps will be considered in the piecewise-linear category, unless otherwise stated.

1. Preliminaries

Let X^n be a compact, connected, piecewise-linear n -manifold and suppose we are given a sequence $X^{(N)} \twoheadrightarrow X^{(N-1)} \twoheadrightarrow \dots \twoheadrightarrow X^{(2)} \twoheadrightarrow X^{(1)} \twoheadrightarrow X^{(0)} = X^n$ such that for each i , $i = 1, 2, \dots, N-1$, $X^{(i+1)} \twoheadrightarrow X^{(i)}$ is the composite of $X^{(i+1)} \twoheadrightarrow \hat{X}^{(i)} \twoheadrightarrow X^{(i)}$, where $\hat{X}^{(i)} \twoheadrightarrow X^{(i)}$ is a finitely sheeted (possibly irregular) connected covering and $X^{(i+1)} \twoheadrightarrow \hat{X}^{(i)}$ is an infinite cyclic connected covering (,that is, a regular connected covering whose covering translation group is infinite cyclic). We use the notation $\hat{Y}$ for a finitely sheeted connected cover of a space Y throughout the paper.

1.1 Definition. For $k \leq 0$ $f(k)$ is the class of arbitrary spaces. $f(1)$ is the class of connected spaces with finitely generated fundamental groups. For $k \geq 2$, $f(k)$ is the class of connected spaces Y with $\pi_1(Y)$ finitely generated and such that the integral group ring $Z[\pi_1(Y)]$ is Noetherian and $H_1(\bar{Y}; Z)$ is a finitely generated left $Z[\pi_1(Y)]$ -module for $i \leq k-1$ and $H_k(\bar{Y}; Q)$ is a finitely generated left $Q[\pi_1(Y)]$ -module. (Throughout the paper, $\bar{Y}$ denotes the universal cover of a space Y .)

1.2 Lemma. $\pi_1(X^{(N)})$ is finitely presented, if it is finitely generated.

This follows rapidly from **Lemma 1.3** below and the induction on N [by considering $X^{(N)} \times S^m$ for a large m , if possible].

1.3 Lemma. Let $n \geq 5$. If $\pi_1(X^{(1)})$ is finitely generated, then there is a map $\phi: X \rightarrow S^1$ such that for a point $p \in S^1$ $\phi^{-1}(p)$ is a connected compact (piecewise-linear, proper bicollared) submanifold of $X^{(1)}$ such that the natural homomorphism $\pi_1(\phi^{-1}(p)) \rightarrow \pi_1(X^{(1)})$ is an isomorphism.

Proof. Let $X^{(1)}$ is obtained from a simplicial map $\phi_1: X \rightarrow S^1$. For a non-vertex point $p \in S^1$ $\phi_1^{-1}(p)$ is a compact submanifold of $X^{(1)}$. Since $\pi_1(X^{(1)})$ is finitely generated, from an argument of J. Stallings [2] we can assume that $\phi_1^{-1}(p)$ is connected. Next, since $n \geq 5$, we can kill the kernel of $\pi_1(\phi_1^{-1}(p)) \rightarrow \pi_1(X^{(1)})$ by a surgery by an argument of M.A.

Gutiérrez[2]. Thus we have a map $\phi: X \rightarrow S^1$ homotopic to ϕ_1 such that $\phi^{-1}(p)$ is connected compact submanifold of $X^{(1)}$ and $\pi_1(\phi^{-1}(p)) \rightarrow \pi_1(X^{(1)})$ is a monomorphism. An argument of L.P. Neuwirth[8], then, implies that this monomorphism must be an isomorphism. This completes the proof.

By Lemma 1.2, if $X^{(N)} \in f(1)$, then $\pi_1(X^{(N)})$ is finitely presented. So, one may note that the class $f(k)$ is closely related to the finiteness condition studied in detail by C.T.C. Wall[3] at least for manifolds of the type of $X^{(N)}$.

It seems difficult to know whether or not the integral group ring of a group is Noetherian. A partial result on this is as follows:

1.4 Lemma. Let G_0 be a subgroup of a group G with finite index or with infinite cyclic quotient group. If $Z[G_0]$ is left Noetherian, then $Z[G]$ is left Noetherian.

Proof. If G/G_0 has a finite index, $Z[G]$ can be considered as a finitely generated left module over $Z[G_0]$. Since $Z[G_0]$ is left Noetherian, it follows immediately that $Z[G]$ is left Noetherian. If G/G_0 is infinite cyclic, then $Z[G]$ can be considered as a polynomial ring with negative exponents and with right coefficients in $Z[G_0]$. Then it follows that $Z[G]$ is left Noetherian by using the proof of the Hilbert basis theorem.

2. A duality theorem

We will state our duality theorem individually on the iterated number N of infinite cyclic coverings.

Duality Theorem 0 (Poincaré Duality). Suppose $\hat{X}^{(0)}$ is orientable. There is a duality

$$\cap \hat{\mu}^{(0)}: H^i(\hat{X}^{(0)}, \partial \hat{X}^{(0)}; \mathbb{Z}) \approx H_{n-i}(\hat{X}^{(0)}; \mathbb{Z}) \text{ for all } i.$$

This is widely known, since $\hat{X}^{(0)}$ is compact.

Duality Theorem 1. Suppose $\hat{X}^{(1)}$ is orientable. If $H_i(\hat{X}^{(1)}, \partial \hat{X}^{(1)}; \mathbb{Z})$ is finitely generated abelian for $i \leq m$ and $\dim_{\mathbb{Q}} H_{m+1}(\hat{X}^{(1)}, \partial \hat{X}^{(1)}; \mathbb{Q}) < +\infty$, then there is a duality

$$\cap \hat{\mu}^{(1)}: H^i(\hat{X}^{(1)}, \partial \hat{X}^{(1)}; \mathbb{Z}) \approx H_{n-1-i}(\hat{X}^{(1)}; \mathbb{Z})$$

for all $i \leq m$ and for $i = m+1$ this map is a monomorphism.

This is a simple version of a known result. (See [4].)

An outline of the proof is as follows: Let $N_p^+ \supset N_{p+1}^+ \supset \dots$ and $N_q^- \supset N_{q+1}^- \supset \dots$ be the neighborhoods of the two ends of $X^{(1)}$ as in [4] or [6]. Let $\hat{N}_p^+ \supset \hat{N}_{p+1}^+ \supset \dots$ and $\hat{N}_q^- \supset \hat{N}_{q+1}^- \supset \dots$ be the lifts, which are still the neighborhoods of ends of $\hat{X}^{(1)}$, since $\hat{X}^{(1)}$ has still two ends. (See D.B.A. Epstein [1].) Using these neighborhoods, from an analogous method of [4] or [6]

we obtain that

$$\cap \hat{\mu}^{(1)}: H^i(\hat{X}^{(1)}, \partial \hat{X}^{(1)}; \mathbb{Z}) \xrightarrow{\cong} \varprojlim_{p, q \rightarrow \infty} H^{i+1}(\hat{X}^{(1)}, \partial \hat{X}^{(1)}; \mathbb{Z}) \cap \hat{N}_p^+ \cap \hat{N}_q^-$$

$= H_c^{i+1}(\hat{X}^{(1)}, \partial\hat{X}^{(1)}; Z) \cap[\hat{X}^{(1)}] \approx H_{n-1-i}(\hat{X}^{(1)}; Z)$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$. This completes the outlined proof.

Duality Theorem 2. Suppose $\hat{X}^{(2)}$ is orientable and $X^{(1)} \in f(m+2)$ and $\partial X^{(1)}$ has at most finitely many components each of which is in $f(m+1)$. If $H_i(\hat{X}^{(2)}, \partial\hat{X}^{(2)}; Z)$ is finitely generated abelian for $i \leq m$ and $\dim_Q H_{m+1}(\hat{X}^{(2)}, \partial\hat{X}^{(2)}; Q) < +\infty$, then there is a duality

$$\cap\mu^{(2)}: H^i(\hat{X}^{(2)}, \partial\hat{X}^{(2)}; Z) \approx H_{n-2-i}(\hat{X}^{(2)}; Z)$$

for all $i \leq m$ and for $i = m+1$ this map is a monomorphism.

To prove Duality Theorem 2, we use the following lemma:

2.1 Lemma. Let $n \geq 6$ and $1 \leq k \leq (n-3)/2$. Suppose $X^{(1)} \in f(k)$ and $\partial X^{(1)}$ has at most finitely many components each of which is in $f(k-1)$. Then there is a proper map $\varphi: X^{(1)} \rightarrow \mathbb{R}^1$ such that for a point $p \in \mathbb{R}^1$ $\varphi^{-1}(p) = M$ is a compact connected (piecewise-linear proper bicollared) submanifold with $\partial X^{(1)} \cup M$ connected and such that $\pi_1(M) \cong \pi_1(X^{(1)})$ and $\pi_i(\partial M) \cong \pi_i(\partial X^{(1)})$ $i = 0, 1$ (if $k \geq 2$) and such that for any cover $(\tilde{X}^{(1)}, \partial\tilde{X}^{(1)})_{\tilde{U}M}$ of $(X^{(1)}, \partial X^{(1)})_{UM}$ $H_i(\tilde{X}^{(1)}, \partial\tilde{X}^{(1)})_{\tilde{U}M} = 0$ with integral coefficients for $i \leq k-1$ or $i \leq k=1$ and with rational coefficients for $i = k \geq 2$. Furthermore, we can have $H_i(\tilde{M}) \cong H_i(\tilde{X}^{(1)})$ with integral coefficients for $i \leq k-1$ or $i \leq k=1$ and with rational coefficients for $i = k \geq 2$. For $k \geq 2$, $H_1(\partial\tilde{M}) \cong H_1(\partial\tilde{X}^{(1)})$

with integral coefficients for $i \leq k-2$ or $i \leq k-1=1$ and with rational coefficients for $i = k-1 \geq 2$.

Proof. By Lemma 1.3 there is a map $\varphi_1: X \rightarrow S^1$ such that $\varphi_1^{-1}(p_1)$ is a compact connected submanifold of $X^{(1)}$ with $\partial X^{(1)} \cup \varphi_1^{-1}(p_1)$ connected and $\pi_1(\varphi_1^{-1}(p_1)) \approx \pi_1(X^{(1)})$ and $\pi_i(\partial \varphi_1^{-1}(p_1)) \approx \pi_i(\partial X^{(1)})$, $i = 0, 1$ (if $k \geq 2$). [Since $\partial X^{(1)}$ has only finite components, each components of $\partial X^{(1)}$ must intersect with $\varphi_1^{-1}(p_1)$. For $k \geq 2$ first apply Lemma 1.3 for each component of $\partial X^{(1)}$.] Let $\tilde{\varphi}: X^{(1)} \rightarrow \mathbb{R}^1$ be a lift of φ_1 . Note that $\varphi^{-1}(p) = \varphi_1^{-1}(p_1)$ for a lift $p \in \mathbb{R}^1$ of p_1 . The rest of the proof follows from Lemma 2.2 below and the following simple assertion: If $H_i(\tilde{X}^{(1)}, \tilde{M}) = 0$ and $H_{i-1}(\partial \tilde{X}^{(1)}, \partial \tilde{M}) = 0$, then $H_i(\tilde{X}^{(1)}, \partial \tilde{X}^{(1)} \cup \tilde{M}) = 0$. This follows from the homology exact sequence of the triple $\tilde{X}^{(1)} \supset \partial \tilde{X}^{(1)} \cup \tilde{M} \supset \tilde{M}$. In fact,

$$\begin{array}{ccccc} H_i(\tilde{X}^{(1)}, \tilde{M}) & \supset & H_i(\tilde{X}^{(1)}, \partial \tilde{X}^{(1)} \cup \tilde{M}) & \supset & H_{i-1}(\partial \tilde{X}^{(1)} \cup \tilde{M}, \tilde{M}) \\ \parallel & & & & \parallel \\ 0 & & & & H_{i-1}(\partial \tilde{X}^{(1)}, \partial \tilde{M}) \\ & & & & \parallel \\ & & & & 0 \end{array}$$

Hence $H_i(\tilde{X}^{(1)}, \partial \tilde{X}^{(1)} \cup \tilde{M}) = 0$.

2.2 Lemma. Let $2 \leq k \leq (n-3)/2$ and $X^{(1)} \in f(k)$. Assume $\pi_1(\varphi^{-1}(p)) \approx \pi_1(X^{(1)})$ for a proper map $\varphi: X^{(1)} \rightarrow \mathbb{R}^1$ with $\varphi^{-1}(p)$ a compact connected submanifold. Then there is a proper map $\varphi': X^{(1)} \rightarrow \mathbb{R}^1$ homotopic to φ by a homotopy with compact support such that $\varphi'^{-1}(p)$ is a compact connected submanifold and $\pi_1(\varphi'^{-1}(p)) \approx \pi_1(X^{(1)})$ and $H_i(\tilde{X}^{(1)}, \tilde{\varphi}'^{-1}(p)) = 0$ with integral coefficients for $i \leq k-1$ and rational coefficients for $i = k$.

Moreover we can have $H_i(\tilde{\varphi}'^{-1}(p)) \simeq H_i(\tilde{X}^{(1)})$ with integral coefficients for $i \leq k-1$ and rational coefficients for $i = k$.

Proof. $H_i(\tilde{X}^{(1)}, \tilde{\varphi}'^{-1}(p)) = 0$ follows from a surgical argument of L.C. Siebenmann [10]. [First, note that $H_2(\bar{X}^{(1)}, \bar{\varphi}'^{-1}(p)) = H_2(\bar{X}_+, \bar{\varphi}'^{-1}(p)) + H_2(\bar{X}_-, \bar{\varphi}'^{-1}(p)) = \pi_2(X_+, \varphi'^{-1}(p)) + \pi_2(X_-, \varphi'^{-1}(p))$ is a finitely generated $\mathbb{Z}[\pi_1(X^{(1)})]$ -module (or $\mathbb{Q}[\pi_1(X^{(1)})]$ -module for $k = 2$), where $\bar{X}^{(1)} = \bar{X}_+ \cup \bar{X}_-$ with $\bar{X}_+ \cap \bar{X}_- = \bar{\varphi}'^{-1}(p)$.

Hence by killing the generators, we may have a proper map

$\varphi_1: X^{(1)} \rightarrow \mathbb{R}^1$ homotopic to φ by a homotopy with compact support such that $\varphi_1^{-1}(p)$ is a compact connected submanifold and

$\pi_1(\varphi_1^{-1}(p)) \simeq \pi_1(X^{(1)})$ and $H_2(\bar{X}^{(1)}, \bar{\varphi}_1^{-1}(p)) = 0$. Similar for $i \geq 2$.] Next, note that $\pi_i(X^{(1)}, \varphi'^{-1}(p)) = 0$ with $\mathbb{Z}$ coefficients for $i \leq k-1$ and $\mathbb{Q}$ coefficients for $i = k$, and $\pi_{k+1}(X^{(1)}, \varphi'^{-1}(p)) = H_{k+1}(\bar{X}^{(1)}, \bar{\varphi}'^{-1}(p))$ with $\mathbb{Q}$ coefficients by the relative

Hurewicz isomorphism theorem (modulo torsion). Consider the exact

sequence of the following part: $H_k(\bar{X}^{(1)}, \bar{\varphi}'^{-1}(p)) \rightarrow H_{k-1}(\bar{\varphi}'^{-1}(p))$

$\xrightarrow{i_*} H_{k-1}(\bar{X}^{(1)}) \rightarrow 0$. Since $H_{k-1}(\varphi'^{-1}(p))$ is finitely generated over $\mathbb{Z}[\pi_1(X^{(1)})]$ and $\mathbb{Z}[\pi_1(X^{(1)})]$ is Noetherian, we obtain

that $\text{Ker } i_*$ is finitely generated over $\mathbb{Z}[\pi_1(X^{(1)})]$. Since

$H_k(\bar{X}^{(1)}, \bar{\varphi}'^{-1}(p)) = H_k(\bar{X}_+, \bar{\varphi}'^{-1}(p)) + H_k(\bar{X}_-, \bar{\varphi}'^{-1}(p)) = \pi_k(X_+, \varphi'^{-1}(p)) + \pi_k(X_-, \varphi'^{-1}(p))$, we can kill the generators of $\text{Ker } i_*$ by a

surgery and hence we can assume that the map $\pi_k(X^{(1)}, \varphi'^{-1}(p)) =$

$H_k(\bar{X}^{(1)}, \bar{\varphi}'^{-1}(p)) \rightarrow H_{k-1}(\bar{\varphi}'^{-1}(p))$ with $\mathbb{Z}$ coefficients is a trivial

homomorphism. This implies that $i_*: H_{k-1}(\varphi'^{-1}(p); \mathbb{Z}) \simeq H_{k-1}(\tilde{X}^{(1)}; \mathbb{Z})$.

With $\mathbb{Q}$ coefficients, the same argument is applicable for the

following part: $H_{k+1}(\bar{X}^{(1)}, \bar{\varphi}'^{-1}(p)) \rightarrow H_k(\bar{\varphi}'^{-1}(p)) \xrightarrow{i_*} H_k(\bar{X}^{(1)}) \rightarrow 0$.

Thus we can assume $H_k(\varphi'^{-1}(p); \mathbb{Q}) \simeq H_k(\tilde{X}^{(1)}; \mathbb{Q})$. This completes the

proof.

By applying Lemma 2.2 for $\partial X^{(1)}$ (if $k \geq 2$) and then for $X^{(1)}$, we complete the proof of Lemma 2.1.

2.3. Proof of Duality Theorem 2. First consider the case that $n \geq 6$ and $m+2 \leq (n-3)/2$. By Lemma 2.1 we have $H_i(\hat{X}^{(2)}, \partial \hat{X}^{(2)} \cup \hat{M}^{(1)}) = 0$ with $\mathbb{Z}$ coefficients for $i \leq m+1$ and $\mathbb{Q}$ coefficients for $i = m+2$. By using a covering translation of $X^{(1)}$, choose a copy M' of M in $X^{(1)}$ so that $M' \cap M = \emptyset$. M and M' separate $X^{(1)}$ into three parts. Let V be the compact part and N_+ , N_- be the others. (See Fig. 1.)

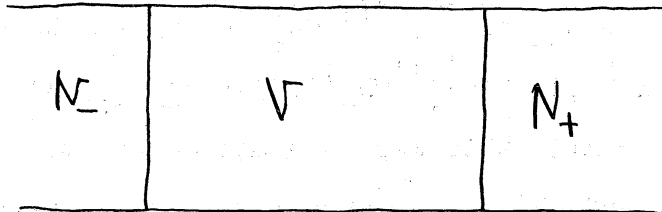


Fig. 1

Note that $X^{(1)}$ can be covered by ascending compact manifolds $V_0 < V_1 < V_2 < V_3 < \dots$ such that each V_i is separated by two copies of M obtained by covering translations of $X^{(1)}$. For $m+2 \geq 1$ we note that M , V , N_+ and N_- are connected and have the fundamental groups isomorphic to $\pi_1(X^{(1)})$ by inclusions. Since $H_i(\hat{X}^{(2)}, \partial \hat{X}^{(2)} \cup \hat{M}^{(1)}) = 0$, from the Mayer-Vietoris sequence we obtain that

$$\delta : H^i(\hat{X}^{(2)}, \partial \hat{X}^{(2)}) \approx H^{i+1}(\hat{X}^{(2)}, \partial \hat{X}^{(2)} \cup \hat{N}_+^{(1)} \cup \hat{N}_-^{(1)})$$

$$\partial : H_{i+1}(\hat{X}^{(2)}, \partial \hat{X}^{(2)} \cup \hat{N}_+^{(1)} \cup \hat{N}_-^{(1)}) \approx H_i(\hat{X}^{(2)}, \partial \hat{X}^{(2)})$$

with Z coefficients for $i \leq m$ and Q coefficients for $i = m+1$. Further the coboundary δ is injective for $i = m+1$ with Z coefficients. By excision, we have $H(\hat{X}^{(2)}, \partial \hat{X}^{(2)} \cup \hat{N}_+^{(1)} \cup \hat{N}_-^{(1)}; Z) = H(\hat{V}^{(1)}, \partial \hat{V}^{(1)}; Z)$, where $H = H^*$ or H_* . Since $H_i(\hat{X}^{(2)}, \partial \hat{X}^{(2)}; Z)$ is finitely generated abelian for $i \leq m$ and $\dim_{Q H_{m+1}}(\hat{X}^{(2)}, \partial \hat{X}^{(2)}; Q) < +\infty$, $H_i(\hat{V}^{(1)}, \partial \hat{V}^{(1)}; Z)$ is finitely generated abelian for $i \leq m+1$ and $\dim_{Q H_{m+2}}(\hat{V}^{(1)}, \partial \hat{V}^{(1)}; Q) < +\infty$. Since $\hat{X}^{(2)}$ is orientable, $\hat{V}^{(1)}$ is orientable. Hence by Duality Theorem 1 we have $\cap \hat{\mu}^{(1)} : H^{i+1}(\hat{V}^{(1)}, \partial \hat{V}^{(1)}; Z) \approx H_{n-2-i}(\hat{V}^{(1)}; Z)$ for $i \leq m$ and for $i = m+1$ this map is a monomorphism, where $\hat{\mu}^{(1)} \in H_{n-1}(\hat{V}^{(1)}, \partial \hat{V}^{(1)}; Z) = H_{n-1}(\hat{X}^{(2)}, \partial \hat{X}^{(2)} \cup \hat{N}_+^{(1)} \cup \hat{N}_-^{(1)}; Z)$. We shall show that $\cap \hat{\mu}^{(1)} : H^{i+1}(\hat{X}^{(2)}, \partial \hat{X}^{(2)} \cup \hat{N}_+^{(1)} \cup \hat{N}_-^{(1)}; Z) \approx H_{n-2-i}(\hat{X}^{(2)}; Z)$ for $i \leq m$ and for $i = m+1$ this map is a monomorphism. Let V' be such that $V' \supset V$ and V' is a compact manifold separated by two copies of M by covering translations. Further let N'_+, N'_- be two components of $\text{cl}(X^{(1)} - V')$ such that $N'_+ \subset N_+$ and $N'_- \subset N_-$ and let $A = \text{cl}(V' - V)$. Consider the following commutative diagram ($i \leq m$):

$$\begin{array}{ccc} H^{i+1}(\hat{X}^{(2)}, \partial \hat{X}^{(2)} \cup \hat{N}_+^{(1)} \cup \hat{N}_-^{(1)}) & \approx & H^{i+1}(\hat{X}^{(2)}, \partial \hat{X}^{(2)} \cup \hat{N}_+^{(1)} \cup \hat{N}_-^{(1)}) \\ \downarrow \approx & & \downarrow \approx \\ H^{i+1}(\hat{V}^{(1)}, \partial \hat{V}^{(1)}) & \approx & H^{i+1}(\hat{V}^{(1)}, \hat{A}^{(1)} \cup \partial \hat{V}^{(1)}) \approx H^{i+1}(\hat{V}^{(1)}, \partial \hat{V}^{(1)}) \end{array}$$

$$\begin{array}{ccc} \cap \hat{\mu}_V^{(w)} \downarrow \approx & \cap \hat{\mu}_{V,A}^{(w)} \downarrow & \cap \hat{\mu}_V^{(w)} \downarrow \approx \\ H_{n-2-i}(\hat{V}^{(1)}) & \xrightarrow{i} H_{n-2-i}(\hat{V}',^{(1)}) & = H_{n-2-i}(\hat{V},^{(1)}) \end{array}$$

This shows that the inclusion $i: \hat{V}^{(1)} \subset \hat{V}',^{(1)}$ induces an isomorphism $i_*: H_{n-2-i}(\hat{V}^{(1)}; Z) \approx H_{n-2-i}(\hat{V}',^{(1)}; Z)$. Using $\lim_{\substack{\rightarrow \\ V}} H_*(\hat{V}^{(1)}; Z) \approx H_*(\hat{X}^{(2)}; Z)$, we obtain that the inclusion $j: \hat{V}^{(1)} \subset \hat{X}^{(2)}$ must induce an isomorphism $j_*: H_{n-2-i}(\hat{V}^{(1)}; Z) \approx H_{n-2-i}(\hat{X}^{(2)}; Z)$. For $i = m+1$ an analogous discussion shows that $j_*: H_{n-3-m}(\hat{V}^{(1)}; Z) \rightarrow H_{n-3-m}(\hat{X}^{(2)}; Z)$ is a monomorphism. [Use additional facts of Lemma 2.1 that $H_m(\hat{N}^{(1)}, \hat{N}',^{(1)}; Z) = H_{m-1}(\partial \hat{N}^{(1)}, \partial \hat{N}',^{(1)}; Z) = 0$ to prove that $H^{m+2}(\hat{V},^{(1)}, \hat{A}^{(1)} \cup \partial \hat{V}^{(1)}; Z) \supset H^{m+2}(\hat{V},^{(1)}, \partial \hat{V},^{(1)}; Z)$ is injective.] Therefore, combined with $\cap \hat{\mu}^{(1)}: H^{i+1}(\hat{V}^{(1)}, \partial \hat{V}^{(1)}; Z) \supset H_{n-2-i}(\hat{V}^{(1)}; Z)$, we have that $\cap \hat{\mu}^{(1)}: H^{i+1}(\hat{X}^{(2)}, \partial \hat{X}^{(2)} \cup \hat{N}_+^{(1)} \cup \hat{N}_-^{(1)}; Z) \supset H_{n-2-i}(\hat{X}^{(2)}; Z)$ is an isomorphism for $i \leq m$ and is a monomorphism for $i = m+1$. Let $\hat{\mu}^{(2)}$ be the image of $\hat{\mu}^{(1)}$ via boundary homomorphism $\partial: H_{n-1}(\hat{X}^{(2)}, \partial \hat{X}^{(2)} \cup \hat{N}_+^{(1)} \cup \hat{N}_-^{(1)}; Z) \supset H_{n-2}(\hat{X}^{(2)}, \partial \hat{X}^{(2)}; Z)$ (of the Mayer-Vietoris sequence). The composite $H^i(\hat{X}^{(2)}, \partial \hat{X}^{(2)}; Z) \xrightarrow{\partial} H^{i+1}(\hat{X}^{(2)}, \hat{X}^{(2)} \cup \hat{N}_+^{(1)} \cup \hat{N}_-^{(1)}; Z) \xrightarrow{\cap \hat{\mu}^{(1)}} H_{n-2-i}(\hat{X}^{(2)}; Z)$ is given by the map $\cap \hat{\mu}^{(2)}$. Thus we have a desired result for the case that $n \geq 6$ and $m+2 \leq (n-3)/2$. For the case that $n \not\geq 6$ or $m+2 \not\leq (n-3)/2$, choose a sufficiently large integer k such that $n+k \geq 6$, $m+2 < (n+k-3)/2$, and consider $S^{(k)} \times \hat{X}^{(2)}$. From the above argument, the map

$$\begin{array}{ccc} \cap \hat{\mu}_k^{(2)}: H^i(S^k \times \hat{X}^{(2)}, S^k \times \partial \hat{X}^{(2)}; Z) & \rightarrow & H_{n+k-2-i}(S^k \times \hat{X}^{(2)}; Z) \\ \parallel & & \parallel \\ H^0(S^k; Z) \times H^i(\hat{X}^{(2)}, \partial \hat{X}^{(2)}; Z) & & H_k(S^k; Z) \times H_{n-2-i}(\hat{X}^{(2)}; Z) \end{array}$$

is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$.

($\times$ denotes the cross product.) Since $H_{n+k-2}(S^k \times \hat{X}^{(2)}, S^k \times \partial \hat{X}^{(2)}; Z) = H_k(S^k; Z) \times H_{n-2}(\hat{X}^{(2)}, \partial \hat{X}^{(2)}; Z)$, $\hat{\mu}_k^{(2)}$ can be written as $[S^k] \times \hat{\mu}^{(2)}$, where $[S^k]$ is a fundamental class of S^k and $\hat{\mu}^{(2)} \in H_{n-2}(\hat{X}^{(2)}, \partial \hat{X}^{(2)}; Z)$. Using the identity $(1 \times u) \cap ([S^k] \times \hat{\mu}^{(2)}) = [S^k] \times (u \cap \hat{\mu}^{(2)})$ for all $u \in H^1(\hat{X}^{(2)}, \partial \hat{X}^{(2)}; Z)$, we have an isomorphism

$$\cap \hat{\mu}^{(2)}: H^1(\hat{X}^{(2)}, \partial \hat{X}^{(2)}; Z) \approx H_{n-2-i}(\hat{X}^{(2)}; Z), \quad i \leq m$$

and a monomorphism for $i = m+1$. This completes the proof.

For $N \geq 3$ some further restrictions on $X^{(N)}$ are needed.

Duality Theorem N ($N \geq 3$). Suppose $\hat{X}^{(N)}$ is orientable and $X^{(N-1)} \in f(m+N)$ and $\partial X^{(N-1)}$ has at most two components such that each component $B^{(N-1)}$ of $\partial X^{(N-1)}$ is in $f(m+N-1)$ and the inclusion $B^{(N-1)} \subset X^{(N-1)}$ induces an isomorphism $\pi_1(B^{(N-1)}) \approx \pi_1(X^{(N-1)})$. If $H_1(\hat{X}^{(N)}; Z)$ and $H_{i-1}(\partial \hat{X}^{(N)}; Z)$ are finitely generated abelian for $i \leq m+N-3$ and $\dim_Q H_{m+N-2}(\hat{X}^{(N)}; Q) < +\infty$ and $\dim_Q H_{m+N-3}(\partial \hat{X}^{(N)}; Q) < +\infty$, then there is a duality $\cap \hat{\mu}^{(N)}: H^1(\hat{X}^{(N)}, \partial \hat{X}^{(N)}; Z) \approx H_{n-N-i}(\hat{X}^{(N)}; Z)$ for $i \leq m$ and for $i = m+1$ this map is a monomorphism.

Proof. Note that $X^{(N-1)} \in f(m+N)$ implies $X^{(1)} \in f(m+N)$ by using Lemma 1.4. By a similar discussion of 2.3 we can assume that the dimension n of X is sufficiently large so that $n \geq 6$ and $m+N \leq (n-3)/2$. We proceed the proof by induction on N . Let $N = 3$ and first suppose $\partial X = \emptyset$. Since $X^{(1)} \in f(m+3)$, it follows from Lemma 2.1 that $H_1(\bar{M}) \approx H_1(\bar{X}^{(1)})$ with Z coefficients for $i \leq m+2$ and Q coefficients for $i = m+3$. Hence $X^{(2)} \in f(m+3)$ implies $M^{(1)} \in f(m+3)$. Let $X^{(1)} = N_- UVUN_+$ as in 2.3, where $\partial V = MUM'$. Consider the following part of Mayer-Vietoris sequence:

$H_i(\bar{M}^{(1)} \cup \bar{M}'^{(1)}) \rightarrow H_i(\bar{N}_+^{(1)} \cup \bar{N}_-^{(1)}) + H_i(\bar{V}^{(1)}) \rightarrow H_i(\bar{X}^{(2)})$. Since $M^{(1)}, M'^{(1)} \in f(m+3)$ and $X^{(2)} \in f(m+3)$, we have $V^{(1)} \in f(m+3)$. By Lemma 2.1 $H_{i+1}(\hat{V}^{(2)}, \partial \hat{V}^{(2)}) = H_{i+1}(\hat{X}^{(3)}, \hat{N}_+^{(2)} \cup \hat{N}_-^{(2)}) \cong H_i(\hat{X}^{(3)})$ and $H^i(\hat{X}^{(3)}) \cong H^{i+1}(\hat{X}^{(3)}, \hat{N}_+^{(2)} \cup \hat{N}_-^{(2)}) = H^{i+1}(\hat{V}^{(2)}, \hat{V}^{(2)})$ with $\mathbb{Z}$ coefficients for $i \leq m+1$ and $\mathbb{Q}$ coefficients for $i = m+2$. In particular, $H_{i+1}(\hat{V}^{(2)}, \partial \hat{V}^{(2)}; \mathbb{Z})$ is finitely generated abelian for $i \leq m$ and $\dim_{\mathbb{Q}} H_{m+2}(\hat{V}^{(2)}, \partial \hat{V}^{(2)}; \mathbb{Q}) < +\infty$. By Duality Theorem 2, $\cap \hat{\mu}^{(2)}: H^{i+1}(\hat{V}^{(2)}, \partial \hat{V}^{(2)}; \mathbb{Z}) \rightarrow H_{n-3-i}(\hat{V}^{(2)}; \mathbb{Z})$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$. According to an analogous method of 2.3, this implies that $\cap \hat{\mu}^{(2)}: H^{i+1}(\hat{X}^{(3)}, \hat{N}_+^{(2)} \cup \hat{N}_-^{(2)}; \mathbb{Z}) \rightarrow H_{n-3-i}(\hat{X}^{(3)}; \mathbb{Z})$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$. Let $\hat{\mu}^{(3)} \in H_{n-3}(\hat{X}^{(3)}; \mathbb{Z})$ be the image of $\hat{\mu}^{(2)}$ by $\partial: H_{n-2}(\hat{X}^{(3)}, \hat{N}_+^{(2)} \cup \hat{N}_-^{(2)}; \mathbb{Z}) \rightarrow H_{n-3}(\hat{X}^{(3)}; \mathbb{Z})$. Then the map $\cap \hat{\mu}^{(3)}: H^i(\hat{X}^{(3)}; \mathbb{Z}) \cong H^{i+1}(\hat{X}^{(3)}, \hat{N}_+^{(2)} \cup \hat{N}_-^{(2)}; \mathbb{Z}) \xrightarrow{\cap \hat{\mu}^{(2)}} H_{n-3-i}(\hat{X}^{(3)}; \mathbb{Z})$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$. Next consider the case that $\partial X \neq \emptyset$. Take the double $D(X^{(2)})$ of $X^{(2)}$. Our assumption implies that $D(X^{(2)}) \in f(m+3)$. [Note that if $\partial X^{(2)}$ is connected, then $\pi_1(D(X^{(2)})) = \pi_1(X^{(2)})$, and if $\partial X^{(2)}$ has two components, then $\pi_1(D(X^{(2)})) = \pi_1(X^{(2)}) \times \mathbb{Z}$.] Using that $H_i(D(\hat{X}^{(3)}); \mathbb{Z})$ is finitely generated abelian for $i \leq m$ and $\dim_{\mathbb{Q}} H_{m+1}(D(\hat{X}^{(3)}); \mathbb{Q}) < +\infty$, from the case of manifolds with empty boundary $\cap \hat{\mu}_D^{(3)}: H^i(D(\hat{X}^{(3)}); \mathbb{Z}) \rightarrow H_{n-3-i}(D(\hat{X}^{(3)}); \mathbb{Z})$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$. Note that $\hat{X}^{(3)}$ is a retract of $D(\hat{X}^{(3)})$. We have the following commutative diagram:

$$0 \rightarrow H^i(\hat{X}^{(3)}, \partial \hat{X}^{(3)}) \rightarrow H^i(D(\hat{X}^{(3)})) \rightarrow H^i(\hat{X}',^{(3)}) \rightarrow 0$$

$$\downarrow \cap \hat{\mu}^{(3)} \quad \downarrow \cap \hat{\mu}_D^{(3)} \quad \downarrow \cap \hat{\mu}'^{(3)}$$

$$0 \rightarrow H_{n-3-i}(\hat{X}^{(3)}) \rightarrow H_{n-3-i}(D(\hat{X}^{(3)})) \rightarrow H_{n-3-i}(\hat{X}',^{(3)}, \partial \hat{X}',^{(3)}) \rightarrow 0$$

, where $\hat{X}',^{(3)}$ is another copy of $\hat{X}^{(3)}$ in $D(\hat{X}^{(3)})$ and the top and bottom sequences are exact. Let $i \leq m$. The middle vertical map $\cap \hat{\mu}_D^{(3)}$ is an isomorphism, and hence the right vertical map is surjective and the left vertical map is injective. (Let $\hat{\mu}^{(3)}$ be the image of $\hat{\mu}_D^{(3)}$ by $H_{n-3-i}(D(\hat{X}^{(3)}); Z) \rightarrow H_{n-3-i}(D(\hat{X}^{(3)}), \hat{X}',^{(3)}; Z) = H_{n-3-i}(\hat{X}^{(3)}, \partial \hat{X}^{(3)}; Z)$.) Also, consider the following commutative diagram:

$$\begin{array}{ccccccc} H^{i-1}(\partial \hat{X}^{(3)}) & \xrightarrow{\delta} & H^i(\hat{X}^{(3)}, \partial \hat{X}^{(3)}) & \xrightarrow{i^*} & H^i(\hat{X}^{(3)}) & \xrightarrow{i^*} & H^i(\partial \hat{X}^{(3)}) \\ \cap \partial \hat{\mu}^{(3)} \downarrow \cap \partial \hat{\mu}^{(3)} & & \downarrow \cap \hat{\mu}^{(3)} & & \downarrow \cap \hat{\mu}^{(3)} & & \downarrow \cap \partial \hat{\mu}^{(3)} \\ H_{n-3-i}(\partial \hat{X}^{(3)}) & \xrightarrow{i_*} & H_{n-3-i}(\hat{X}^{(3)}) & \xrightarrow{j_*} & H_{n-3-i}(\hat{X}^{(3)}, \partial \hat{X}^{(3)}) & \xrightarrow{\partial} & H_{n-4-i}(\partial \hat{X}^{(3)}) \end{array}$$

, where the top and bottom sequences are exact and the map $\cap \partial \hat{\mu}^{(3)}: H^{i-1}(\partial \hat{X}^{(3)}; Z) \rightarrow H_{n-3-i}(\partial \hat{X}^{(3)}; Z)$ is an isomorphism for $i \leq m$ and a monomorphism for $i = m+1$ by the case of manifolds with empty boundary. Since $\cap \hat{\mu}^{(3)}: H^i(\hat{X}^{(3)}, \partial \hat{X}^{(3)}; Z) \rightarrow H_{n-3-i}(\hat{X}^{(3)}; Z)$ is a monomorphism for $i \leq m$, it follows that $\cap \hat{\mu}^{(3)}: H^i(\hat{X}^{(3)}; Z) \rightarrow H_{n-3-i}(\hat{X}^{(3)}, \partial \hat{X}^{(3)}; Z)$ is a monomorphism and hence an isomorphism for $i \leq m$. By the five lemma, this implies that $\cap \hat{\mu}^{(3)}: H^i(\hat{X}^{(3)}, \partial \hat{X}^{(3)}; Z) \rightarrow H_{n-3-i}(\hat{X}^{(3)}; Z)$ is an isomorphism for $i \leq m$. This map is also a monomorphism for $i = m+1$, since $\cap \hat{\mu}_D^{(3)}: H^{m+1}(D(\hat{X}^{(3)}); Z) \rightarrow H_{n-3-(m+1)}(D(\hat{X}^{(3)}); Z)$ is injective. This completes the proof of

$N = 3$.

By assuming Duality Theorem N-1, we must show Duality Theorem N. However, the proof is quite parallel to the proof of $N = 3$ and left to the reader. This completes the proof.

3. An application

Consider a closed, connected n -manifold X^n satisfying the following (1) and (2):

(1) $\pi_1(X)$ admits a tower of subgroup $\pi_1(X) = G_0 > G_1 > \dots > G_r = \{1\}$ such that for each $i, i=1,2,\dots,r$, G_{i-1}/G_i has a finite index or is an infinite cyclic group,

(2) $\pi_i(X), 2 \leq i \leq (n-1)/2$, is finitely generated abelian and, when n is even, $\dim_{\mathbb{Q}} \pi_{n/2}(X) \otimes \mathbb{Q} < +\infty$.

The class of such manifolds X is denoted by $\mathcal{M}$.

3.1 Definition. Let $X \in \mathcal{M}$. $\pi_1(X)$ is said to have rank R , if infinite cyclic quotient groups occur at R times in a tower of $\pi_1(X)$.

We must prove

3.2 Lemma. R does not depend on a choice of towers of $\pi_1(X)$.

3.3 Definition. $\rho = \rho(X) = n - R$ is called the (topological) Kodaira dimension of $X^n \in \mathcal{M}$.

We shall show the following:

3.4 Theorem. ρ is a non-negative integer except for one.

3.5. Proof of Lemma 3.2 and Theorem 3.4. First note that $Z[\pi_1(X)]$ is Noetherian by Corollary 1.5. We apply Duality Theorem N ($N \geq 0$). Let n be even, say, $n = 2n'$. For all $i \leq n'-R$ we have $H^i(\bar{X}; Z) \approx H_{n-R-i}(\bar{X}; Z)$. When $i \leq n'-R$, $n-R-i \geq n-R - (n'-R) = n - n' = n'$. Thus, $H_i(\bar{X}; Z)$ is finitely generated abelian for $i \geq n'$, hence for all i . Let n' be odd, say, $n = 2n'+1$. For all $i \leq n'-R$, we have $H^i(\bar{X}; Z) \approx H_{n-R-i}(\bar{X}; Z)$. When $i \leq n'-R$, we have $n-R-i \geq n-R-(n'-R) = n-n' = n'+1$. Thus, $H_i(\bar{X}; Z)$ is finitely generated abelian for $i \geq n'+1$, hence for all i . As a result, for each $n \geq 1$ there is a duality $H^i(\bar{X}; Z) \approx H_{n-R-i}(\bar{X}; Z)$ for all i . This implies that R does not depend on any choice of towers of $\pi_1(X)$. By taking $i = 0$, $\rho = n-R \geq 0$. Since $\bar{X}$ is simply connected, $\rho \neq 1$. This completes the proof.

We further state individually results for X in $\mathcal{M}$ on each Kodaira dimension ρ .

3.6. $\rho = 0$. In this case, $H_*(\bar{X}; Z) = 0$, hence $\bar{X}$ is contractible, and X is $K(\pi, 1)$. In particular, $\pi_1(X)$ is torsion-free. If $\pi_1(X)$ is abelian and $n \geq 5$, X is topologically homeomorphic to an n -torus $S^1 \times \cdots \times S^1$. More generally, if $\pi_1(X)$ is a poly-infinite-cyclic group and $n \geq 5$, then according to C.T.C. Wall [4] $\bar{X}$ is homeomorphic to $\mathbb{R}^n$ and

for any closed n -manifold X' with $\pi_1(X') \approx \pi_1(X)$, any homotopy equivalence $X' \rightarrow X$ is homotopic to a topological homeomorphism.

3.7. $\rho = 2$. Then $H_*(\bar{X}; Z) \approx H_*(S^2; Z)$. Hence $\bar{X}$ is homotopy equivalent to S^2 . Assume $\pi_1(X)$ admits a special tower $\pi_1(X) = G_0 > G_1 > \cdots > G_R > \{1\}$ such that for each i , $i=1, 2, \dots, R$, G_{i-1}/G_i is an infinite cyclic group and G_R is a finite group. Then G_R is $\{1\}$ or Z_2 . To see this, consider the cover $X^{(R)}$ corresponding to G_R . First, suppose $X^{(R)}$ is orientable. If $G_R \neq \{1\}$, then let Z_p be a cyclic subgroup of G_R and $\hat{X}^{(R)}$ be the corresponding cover. By Duality Theorem R , there is a duality $0 = H^1(\hat{X}^{(R)}; Z) \approx H_1(\hat{X}^{(R)}; Z) = Z_p$, which is a contradiction. [Note that $H_*(\hat{X}^{(R)}; Z)$ is finitely generated abelian, since $X^{(R)}$ is homotopy equivalent to a complex of finite type (i.e., skeleton-finite complex).] Hence $G_R = \{1\}$. If $X^{(R)}$ is non-orientable, by the orientable case, the orientation cover of $X^{(R)}$ is simply connected. This implies $G_R = Z_2$. As a simple consequence, if $\pi_1(X)$ is abelian, then $\pi_1(X)$ is isomorphic to Z^{n-2} or $Z^{n-2} + Z_2$.

3.8. $\rho = 3$. Since $H_*(\bar{X}; Z) \approx H_*(S^3; Z)$, X is homotopy equivalent to S^3 . Assume $\pi_1(X)$ admits a special tower $\pi_1(X) = G_0 > G_1 > \cdots > G_R > \{1\}$ as in 3.7. We shall show that every abelian subgroup of G_R is cyclic. Hence G_R has a period > 1 . In particular, if $\pi_1(X)$ is abelian, then $\pi_1(X) \approx Z^{n-3} + Z_m$ ($m \geq 1$). To see this, first assume $X^{(R)}$ is orientable. Let A be an abelian subgroup of G_R . Let $\hat{X}^{(R)}$ be the corresponding

cover. By Duality Theorem R, we have $H^i(\hat{X}^{(R)}; Z) \approx H_{3-i}(\hat{X}^{(R)}; Z)$ for all i . Since $H_1(\hat{X}^{(R)}; Z)$ is a finite group, $H_2(\hat{X}^{(R)}; Z) = 0$. This implies $H_2(A; Z) = 0$; so, A is cyclic. [Use a cyclic decomposition of A .] Now we must show that $X^{(R)}$ is necessarily orientable. From a successive use of the Novikov-Siebenmann splitting theorem (See L.C.Siebenmann [C].), we have $S^3 \times X^{(R)} \cong M^6 \times R^{(R)}$ for a closed 6-manifold M^6 . [Note that the Wall-Siebenmann obstruction $\tau(S^3 \times X^{(i)})$, $i=1, 2, \dots, R$, in the reduced projective class group $\tilde{K}_0(Z[\pi_1(S^3 \times X^{(i)})])$ vanishes, since the Euler characteristic $\chi(S^3)$ of S^3 is 0.] Hence $H_1(S^3 \times X^{(R)}; Z) = 0$, $i \geq 7$. $X^{(R)}$ is non-orientable if and only if $H_6(S^3 \times X^{(R)}; Z) = 0$. In other words, $H_1(X^{(R)}; Z) = 0$, $i \geq 4$, and $X^{(R)}$ is non-orientable if and only if $H_3(X^{(R)}; Z) = 0$. Note that there is a duality $\cap \mu : H^i(S^3 \times X^{(R)}; Z_2) \approx H_{6-i}(S^3 \times X^{(R)}; Z_2)$ for all i . As an analogy of a fact shown in 2.3, this duality can be interpreted as $\cap \mu : H^i(X^{(R)}; Z_2) \approx H_{3-i}(X^{(R)}; Z_2)$ for all i . This shows that the Euler characteristic $\chi(X^{(R)}; Z_2)$ over the coefficient field Z_2 is 0. We need the following lemma:

3.8.1 Lemma. Suppose a space K has a finitely generated integral homology group $H_*(K; Z)$. Then the Euler characteristic of K is independent of a coefficient field which is used.

From this lemma, the usual Euler characteristic (, that is, the Euler characteristic over Q) $\chi(X^{(R)}) = 0$. Suppose $X^{(R)}$ is non-orientable. We count the Betti numbers of $H_*(X^{(R)}; Z)$. We have $0 = \chi(X^{(R)}) = \beta_0(X^{(R)}) - \beta_1(X^{(R)}) + \beta_2(X^{(R)}) = 1 - 0 + \beta_2(X^{(R)})$

≥ 1 , which is a contradiction. [Note that $H_1(X^{(R)}; \mathbb{Z})$ is a finite group. Therefore, $X^{(R)}$ is orientable.

3.8.2. Proof of Lemma 3.8.1. Since $H_*(K; \mathbb{Z})$ is finitely generated, by the proof of E.H. Spanier[//], Lemma 9, p 246, there is a finitely generated free chain complex C chain equivalent to the free geometric chain complex $C(K; \mathbb{Z})$. Then the assertion follows from the Euler-Poincaré formula. In fact, for a field F

$$\begin{aligned}\chi(K; F) &= \sum_i (-1)^i \dim_F H_i(K; F) \\ &= \sum_i (-1)^i \dim_F H_i(C; F) \\ &= \sum_i (-1)^i \dim_F C_i \otimes F\end{aligned}$$

and $\dim_F C_i \otimes F$ is independent of a choice of fields F . This completes the proof.

3.9. $\varrho = 4$. In this case $\bar{X}$ is only a simply connected Poincaré 4-complex. $\bar{X}$ is homotopy equivalent to the adjunction space of a 4-cell B^4 to a bouquet $S^2 \vee \dots \vee S^2$ by a map $a : \partial B^4 \rightarrow S^2 \vee \dots \vee S^2$. From this, one can see that the homotopy type of $\bar{X}$ is characterized by the symmetric inner product $H^2(\bar{X}; \mathbb{Z}) \times H^2(\bar{X}; \mathbb{Z}) \xrightarrow{U} H^4(\bar{X}; \mathbb{Z}) = \mathbb{Z}$. (cf. J.W. Milnor-D. Husemoller[7].)

3.10. $\varrho \geq 5$. $\bar{X}$ is a simply connected Poincaré ϱ -complex. Let precisely $\pi_1(X) = G_0 \supset \hat{G}_0 \supset G_1 \supset \hat{G}_1 \supset \dots \supset G_R \supset \hat{G}_R = \{1\}$, where for each i , $i=0, 1, \dots, R$ $G_i/\hat{G}_i$ has a finite index and for each i , $i=1, 2, \dots, R$, $\hat{G}_{i-1}/G_i$ is an infinite cyclic group. If the Wall-Siebenmann obstruction $\mathcal{O}(X^{(i)})$ in $\tilde{K}_0(\mathbb{Z}[G_i])$ vanishes, then $\bar{X}$ is piecewise-linearly homeomorphic to the splitting $M \times \mathbb{R}^R$ for a piecewise-linear closed manifold M by the Novikov-

Siebenmann splitting theorem. In particular, if G contains a poly-infinite-cyclic group (of rank R) as a subgroup with finite index (for example, G is abelian), then $\bar{X}$ splits: $\bar{X} \cong M \times \mathbb{R}^R$, because $\tilde{K}_0(\mathbb{Z}[P]) = 0$ for all poly-infinite-cyclic groups P of finite rank. (cf. W.C.Hsiang [3].)

We consider low-dimensional consequences. For example, closed 3-manifolds with abelian fundamental groups are contained in $\mathcal{M}$. It follows that the possible group as the fundamental groups is $\mathbb{Z}^3$, $\mathbb{Z} + \mathbb{Z}_2$, $\mathbb{Z}$ or $\mathbb{Z}_m$. This is also a classical result due to K. Reidemeister [9]. Certainly, the universal cover is contractible (more precisely, $\mathbb{R}^3$ modulo Poincaré conjecture) or has the homotopy type of S^2 or S^3 according as the Kodaira dimension $\varrho = 0$ or 2 or 3. Next, for example, consider a closed 4-manifold M with $\pi_1(M) = \mathbb{Z}^r$, $r \geq 5$. [Note that such a manifold M does exist.] As a simple consequence of Theorem 3.4, $\pi_2(M) \otimes \mathbb{Q}$ is necessarily infinitely generated over $\mathbb{Q}$.

4. Further discussions

4.1. Although we established the duality theorem with integral coefficients, under the same hypotheses we can have every torsion-free group as a coefficient of the duality. This fact is based on A.Kawauchi [4]. Further, from [4], if our class $f(k)$, $k \geq 1$, is replaced by Wall's class NF_k in [13] and $\dim_{\mathbb{Q}} H_k(\hat{X}^{(N)}, \partial \hat{X}^{(N)}; \mathbb{Q}) < +\infty$ (or $\dim_{\mathbb{Q}} H_k(\hat{X}^{(N)}; \mathbb{Q}) < +\infty$ or $\dim_{\mathbb{Q}} H_k(\hat{X}^{(N)}, \partial \hat{X}^{(N)}; \mathbb{Q}) < +\infty$)

$\partial \hat{X}^{(N)}; \mathbb{Q}) < +\infty$) is replaced by a condition that $H_k(\hat{X}^{(N)}, \partial \hat{X}^{(N)}; \mathbb{Z})$ (or $H_k(\hat{X}^{(N)}; \mathbb{Z})$ or $H_k(\partial \hat{X}^{(N)}; \mathbb{Z})$) is finitely generated abelian, then Duality Theorem N(≥ 1) holds for an arbitrary coefficient group.

4.2. One can obtain a duality of a type in Kawauchi[5] for iterated infinite cyclic coverings under rather simple hypotheses, but details remain open.

4.3. In this paper, we worked in the piecewise-linear category. However, a manifold may be a topological manifold, because given m , it suffices to establish an i -duality, $i \leq m$, of a manifold with a sufficiently large dimension n in contrast with m , and a **transverse-regularity** of topological manifolds in high-codimension and a surgery on low-dimensional handles in high-dimensional topological manifolds can be done just as piecewise-linear manifolds.

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