

MAXIMAL EDGE-TRAVERSAL TIME IN FIRST PASSAGE PERCOLATION

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ABSTRACT. In this paper, we study the maximal edge-traversal time (simply we call maximal weight hereafter) on the optimal paths in the first passage percolation for several edge distributions, including the Pareto and Weibull distributions. It is known to be unbounded when the edge distribution has unbounded support [J. van den Berg and H. Kesten. Inequalities for the time constant in first-passage percolation. Ann. Appl. Probab. 56-80, 1993]. We determine the order of the growth depending on the tail of the edge distribution.

1. INTRODUCTION

First Passage Percolation (FPP) is a model of the spread of a fluid through a random medium which was first introduced by Hammersley and Welsh in 1965. In FPP, a graph with random weights is given and we consider the optimization problem of the passage time between two fixed vertices. The minimum value is called the first passage time and it represents the time when the fluid reaches from one point to the other. From the viewpoint of an optimization problem, properties of optimal paths that attain the minimum value are also of interest. Theoretical physicists predicted that the front of spread in FPP asymptotically satisfies KPZ-equation [4] in some sense. Moreover, they have found the relationship between the fluctuation of surface and the deviation of optimal paths, the so-called *scaling relation* [5]. Over 50 years, as mathematical techniques have been developed for these problems, there has been a significant progress especially about the asymptotics and fluctuation of the first passage time and the surface growth. On the other hand, not much is known about the properties of optimal paths. The above-mentioned scaling relation concerns the geometry of optimal paths but it has not been proved fully rigorously. This paper studies the maximal weight of edges on optimal paths aiming to provide a better understanding of how the medium along optimal paths looks. For more on the background and known results about FPP, we refer the reader to [1].

1.1. The setting of the model. In this paper, we consider the first passage percolation on $\mathbb{Z}^d$. The model is defined as follows. An element of $\mathbb{Z}^d$ is called a vertex. Denote by E^d the set of non-oriented edges of the lattice $\mathbb{Z}^d$:

$$E^d = \{\langle v, w \rangle \mid v, w \in \mathbb{Z}^d, |v - w|_1 = 1\},$$

where $|u|_1 = \sum_{i=1}^d |u_i|$ for $u \in \mathbb{Z}^d$. We say that v and w are adjacent if $|v - w|_1 = 1$. With a slight abuse of notation, an edge $e = \langle v, w \rangle$ is considered as a subset of $\mathbb{Z}^d$ such as $e = \{v, w\}$. We assign a non-negative random variable τ_e on each edge e . Assume that the collection $\tau = \{\tau_e\}_{e \in E^d}$ is independent and identically distributed with common distribution F . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space and denote by $\mathbb{E}$ the expectation. A *path* Γ is a finite sequence $(x_0, \dots, x_l)$ of $\mathbb{Z}^d$ such that for any $1 \leq i \leq l$, x_i and x_{i-1} are adjacent. If $x_0 = v$ and $x_l = w$ for a path $\Gamma = (x_0, \dots, x_l)$, we write $\Gamma : v \rightarrow w$ and then Γ is said to be a path from v to w . It is sometimes convenient to regard a path $\Gamma = (x_0, \dots, x_l)$ as a sequence of edges such as $(\langle x_0, x_1 \rangle, \dots, \langle x_{l-1}, x_l \rangle)$. Thus we will use the convention with some abuse of notation.

Given a finite path Γ , we define the *passage time* of Γ as

$$t(\Gamma) = \sum_{e \in \Gamma} \tau_e.$$

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Given two vertices $v, w \in \mathbb{Z}^d$, we define the *first passage time* from v to w by

$$t(v, w) = \inf_{\Gamma: v \rightarrow w} t(\Gamma),$$

where the infimum is take over all paths from v to w . We say that $\Gamma : v \rightarrow w$ is an *optimal* path if $t(\Gamma) = t(v, w)$. Denote by $\mathbb{O}(v, w)$ the set of optimal paths from v to w :

$$\mathbb{O}(v, w) = \{\Gamma : v \rightarrow w \mid t(\Gamma) = t(v, w)\}.$$

It is easy to see that $t : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{R}_{\geq 0}$ is pseudometric. Hence optimal paths are sometimes called *geodesics*. Given a path Γ , we set the *maximal weight* of Γ as

$$\mathcal{M}(\Gamma) = \max_{e \in \Gamma} \tau_e.$$

An edge e is said to be a maximal edge for Γ if e belongs to Γ and it attains the maximal weight of Γ . In this paper, we investigate the growth rate of the maximum weight of optimal paths.

1.2. Overview. Van den Berg and Kesten proved in [3] that maximal weight of optimal paths goes to infinity as the endpoint goes to infinity if the distribution F is unbounded, as a special case of more general theorem. Then the problem of the speed of the divergence is natural to consider and appears in [1] as Open problem 2.

Since the formulation of our results will be a bit complicated, we state some illustrative consequence first when the distribution is continuous, i.e., $\mathbb{P}(\tau_e = a) = 0$ for any $a \in \mathbb{R}$. (See Thoerem 1 and Remark 1.) If $1 - F(t) = \exp\{-t^{r+o(1)}\}$ with $r > 0$ for any sufficiently large $t \in \mathbb{R}$, then for any $x \in \mathbb{Z}^d$, the following occurs with high probability: for any optimal path Γ from 0 to x ,

$$(1.1) \quad \mathcal{M}(\Gamma) = \begin{cases} (\log |x|_1)^{\frac{1}{1+r}+o(1)}, & \text{if } r \leq d-1, \\ (\log |x|_1)^{\frac{1}{d}+o(1)}, & \text{if } d-1 < r \leq d, \\ (\log |x|_1)^{\frac{1}{r}+o(1)}, & \text{if } d < r. \end{cases}$$

It should be noted that if, in addition, we remove $o(1)$ in the above assumptions, we get the same results without $o(1)$ except when $r = d$ or $r = d-1$. And, if $\beta_1 t^{-\alpha} \leq 1 - F(t) \leq \beta_2 t^{-\alpha}$ for any sufficiently large $t > 0$ with some constants $\alpha > 2$ and $\beta_1, \beta_2 > 0$, then the order becomes

$$\mathcal{M}(\Gamma) \asymp \frac{\log |x|_1}{\log \log |x|_1}.$$

These transitions itself are interesting and closely related to that of the large deviation of the first passage time [2].

In the proofs, one can see that different geometric pictures around the maximal edge appear. It would be interesting if there is a transition of some quantity about the configurations around the maximal edge, such as the average weight around the maximal edge.

1.3. Main results. We only consider the optimal paths from 0 to $N\mathbf{e}_1$, though all of the results also hold for any direction. For the sake of the simplicity, we write $\mathbb{O}_N = \mathbb{O}(0, N\mathbf{e}_1)$. We set $f_{d,r}(N)$ as

$$(1.2) \quad f_{d,r}(N) = \begin{cases} (\log N)(\log \log N)^{-1}, & \text{if } r = 0 \\ (\log N)^{\frac{1}{1+r}}, & \text{if } 0 < r < d-1 \\ (\log N)^{\frac{1}{d}}(\log \log N)^{\frac{d-2}{d}}, & \text{if } r = d-1 \\ (\log N)^{\frac{1}{d}}, & \text{if } d-1 < r < d \\ (\log N)^{\frac{1}{d}}(\log \log N)^{-\frac{1}{d}}, & \text{if } r = d \\ (\log N)^{\frac{1}{r}}, & \text{if } d < r. \end{cases}$$

First we state the upper bound for maximal weight.

Theorem 1. *Let $d \geq 2$. Suppose that there exist constants $r \in (0, \infty)$, $b, b' > 0$ such that for $t \geq 0$,*

$$(1.3) \quad \mathbb{P}(\tau_e \geq t) \leq b' e^{-bt^r},$$

and $F(0) < p_c(d)$. Then, there exists a positive constant C such that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\max_{\Gamma \in \mathcal{O}_N} \mathcal{M}(\Gamma) \leq C f_{d,r}(N) \right) = 1.$$

Theorem 2. *Let $d \geq 2$. Suppose that $\mathbb{E}\tau_e^2 < \infty$ and $F(0) < p_c(d)$. Then, there exists a positive constant C such that*

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\max_{\Gamma \in \mathcal{O}_N} \mathcal{M}(\Gamma) \leq C f_{d,0}(N) \right) = 1.$$

Next we move on to the lower bound. Let us restrict our attention to the following useful distributions, which is introduced in [3]. Note that if F is continuous, then F is useful.

Definition 1. *Given a distribution F with $\text{supp}(F)$ ($=$ support of F) $\subset [0, \infty)$, let F^- be the infimum of the support. We say that F is useful if either holds:*

$$(1) \ F^- = 0 \text{ and } F(0) < p_c(d) \text{ or } (2) \ F^- > 0 \text{ and } F(F^-) < \vec{p}_c(d),$$

where $p_c(d)$ and $\vec{p}_c(d)$ stand for the critical probabilities for d dimensional percolation and oriented percolation model, respectively.

Theorem 3. *Let $d \geq 2$. Suppose that the following three hold:*

- (i) F is useful,
- (ii) for any positive integer m , $\mathbb{E}[\tau_e^m] < \infty$,
- (iii) there exist $r \in (0, \infty)$, $\alpha, \beta, \kappa > 0$ and $\gamma > 1$ such that for any $t > \kappa$,

$$\mathbb{P}(t < \tau_e < \gamma t) \geq \beta e^{-\alpha t^r}.$$

Then, there exists a positive constant c such that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\min_{\Gamma \in \mathcal{O}_N} \mathcal{M}(\Gamma) \geq c f_{d,r}(N) \right) = 1.$$

Remark 1. Let us comment how to deduce the lower bound of (1.1). We now consider a weaker assumption that γ depends on t such that $\gamma = t^\delta$ with $\delta > 0$. Essentially the same proof yields the following result: for any $\epsilon > 0$, there exists $\delta > 0$ such that under the above condition (i)–(iii) with $\gamma = t^\delta$,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\min_{\Gamma \in \mathcal{O}_{\text{pt}_N}} \mathcal{M}(\Gamma) \geq c f_{d,r}(N)^{1-\epsilon} \right) = 1.$$

Note that if $1 - F(t) = \exp(-t^{r+o(1)})$, then it is easy to check that for any $\delta > 0$,

$$\mathbb{P}(t \leq \tau_e \leq t^{1+\delta}) \geq e^{-t^{r+o(1)}},$$

for sufficiently large t and that the other conditions hold. Therefore, the lower bound of (1.1) follows.

Theorem 4. *Let $d \geq 2$. Suppose that the following three hold:*

- (i) F is useful,
- (ii) $\mathbb{E}[\tau_e^2] < \infty$,
- (iii) there exist $\alpha, \beta, \kappa > 0$ and $\gamma > 1$ such that for any $t > \kappa$,

$$\mathbb{P}(t < \tau_e < \gamma t) \geq \beta t^{-\alpha}.$$

Then, there exists a positive constant c such that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\min_{\Gamma \in \mathcal{O}_N} \mathcal{M}(\Gamma) \geq c f_{d,0}(N) \right) = 1.$$

Remark 2. If we consider a Box-to-Box first passage time $t(D(0), D(N\mathbf{e}_1))$ where $D(x) = x + [-L_N, L_N]^d$ for $x \in \mathbb{R}^d$ and L_N will be specified below, instead of $t(0, N\mathbf{e}_1)$ and the maximal weight of corresponding optimal paths, the above four results hold not only in probability, but with probability one. More precisely, the following results hold:

Proposition 1. *Let $L_N = \log N$. Under the condition of Theorem 1, the following happens with probability one: there exists a positive constant C such that for any $N \in \mathbb{N}$,*

$$(1.4) \quad \max_{\pi_N \in \mathbb{O}(D(0), D(N\mathbf{e}_1))} \mathcal{M}(\pi_N) \leq C f_{d,r}(N).$$

If only we assume the condition of Theorem 2, (1.4) holds with $r = 0$.

Proposition 2. *Take a positive constant κ and set $L_N = (\log N)^{1+\kappa}$. Under the condition of Theorem 3, the following happens with probability one: there exists a positive constant c such that for any $N \in \mathbb{N}$,*

$$(1.5) \quad \max_{\pi_N \in \mathbb{O}(D(0), D(N\mathbf{e}_1))} \mathcal{M}(\pi_N) \geq c f_{d,r}(N).$$

If we assume the condition of Theorem 4 instead, (1.5) holds with $r = 0$.

Remark 4 and 5 explain the necessary modifications to show the almost surely results.

Remark 3. In order to get the exact order of the growth of the maximal weight, in general, we need the assumption of the distribution such as $1 - F(t) \sim \exp\{-f(t)\}$, where $f(t)$ is regularly varying function, as we do for an extreme value problem of independent and identical distributed random variables. Therefore, our assumption on distributions is natural one.

1.4. Notation and terminology. This subsection introduces useful notations and terminologies for the proof.

- Given two vertices $v, w \in \mathbb{Z}^d$ and a subset $D \subset \mathbb{Z}^d$, we set the *restricted* first passage time as

$$t_D(v, w) = \inf_{\Gamma \subset D} t(\Gamma),$$

where the infimum is over all paths Γ from v to w and $\Gamma \subset D$. If such a path does not exist, we set the infinity instead.

- We use c and C for positive constants. They may change from line to line. Typically, c is used for a small constant and C for a large one.
- $\lfloor \cdot \rfloor$ is a floor function, i.e. $\lfloor x \rfloor$ is the greatest integer less than or equal to x .
- Given $x, y \in \mathbb{R}^d$, denote by $d_p(\cdot, \cdot)$ the p -norm. We only use $p = 1$ or $p = \infty$ in this article. It is useful to extend the definition as

$$d_p(A, B) = \inf\{d_p(x, y) \mid x \in A, y \in B\} \quad \text{for } A, B \subset \mathbb{R}^d.$$

When $A = \{x\}$, we write $d_p(x, B)$.

- Given a set D of $\mathbb{Z}^d$, let us define the inner boundary of D as $\partial D = \{v \in \mathbb{Z}^d \mid v \in D \text{ and } \exists w \notin D \text{ such that } |v - w|_1 = 1\}$. We define $\iota(D) = D \setminus \partial D$.
- Given another configuration $\tau^* = \{\tau_e^*\}_{e \in E^d}$, we denote the corresponding first passage time and the set of optimal paths by $t^*(v, w)$ and $\mathbb{O}^*(v, w)$, respectively. Define $\tilde{t}(v, w)$ and $\tilde{\mathbb{O}}_N$ similarly for another configuration $\tilde{\tau} = \{\tilde{\tau}_e\}_{e \in E^d}$.
- Given a set $D \subset \mathbb{Z}^d$ and $x, y \in D$, we write $x \sim_D y$ if there exists a path from a to b which lies only on D . Let us denote the connected component of D containing of a as $\text{Conn}(x, D) = \{y \in D \mid x \sim_D y\}$.

1.5. Reader's guide.

For the proof of the upper bound, to each edge, we will consider a condition, where if the edge has a large weight and a path passes through it, then one can make a detour from this path to get a smaller passage time. Then we will check that all edges related to optimal paths satisfy this condition with high probability, and thus optimal paths do not have too large weights. This condition appears in Lemma 2. The easiest case for the upper bound is $r = 1$ and proved in

Subsection 2.1.

For the proof of the lower bound, we will use the resampling argument introduced in [3]. Then we resample the local configurations to make the optimal paths pass through this region and take a sufficiently large weight. (See the conditions A_1 – A_3 in Definitions 6, 8, 9 and Propositions 5, 7, 10.) It needs the detailed information of optimal paths near the maximal edge, which heavily depends on the tail of distributions. The easiest case for the lower bound is $0 < r < d - 1$ and proved in Subsection 3.1 and 3.2.

2. PROOF FOR THE UPPER BOUND

2.1. Upper bound for $r \leq 1$. Given an edge $e = \langle v, w \rangle$, we define $v_e \in e$ such that $|v_e|_1 = \min\{|v|_1, |w|_1\}$ (such v_e is uniquely determined) and denote the k -th boundary and the set of its edges by $C_k^{(e)}$ and $\tilde{C}_k^{(e)}$ (See Figure 1):

$$C_k^{(e)} = \{z \in \mathbb{Z}^d : |v_e - z|_\infty = k\},$$

$$\tilde{C}_k^{(e)} = \{\langle x, y \rangle : x, y \in C_k^{(e)} \text{ and } |x - y|_1 = 1\}.$$

Note that $\tilde{C}_k^{(e)} \cap \tilde{C}_{k'}^{(e)} = \emptyset$ if $k \neq k'$ and each face is the square with size $2k + 1$ and the dimension $d - 1$. Thus there exists $C(d) > 0$ which depends only on the dimension d such that

$$(2.1) \quad \#\{v \in \mathbb{Z}^d \mid v \in C_k^{(e)}\} \leq C(d)k^{d-1}.$$

In fact we can take $C(d) = 4^d d$ since $\#\{v \in \mathbb{Z}^d \mid v \in C_k^{(e)}\} \leq 2d(2k + 1)^{d-1}$.

Definition 2. We say that e is good if there exists $1 \leq k \leq f_{d,r}(N)$ such that for any $v, w \in C_k^{(e)}$,

$$t_{C_k^{(e)}}(v, w) \leq 2M f_{d,r}(N),$$

where M will be chosen later.

It will be proved in Lemma 2 that for any path Γ , if $e \in \Gamma$ and $\tau_e > 2M f_{d,r}(N)$, the goodness of e can make Γ detour with a smaller passage time.

First, we prove the case $r = 1$. Then $f_{d,r}(N) = \sqrt{\log N}$ and there exists $\alpha > 0$ such that $\mathbb{E}e^{\alpha\tau_e} < \infty$. In this case, we take

$$(2.2) \quad M = \alpha^{-1} \max\{8d, 8d^2 \log \mathbb{E}e^{\alpha\tau_e}, 2\alpha\}.$$

Given two vertices $v, w \in C_k^{(e)}$, we take a path $\gamma_v^w : v \rightarrow w$ lying on $C_k^{(e)}$ whose length is at most $4d^2 f_{d,r}(N) = 4d^2 \sqrt{\log N}$. To calculate the probability that e is good, fix $v, w \in C_k^{(e)}$. Then by

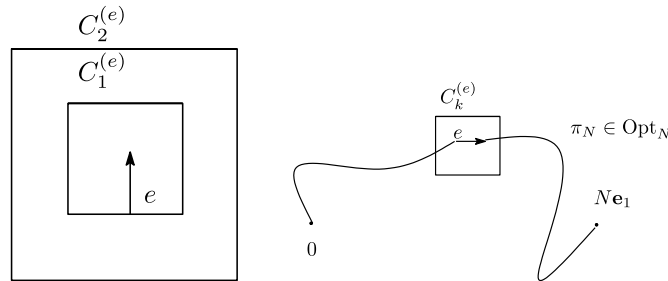


FIGURE 1

Left: The figure of $C_k^{(e)}$ and $\tilde{C}_k^{(e)}$.

Right: We make a better path from the original one.

(2.2), we have

$$\begin{aligned}
 (2.3) \quad \mathbb{P} \left(\sum_{\eta \in \gamma_v^w} \tau_\eta > M\sqrt{\log N} \right) &\leq \exp(-\alpha M\sqrt{\log N}) \prod_{\eta \in \gamma_v^w} \mathbb{E} e^{\alpha \tau_\eta} \\
 &\leq \exp(-\alpha M\sqrt{\log N}) (\mathbb{E} e^{\alpha \tau_e})^{4d^2 \sqrt{\log N}} \\
 &\leq \exp(-\alpha M\sqrt{\log N}/2).
 \end{aligned}$$

Note that if $k \neq k'$, $\{\tau_e\}_{e \in \tilde{C}_k^{(e)}}$ and $\{\tau_e\}_{e \in \tilde{C}_{k'}^{(e)}}$ are independent. It follows that by (2.1) and (2.2), for sufficiently large N ,

$$\begin{aligned}
 (2.4) \quad \mathbb{P}(e \text{ is not good}) &\leq \prod_{k \leq \sqrt{\log N}} \mathbb{P} \left(\text{there exist } v, w \in C_k^{(e)} \text{ such that } \sum_{\eta \in \gamma_v^w} \tau_\eta > M\sqrt{\log N} \right) \\
 &\leq \prod_{k \leq \sqrt{\log N}} C(d)^2 (\log N)^{d-1} \max_{v, w \in C_k^{(e)}} \mathbb{P} \left(\sum_{\eta \in \gamma_v^w} \tau_\eta > M\sqrt{\log N} \right) \\
 &\leq \left(\exp \left(-\frac{\alpha}{4} M\sqrt{\log N} \right) \right)^{\sqrt{\log N}} \leq N^{-2d}.
 \end{aligned}$$

Next, we define the following boxes.

Definition 3. *Given $K > 0$, we define*

$$\mathcal{B}_{KN} = [-KN, KN]^d \cap \mathbb{Z}^d,$$

$$D(0) = [-M^2 f_{d,r}(N), M^2 f_{d,r}(N)]^d \cap \mathbb{Z}^d,$$

$$D(N\mathbf{e}_1) = (N\mathbf{e}_1 + [-M^2 f_{d,r}(N), M^2 f_{d,r}(N)]^d) \cap \mathbb{Z}^d.$$

Lemma 1. *Suppose $\mathbb{E}\tau_e^2 < \infty$ and $F(0) < p_c(d)$. Then there exist $C, K > 0$ such that for any $N \in \mathbb{N}$,*

$$(2.5) \quad \mathbb{P} \left(\left(\bigcup_{\pi_N \in \mathbb{O}_N} \pi_N \right) \cap (\mathcal{B}_{KN})^c \neq \emptyset \right) \leq CN^{-2d}.$$

Proof. From Proposition 5.8 in [6], there exist $C_1, C_2, C_3 > 0$ such that for any $r > 0$

$$(2.6) \quad \mathbb{P}(\exists \text{ self avoiding path } \Gamma \text{ from } 0 \text{ with } |\Gamma| \geq r \text{ and } t(\Gamma) < C_1 r) < C_2 \exp(-C_3 r).$$

We take $K_1 > 2\mathbb{E}[\tau_e]/C_1$. Then for any $s > K_1$,

$$\begin{aligned}
 (2.7) \quad &\mathbb{P}(\exists \Gamma \in \mathbb{O}_N \text{ such that } |\Gamma| \geq sN) \\
 &\leq \mathbb{P}(\exists \Gamma \in \mathbb{O}_N \text{ such that } |\Gamma| \geq sN \text{ and } t(0, N\mathbf{e}_1) < C_1 sN) + \mathbb{P}(t(0, N\mathbf{e}_1) \geq C_1 sN) \\
 &\leq C_2 \exp(-C_3 sN) + \mathbb{P}(t(0, N\mathbf{e}_1) \geq C_1 sN).
 \end{aligned}$$

where we have used (2.6) in the second inequality. Now we consider $2d$ disjoint paths $\{r_i\}_{i=1}^{2d}$ from 0 to $N\mathbf{e}_1$ so that

$$\max\{|r_i| \mid i = 1, \dots, 2d\} \leq N + 8$$

as in [6, p 135]. By the Chebyshev inequality, we have that there exists $C_4 = C_4(d, F, C_1) > 0$ such that

$$\begin{aligned}
 \mathbb{P}(t(0, N\mathbf{e}_1) \geq C_1 s N) &\leq \prod_{i=1}^{2d} \mathbb{P}(t(r_i) \geq C_1 s N) \\
 &\leq \prod_{i=1}^{2d} \mathbb{P}(|t(r_i) - \mathbb{E}[t(r_i)]| \geq C_1 s N/2) \\
 &\leq \prod_{i=1}^{2d} ((C_1 s N/2)^{-2} (N+8) \mathbb{E}[\tau_e^2]) \\
 &\leq C_4 s^{-4d} N^{-2d}.
 \end{aligned}
 \tag{2.8}$$

Thus we have for $s > K_1$,

$$\mathbb{P}(\exists \pi_N \in \mathbb{O}_N \text{ such that } |\pi_N| \geq sN) \leq 2C_4 s^{-4d} N^{-2d}.$$

Let $K = 2K_1$. Then since $\left(\bigcup_{\pi_N \in \mathbb{O}_N} \pi_N\right) \cap (\mathcal{B}_{KN})^c \neq \emptyset$ implies $\max_{\pi_N \in \mathbb{O}_N} \sharp \pi_N \geq N^2$,

$$\mathbb{P}\left(\bigcup_{\pi_N \in \mathbb{O}_N} \pi_N \cap (\mathcal{B}_{KN})^c \neq \emptyset\right) \leq 2^{-4d+1} C_4 N^{-2d}.
 \tag{2.9}$$

□

We take $K > 0$ as in Lemma 1. We define three events as

$$\begin{aligned}
 A_1 &= \{\forall e \in E^d \text{ with } e \cap \mathcal{B}_{KN} \neq \emptyset, e \text{ is good}\}, \\
 A_2 &= \{\forall \pi_N \in \mathbb{O}_N, \pi_N \subset \mathcal{B}_{KN}\}, \\
 A_3 &= \{\forall e \in E^d \text{ with } e \cap (D(0) \cup D(N\mathbf{e}_1)) \neq \emptyset, \tau_e \leq M\sqrt{\log N}\}.
 \end{aligned}
 \tag{2.10}$$

Then by the union bound, we have

$$\begin{aligned}
 &\mathbb{P}(A_1 \cap A_2 \cap A_3) \\
 &\geq 1 - \sum_{i=1}^3 \mathbb{P}(A_i^c) \\
 &\geq 1 - 2d(2KN)^d \cdot N^{-2d} - CN^{-2d} - 2d\sharp(D(0) \cup D(N\mathbf{e}_1))b' \exp(-b(M\sqrt{\log N})) \\
 &\geq 1 - (2^{d+1}dK + C)N^{-d} - 2dM(\log N)^d e^{-\alpha M\sqrt{\log N}},
 \end{aligned}
 \tag{2.11}$$

and $\lim_{N \rightarrow \infty} \mathbb{P}(A_1 \cap A_2 \cap A_3) = 1$. Combining it with the following lemma, we complete the proof of the case $r = 1$.

Lemma 2. *On the event $A_1 \cap A_2 \cap A_3$, for any optimal path $\pi_N \in \mathbb{O}_N$ and any edge $e \in \pi_N$,*

$$\tau_e \leq M\sqrt{\log N}.$$

Proof. Take an arbitrary optimal path π_N and write $\pi_N = \{x_1, \dots, x_K\}$. Suppose that there exists $1 \leq t \leq K-1$ such that $\tau_{\langle x_t, x_{t+1} \rangle} > M\sqrt{\log N}$. We set $e = \langle x_t, x_{t+1} \rangle$. Let x_m and x_l be the first and final intersecting point between π_N and $C_k^{(e)}$: $m = \inf\{s \mid x_s \in C_k^{(e)}\}$ and $l = \sup\{s \mid x_s \in C_k^{(e)}\}$. Since e is good, there exists $k \leq \sqrt{\log N}$ such that for any $v, w \in C_k^{(e)}$, there exists a path $\ell : v \rightarrow w$ on $C_k^{(e)}$ such that

$$\sum_{\eta \in \ell} \tau_\eta \leq M\sqrt{\log N}.$$

We take such a k . Then due to the definition of A_3 , $e \cap (D(0) \cup D(N\mathbf{e}_1)) = \emptyset$, in particular $0 < l < t < m < \infty$. Thus we have $t(x_m, x_l) \leq M\sqrt{\log N}$, which is a contradiction. □

Next we consider the case $0 < r < 1$. Then, by (4.2) of [8], (2.3) is replaced by

$$(2.12) \quad \mathbb{P} \left(\sum_{\eta \in \gamma_v^w} \tau_\eta > M f_{d,r}(N) \right) \leq e^{-c M^r f_{d,r}(N)^r},$$

with some constant $c > 0$ that depends only on the distribution F . The rest is the same as before.

Finally, suppose only $\mathbb{E} \tau_e^2 < \infty$. In this case, e is said to be good if there exists $1 \leq k \leq M f_{d,0}(N)$ such that for any $v, w \in C_k^{(e)}$,

$$t_{C_k^{(e)}}(v, w) \leq 8d^2 M^2 f_{d,0}(N),$$

We consider $2(d-1)$ disjoint paths $\{r_i\}_{i=1}^{2(d-1)}$ from v to w on $C_k^{(e)}$ so that

$$\max\{|r_i| \mid i = 1, \dots, 2(d-1)\} \leq 4d^2 M f_{d,0}(N)$$

as in [6, p 135]. Then the Markov inequality yields that for any $i \in \{1, \dots, 2(d-1)\}$,

$$(2.13) \quad \mathbb{P} \left(\sum_{e \in r_i} \tau_e > 4d^2 M^2 f_{d,0}(N) \right) \leq (2d^2 M^2 f_{d,0}(N))^{-2} \mathbb{E} \left[\left(\sum_{e \in r_i} (\tau_e - \mathbb{E} \tau_e) \right)^2 \right] \\ \leq M^{-2} f_{d,0}(N)^{-1}.$$

Thus we have

$$\mathbb{P}(t_{C_k^{(e)}}(v, w) > 4d^2 M^2 f_{d,0}(N)) \leq M^{-4(d-1)} f_{d,0}(N)^{2(d-1)}.$$

Fix an arbitrary vertex $v_k \in C_k^{(e)}$ for $k \in \mathbb{N}$. Then for any $v, w \in C_k^{(e)}$, $t(\gamma_v^w) \leq t(\gamma_{v_k}^{v_k}) + t(\gamma_{v_k}^w)$. Thus, if there exist $v, w \in C_k^{(e)}$ such that $t_{C_k^{(e)}}(v, w) > 8d^2 M f_{d,0}(N)$, there exists $w \in C_k^{(e)}$ such that $\sum_{\eta \in \gamma_{v_k}^w} \tau_\eta > 4d^2 M f_{d,0}(N)$. Note that if $k \leq M f_{d,0}(N)$, $\#C_k^{(e)} \leq C(d)(M f_{d,0}(N))^{d-1}$. As in the proof of (2.4), if M is sufficiently large, we have

$$(2.14) \quad \mathbb{P}(e \text{ is not good}) \leq \prod_{k \leq M f_{d,0}} \mathbb{P} \left(\text{there exists } w \in C_k^{(e)} \text{ such that } \sum_{\eta \in \gamma_{v_k}^w} \tau_\eta > 4d^2 M^2 f_{d,0}(N) \right) \\ \leq \prod_{k \leq M f_{d,0}} C(d)^2 (M f_{d,0}(N))^d - 1 \max_{w \in C_k^{(e)}} \mathbb{P} \left(\sum_{\eta \in \gamma_{v_k}^w} \tau_\eta > 4d^2 M^2 f_{d,0}(N) \right) \\ \leq f_{d,0}(N)^{-(d-1)M[f_{d,0}(N)]} \\ \leq N^{-2d}.$$

Recall that $D(0) = [-M^2 f_{d,0}(N), M^2 f_{d,0}(N)]^d \cap \mathbb{Z}^d$ and $D(N\mathbf{e}_1) = N\mathbf{e}_1 + D(0)$. If there exist an optimal path π_N and an edge $e \in \pi_N \cap D(0)$ such that $\tau_e \geq 4d^2 M^4 f_{d,0}(N)$, then there exists $v \in \partial D(0)$ such that $t(0, v) \geq 4d^2 M^4 f_{d,0}(N)$. On the other hand, as in (2.14), we obtain

$$\mathbb{P}(\exists v \in \partial D(0) \text{ such that } t(0, v) \geq 4d^2 M^4 f_{d,0}(N)) \leq 4d^2 (4M^2 f_{d,0}(N))^{d-1} f_{d,0}(N)^{-2d}.$$

A similar result holds for $D(N\mathbf{e}_1)$. We define

$$A'_3 = \{\forall v \in \partial D(0), t(0, v) \leq 4d^2 M^4 f_{d,0}(N) \text{ and } \forall v \in \partial D(N\mathbf{e}_1), t(v, N\mathbf{e}_1) \leq 4d^2 M^4 f_{d,0}(N)\}.$$

It follows that $\mathbb{P}(A'_3) \rightarrow 1$ as $N \rightarrow \infty$. Then with $A_1 \cap A_2 \cap A'_3$, Lemma 2 and (2.11) hold by the similar argument of the previous proofs and thus the proof is completed.

Remark 4. Let us comment on how to prove Proposition 1. Then we replace $A_1 \cap A_2 \cap A_3$ by $A_1 \cap A_2$. Indeed, Lemma 2 can be proved by the exactly same argument. Moreover, (2.11) yields that $\mathbb{P}(A_1 \cap A_2) \geq 1 - CN^{-d}$ and the Borel-Cantelli lemma leads us to the conclusion.

2.2. Upper bound for $r > 1$. When $r > d$, the upper bound is rather trivial, since $\mathbb{P}(A_2^c) \rightarrow 0$ and for sufficiently large M ,

$$\mathbb{P}(\exists e \in E^d \text{ such that } e \subset [-KN, KN] \text{ and } \tau_e > Mf_{d,r}(N)) \leq N^{-2},$$

where K is in Lemma 1.

Next, we will show the upper bound when $1 < r \leq d$. Note that in the above argument, the estimates of $\mathbb{P}(e \text{ is not good})$ rely on simple (sub-)exponential large deviation bounds for fixed paths, see (2.3)–(2.4) for example. It turns out that when $1 < r \leq d$ we need the following *super-exponential* tail estimates on the passage time.

Proposition 3. *Let $d \geq 1$. Suppose that the condition of Theorem 1 holds with $r > 1$. For any $M_1 > 0$ there exists $M_2 > 0$ such that for any $L \geq 1$, $0 \leq L_1 \leq 2L + 1$ and $v, w \in [0, L_1]^d$,*

$$(2.15) \quad \mathbb{P}(t_{[0, L_1]^d}(v, w) > M_2 L) \leq \exp(-g(r, d, L, L_1)M_1),$$

where

$$(2.16) \quad g(r, d, L, L_1) = \begin{cases} L^r & \text{if } 1 < r < d \\ \frac{L^d}{(\log L)^{d-1}} & \text{if } r = d \\ L^d & \text{if } r \in (d, d+1) \\ L^{d+1}/L_1 & \text{if } r = d+1. \end{cases}$$

A similar result for $r > d \geq 2$ or $r = d = 2$ were proved in [2] and the above is a kind of generalization of their results. This will be proved in the next subsection.

We apply the above proposition with the dimension $d - 1$ because each face of $C_k^{(e)}$ has the dimension $d - 1$. Assuming this proposition, we first complete the proof. We take sufficiently large $M_1 > 0$ and M_2 so that Proposition 3 holds. By (2.1), with some constant $C(d)$, we have

$$(2.17) \quad \begin{aligned} & \mathbb{P}(\exists v, w \in C_k^{(e)} \text{ such that } t_{C_k^{(e)}}(v, w) > M_2 f_{d,r}(N)) \\ & \leq C(d)k^{2d} \sup_{v, w \in C_k^{(e)}} \mathbb{P}(t_{C_k^{(e)}}(v, w) > M_2 f_{d,r}(N)). \end{aligned}$$

To bound the supremum above, fix $v, w \in C_k^{(e)}$. Considering a shortest path from v to w , one can see that there exist $v(0) = v, v(1), \dots, v(l) = w \in C_k^{(e)}$ with some $l \leq 2d$ such that for any $1 \leq i \leq l$, $v(i)$ and $v(i-1)$ are on the same face of $C_k^{(e)}$ and for any $1 \leq i \leq l-1$, $v(i)$ lies on at least two faces. Let W_i be the face which contains $v(i)$ and $v(i-1)$. Since each face of $C_k^{(e)}$ is congruent with $[0, 2k+1]^{d-1}$, Proposition 3 with $L = [f_{d,r}(N)]$ and $L_1 = 2k+1$ leads to

$$(2.18) \quad \begin{aligned} & \mathbb{P}(t_{C_k^{(e)}}(v, w) > 2dM_2 f_{d,r}(N)) \\ & \leq \mathbb{P}(1 \leq \exists i \leq l \text{ such that } t_{W_i}(v(i-1), v(i)) > M_2 f_{d,r}(N)) \\ & \leq 2d \exp(-g(r, d-1, f_{d,r}(N), k)M_1). \end{aligned}$$

An edge e is said to be good if there exists $1 \leq k \leq f_{d,r}(N)$ such that for any $v, w \in C_k^{(e)}$, $t_{C_k^{(e)}}(v, w) \leq 2dM_2 f_{d,r}(N)$. Then for any sufficiently large $N \in \mathbb{N}$,

$$(2.19) \quad \begin{aligned} \mathbb{P}(e \text{ is not good}) & \leq \prod_{1 \leq k \leq f_{d,r}(N)} \mathbb{P}(\exists v, w \in C_k^{(e)} \text{ such that } t_{C_k^{(e)}}(v, w) > Mf_{d,r}(N)) \\ & \leq (2dC(d)^2 f_{d,r}(N)^{2d})^{f_{d,r}(N)} \exp\left(-\prod_{k=1}^{[f_{d,r}(N)]} g(d-1, f_{d,r}(N), k)M_1\right) \\ & \leq N^{-2d}, \end{aligned}$$

where when $r = d$, we have used the following fact:

$$\sum_{k=1}^{\lfloor f_{d,r}(N) \rfloor} \frac{1}{k} \geq \frac{1}{2d} \log \log N.$$

The rest is the same as before.

2.3. Large deviation bound for the first passage time. In (2.4) and (2.12) for $r \leq 1$, in order to estimate $\mathbb{P}(t_{[0,k]^d}(v, w) < M f_{d,r}(N))$, we followed the strategy to consider a path γ_v^w on $[0, k]^d$ and use the large deviation bound for $\sum_{e \in \gamma_v^w} \tau_e$. The same strategy would prove $\exp\{-cL\}$ bound instead of Proposition 3 but it is not sufficient for our purpose. To make the estimate sharper, we need to take into account that $t_{[0,K]^d}(v, w)$ is the infimum taken over many paths. To this end, we will construct the tree-like structure, which is similar to the so-called *Generalized Random Energy Model*, from the FPP and then we apply the arguments in [2]. This construction will be done in (2.20).

We start with the large deviation bound for $\mathbf{e}_1$ direction. For $k \in \mathbb{N}$ and $v, w \in \mathbb{Z}^d$, we set

$$\bar{t}^k(v, w) = t_{[0,k] \times [0,k/2]^{d-1}}(v, w).$$

Proposition 4. *Let $d \geq 1$ and $r \in (1, d+1]$. Suppose that the condition of Theorem 1 holds. For any $M_1 > 0$ there exists $M_2 > 0$ such that for any $m \leq 2L+1 \in \mathbb{N}$,*

$$\mathbb{P}(\bar{t}^m(0, m\mathbf{e}_1) > M_2 L) \leq \exp(-g(r, d, L, m)M_1).$$

Proof. If $m \leq 12d$, by considering the straight line from 0 to $m\mathbf{e}_1$, we have

$$\mathbb{P}(\bar{t}^m(0, m\mathbf{e}_1) > M_2 L) \leq 12dP(\tau_e > M_2 L/(12d)),$$

and in this case, it is trivial. Let $12d < m$ and $n = \lfloor \log_2(m/6d) \rfloor$. Our first goal is to define the following family of the set of edges:

$$(2.20) \quad \{\Xi^{i_1, \dots, i_j, j}\}_{i_1, \dots, i_j \in \{0,1\}^{d-1}, 1 \leq j \leq n} \text{ and } \Xi^{0,0}$$

which satisfies the following conditions:

(Ξ -1) $\Xi^{i_1, \dots, i_j, j} \cap \Xi^{i'_1, \dots, i'_j, j'} = \emptyset$ unless $j = j'$ and $i_1 = i'_1, \dots, i_j = i'_j$.

(Ξ -2) For any $1 \leq j \leq n$, $e \in \Xi^{i_1, \dots, i_j, j}$, $e \subset [0, m/3] \times [0, m/2]^{d-1}$

(Ξ -3) $\#\{e \in \Xi^{i_1, \dots, i_j, j}\} \leq 2^{j+d+1}$.

(Ξ -4) For any $i_1, \dots, i_n \in \{0,1\}^{d-1}$, there exists a path from 0 to $(d2^{n+1}, \sum_{j=1}^n 2^{n-j+1}i_j)$, which is a subset of $\cup_{1 \leq j \leq n} \Xi^{i_1, \dots, i_j, j} \cup \Xi^{0,0}$.

To construct this, for any $v = (v_1, \dots, v_d) \in \mathbb{N}^d$, we define the sequence $v(i)$ for $i \in \{0, 1, \dots\}$ as follows: let $v(0) = v$. When $i \leq 2 \sum_{j=2}^d v_j$, we take $k < d$ so that $2 \sum_{j=2}^k v_j < i \leq 2 \sum_{j=2}^{k+1} v_j$ and let

$$v(i) = \left(v_1 + [(i+1)/2], 2v_2, \dots, 2v_k, v_{k+1} + \left\lfloor \frac{i - 2 \sum_{j=2}^k v_j}{2} \right\rfloor, v_{k+2}, \dots, v_d \right).$$

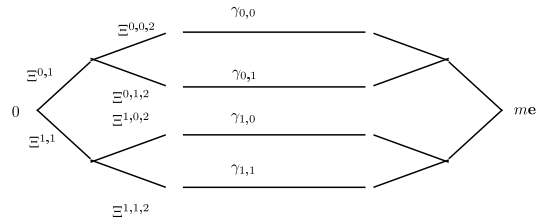


FIGURE 2

A schematic picture of Ξ and γ used in the proof of Proposition 4.

When $2 \sum_{j=2}^d v_j < i$, let $v(i) = (v_1 + i - \sum_{j=2}^d v_j, 2v_2, \dots, 2v_d)$. Given a vertex $v \in \mathbb{Z}^d$, we define

$$\Xi^{v,0} = \left\{ \langle w_1, w_2 \rangle \left| w_1, w_2 \in \left\{ v + \sum_{j=2}^d a_j \mathbf{e}_j \mid a_j \in \{0, 1\} \right\}, |w_1 - w_2|_1 = 1 \right. \right\},$$

and for $1 \leq k \in \mathbb{N}$,

$$\Xi^{v,k} = \{ \langle v(i), v(i+1) \rangle \mid i \in \{1, 2, \dots\} \text{ with } v(i+1) \in [0, d2^{k+1}] \times \mathbb{Z}^{d-1} \} \cup \Xi^{v(\ell^k(v)),0},$$

where $\ell^k(v) = \max\{\ell \in \mathbb{N} \mid v(\ell) \in [0, d2^{k+1}] \times \mathbb{Z}^{d-1}\}$. Then we define

$$\Xi^{i_1, \dots, i_k, k} = \Xi^{(d2^{k-1}, \sum_{s=1}^k 2^{k-s} i_s), k}.$$

Note that Ξ has the hierarchical structure, and

$$(2.21) \quad |\Xi^{i_1, \dots, i_k, k}| \leq 2^{k+d+1} \text{ and } |\Xi^{0,0}| \leq 2^{d+1}.$$

Moreover, Ξ is our suitable object.

Lemma 3. $\{\Xi^{i_1, \dots, i_j, j}\}_{i_1, \dots, i_j, j}$ and $\Xi^{0,0}$ satisfies the above four conditions.

The proof is postponed by Appendix.

Lemma 4. Let $\{X_i\}_{i \in \mathbb{N}}$ be independent and have identical distribution F . There exist $C_1, C_2 > 0$ such that for any $L \in \mathbb{N}$ and $t > C_1$.

$$\mathbb{P} \left(\sum_{i=1}^L X_i \geq Lt \right) \leq \exp(-C_2 t^r L).$$

Proof. The proof is based on the Chernoff bound. For any $\lambda > 0$,

$$(2.22) \quad \begin{aligned} \mathbb{P} \left(\sum_{i=1}^L X_i \geq Lt \right) &\leq \exp(-\lambda t L) \mathbb{E}[\exp(\lambda X_1)]^L \\ &\leq \exp(-\lambda t L) \left\{ \int_0^\infty b' \lambda \exp(\lambda s) \exp(-bs^r) ds \right\}^L. \end{aligned}$$

By Young's inequality, we have

$$(2.23) \quad \begin{aligned} s\lambda &= \left(\frac{br}{2} \right)^{1/r} s \left(\frac{br}{2} \right)^{-1/r} \lambda \\ &\leq \frac{1}{r} \left(\frac{br}{2} \right) s^r + \frac{r-1}{r} \left(\frac{br}{2} \right)^{1/(r-1)} \lambda^{r/(r-1)} \\ &\leq \frac{b}{2} s^r + C \lambda^{r/(r-1)}. \end{aligned}$$

It follows from the fact $\lambda \exp(C \lambda^{r/(r-1)}) \leq \exp(C' \lambda^{r/(r-1)})$ with some $C' > 0$ independent of λ that

$$\lambda \exp(\lambda s) \exp(-bs^r) \leq \exp(C \lambda^{r/(r-1)} - bs^r/2).$$

Thus, (2.22) is further bounded from above by

$$(2.24) \quad C^L \exp(-\lambda t L) \exp(CL \lambda^{r/(r-1)}).$$

Since $\lambda > 0$ is arbitrary, we take the infimum over $\lambda > 0$. Since $-\lambda t L + CL \lambda^{r/(r-1)}$ is taken the infimum at $\lambda = \left(\frac{t(r-1)}{cr} \right)^{r-1}$, which equals $-Lr^{-r} \left(\frac{r-1}{C} \right)^{r-1} t^r$, we have that there exists $C_2 > 0$ such that for sufficiently large $t > 0$,

$$(2.25) \quad \begin{aligned} \mathbb{P} \left(\sum_{i=1}^L X_i \geq Lt \right) &\leq C^L \exp(\inf_{\lambda} (-\lambda t L + CL \lambda^{r/(r-1)})) \\ &= C^L \exp \left(-Lr^{-r} \left(\frac{r-1}{C} \right)^{r-1} t^r \right) \\ &\leq \exp(-C_2 L t^r). \end{aligned}$$

□

Let

$$V_0 = \sum_{e \in \Xi^{0,0}} \tau_e \text{ and } V_k = \sum_{i_1, \dots, i_k \in \{0,1\}^{d-1}} \sum_{e \in \Xi^{i_1, \dots, i_k}} \tau_e.$$

The following lemmas correspond to Lemmas 3.3 and 3.4 in [2], respectively.

Lemma 5. *There exist $C, M_2 > 0$ such that for any $M > M_2$, $0 \leq k \leq n$ and $x > 2^k$,*

$$\mathbb{P}\left(V_k \geq M2^{(d-1)k}x\right) \leq \exp\left(-M\left(\frac{x}{2^k}\right)^r 2^{dk}\right).$$

Proof. Applying Lemma 4 with $L = 2^{dk+d+1}$ and $t = Mx/2^{k+d+1}$, we have

$$\mathbb{P}\left(\sum_{i=1}^{2^{dk+d+1}} X_i \geq M2^{(d-1)k}x\right) \leq \exp\left(-C_2 2^{(1-r)(d+1)} M^r \left(\frac{x}{2^k}\right)^r 2^{dk}\right).$$

Because of the condition $(\Xi-3)$ of $(\Xi^{i_1, \dots, i_k})$ and $\mathbb{P}(\tau_e \geq 0) = 1$, the proof is completed. □

Lemma 6. *there exists $M_2 > 0$ such that for any $M > M_2$,*

$$\mathbb{P}\left(\sum_{k=0}^n 2^{-(d-1)k} V_k \geq 2ML\right) \leq \exp(-g(r, d, L, m)M).$$

Proof. Let $W_k = V_k 2^{-(d-1)k} \mathbb{I}_{\{V_k > M2^{dk}\}}$. If $\sum_{k=0}^n W_k < ML$, then by $2^{n+1} = m/3d \leq L$, we have

$$\begin{aligned} \sum_{k=0}^n 2^{-(d-1)k} V_k &\leq \sum_{k=0}^n (W_k + M2^k) \\ (2.26) \quad &\leq ML + M2^{n+1} \\ &\leq 2ML. \end{aligned}$$

Therefore, $\sum_{k=0}^n 2^{-(d-1)k} V_k \geq 2ML$ implies $\sum_{k=0}^n W_k \geq ML$. Thus, for any $\lambda > 0$,

$$\begin{aligned} \mathbb{P}\left(\sum_{k=0}^n 2^{-(d-1)k} V_k \geq 2ML\right) &\leq \mathbb{P}\left(\sum_{k=0}^n W_k \geq ML\right) \\ (2.27) \quad &\leq \exp(-\lambda ML) \prod_{k=0}^n \mathbb{E}[\exp(\lambda W_k)] \\ &\leq \exp(-\lambda ML) \prod_{k=0}^n \lambda \int_0^\infty \exp(\lambda t - Mt^r 2^{(d-r)k}) dt, \end{aligned}$$

where we have used Lemma 5 in the last inequality. Since there exist $C_1, C_2 > 0$ such that for any $t \geq 0$,

$$\lambda t - \frac{Mt^r 2^{(d-r)k}}{2} \leq C_1 \lambda^{r/(r-1)} 2^{-(d-r)k/(r-1)},$$

and

$$(2.28) \quad \int_0^\infty \exp(-Mt^r 2^{(d-r)k}/2) dt \leq 2^{k/r} \int_0^\infty \exp(-Mt^r/2) < 2^{k/r} C_2,$$

it is further bounded from above by

$$\begin{aligned} &C2^{n^2/r} \lambda \exp\left(-\lambda ML + C\lambda^{r/(r-1)} \sum_{k=0}^n 2^{-(d-r)k/(r-1)}\right) \\ (2.29) \quad &\leq \begin{cases} C \exp(Cn^2) \exp(-\lambda ML + C\lambda^{r/(r-1)}) & \text{if } r < d, \\ C \exp(Cn^2) \exp(-\lambda ML + C\lambda^{r/(r-1)}n) & \text{if } r = d, \\ C \exp(Cn^2) \exp(-\lambda ML + C\lambda^{r/(r-1)}2^{-(d-r)n/(r-1)}) & \text{if } r \in (d, d+1]. \end{cases} \end{aligned}$$

Taking the infimum over $\lambda > 0$, by $2^n \leq m \leq k \leq 2L+1$, we have

$$\begin{aligned}
(2.30) \quad & \mathbb{P} \left(\sum_{k=0}^n 2^{-(d-1)k} V_k \geq 2ML \right) \\
& \leq \begin{cases} \exp(Cn^2) \exp(-M^r r^{-r} ((r-1)/C)^{r-1} L^r) & \text{if } r < d, \\ \exp(Cn^2) \exp(-M^r r^{-r} ((r-1)/C)^{r-1} L^r n^{1-r}) & \text{if } r = d, \\ \exp(Cn^2) \exp(-M^r r^{-r} ((r-1)/C)^{r-1} L^r 2^{(d-r)n}) & \text{if } r \in (d, d+1], \end{cases} \\
& \leq \exp(-g(r, d, L, m)M).
\end{aligned}$$

□

Let Z be a uniformly distributed random variable on $(\{0, 1\}^{d-1})^n$ and independent of τ . We write $Z = (z_1, \dots, z_n)$ and denote the probability and expectation with respect to Z by $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{E}}$. The following lemma corresponds to Corollary 3.2 in [2].

Lemma 7.

$$\tilde{\mathbb{P}} \left(V_0 + \sum_{k=1}^n \sum_{e \in \Xi^{z_1, \dots, z_k, k}} \tau_e \geq 10 \sum_{k=0}^n 2^{-(d-1)k} V_k \right) \leq \frac{1}{10} \text{ } \mathbb{P}\text{-a.s.}$$

Proof. Since $\tilde{\mathbb{E}} [V_0 + \sum_{k=1}^n \sum_{e \in \Xi^{z_1, \dots, z_k, k}} \tau_e] = \sum_{k=0}^n 2^{-(d-1)k} V_k$, by Markov inequality, we have that for any $R > 0$,

$$\begin{aligned}
(2.31) \quad & \tilde{\mathbb{P}} \left(V_0 + \sum_{k=1}^n \sum_{e \in \Xi^{z_1, \dots, z_k, k}} \tau_e \geq R \sum_{k=0}^n 2^{-(d-1)k} V_k \right) \\
& \leq \left(R \sum_{k=0}^n 2^{-(d-1)k} V_k \right)^{-1} \tilde{\mathbb{E}} \left[V_0 + \sum_{k=1}^n \sum_{e \in \Xi^{z_1, \dots, z_k, k}} \tau_e \right] \\
& \leq 1/R.
\end{aligned}$$

Thus, taking $R = 10$, the claim holds. □

Let $G_n = \{\sum_{j=1}^n 2^{n-j+1} i_j \mid i_j \in \{0, 1\}^{d-1}, j = 1, \dots, n\} \subset \mathbb{Z}^{d-1}$. We write $(v_1, \bar{v}) = (v_1, v_2, \dots, v_d)$. Then, the above lemma yields that

$$\begin{aligned}
(2.32) \quad & \mathbb{P} \left(\#\{\bar{v} \in G_n \mid \bar{t}^m(0, (d2^n, \bar{v}) \leq 20ML\} \leq \frac{9}{10} 2^{(d-1)n} \right) \\
& \leq \mathbb{P} \left(\#\left\{ (i_1, \dots, i_n) \in (\{0, 1\}^{d-1})^n \mid V_0 + \sum_{k=1}^n \sum_{e \in \Xi^{i_1, \dots, i_k, k}} \tau_e \leq 20ML \right\} \leq \frac{9}{10} 2^{(d-1)n} \right) \\
& \leq \mathbb{P} \left(\sum_{k=0}^n 2^{-(d-1)k} V_k \geq 2ML \right) \\
& \leq \exp(-g(r, d, L, m)M),
\end{aligned}$$

where we have used the condition $(\Xi-4)$ in the first inequality, Lemma 7 in the second inequality and Lemma 6 in the last inequality. Similarly,

$$(2.33) \quad \mathbb{P} \left(\#\{\bar{v} \in G_n \mid \bar{t}^m((m - d2^n, \bar{v}), m\mathbf{e}_1) \leq 2M^2 L\} \leq \frac{9}{10} 2^{(d-1)n} \right) \leq \exp(-g(r, d, L, m)M).$$

Let $\gamma_{\bar{v}}$ be a straight path from $(d2^n, \bar{v})$ to $(m - d2^n, \bar{v})$, i.e.

$$\gamma_{\bar{v}} = \{(k + d2^n, \bar{v}) \mid k = 0, \dots, m - d2^{n+1}\}.$$

Then for sufficiently large M , with C_2 in Lemma 4, it follows that

$$\begin{aligned}
 & \mathbb{P}(\bar{t}^m(0, m\mathbf{e}_1) > 10M^2L) \\
 & \leq \mathbb{P}\left(\#\{\bar{v} \in G_n \mid t(\gamma_{\bar{v}}) \leq 2M^2L\} \leq \frac{1}{5}2^{(d-1)n}\right) + 2\exp(-g(r, d, L, m)M) \\
 (2.34) \quad & \leq 2^{|G_n|}(\mathbb{P}(t(\gamma_{\bar{v}}) > 2M^2L))^{\frac{4}{5}2^{(d-1)n}} + 2\exp(-g(r, d, L, m)M) \\
 & \leq 2^{2^{n(d-1)}} \exp\left(-\frac{4}{5}C_2(2M^2L/m)^r m2^{(d-1)n}\right) + 2\exp(-g(r, d, L, m)M) \\
 & \leq 3\exp(-g(r, d, L, m)M),
 \end{aligned}$$

where we have applied Lemma 4 in the third inequality, and the proof is completed. $\square$

Proof of Proposition 3. Let $v(0) = v$ and $v(i) = (w_1, \dots, w_i, v_{i+1}, v_{i+2}, \dots, v_d)$. Then we have

$$(2.35) \quad \mathbb{P}(t_{[0, L_1]^d}(v, w) > dM_2L) \leq \sum_{i=1}^d \mathbb{P}(t_{[0, L_1]^d}(v(i-1), v(i)) > M_2L).$$

We first consider $i = 1$. Let $m = |v_1 - w_1| = |v(1) - v(0)|_1$. Without loss of generality, we suppose that $v_1 < w_1$ and $v + [0, m] \times [0, m/2]^d \subset [0, L_1]^d$. Then from Proposition 4,

$$\begin{aligned}
 (2.36) \quad & \mathbb{P}(t_{[0, L_1]^d}(v(0), v(1)) > M_2L) \leq \mathbb{P}(\bar{t}^m(v, v + m\mathbf{e}_1) > M_2L) \\
 & \leq \exp(-g(r, d, L, m)M_1) \leq \exp(-g(r, d, L, L_1)M_1).
 \end{aligned}$$

Considering the rotation, other cases can be treated the same way and we have

$$(2.37) \quad \mathbb{P}(t_{[0, L_1]^d}(v, w) > dM_2L) \leq d\exp(-g(r, d, L, L_1)M_1).$$

$\square$

3. PROOF FOR THE LOWER BOUND

3.1. From the means to the lower bound. Suppose that the condition of Theorem 3 holds. We take $\tilde{\tau}_e$ such that if $\tau_e < cf_{d,r}(N) - 1$, $\tilde{\tau}_e = \tau_e$ and otherwise, $\tilde{\tau}_e = \tau_e + 1$. The following statement will be proved in the next subsections.

Lemma 8. *For any $\delta > 0$, there exists $c > 0$ such that*

$$(3.1) \quad \mathbb{E} \left[\min_{\pi_N \in \tilde{\mathbb{O}}_N} \#\{e \in \pi_N \mid \tilde{\tau}_e \geq cf_{d,r}(N)\} \right] \geq cN^{1-\delta}.$$

Since

$$\tilde{t}(0, N\mathbf{e}_1) \geq t(0, N\mathbf{e}_1) + \min_{\pi_N \in \tilde{\mathbb{O}}_N} \#\{e \in \pi_N \mid \tilde{\tau}_e \geq cf_{d,r}(N)\},$$

this lemma yields

$$(3.2) \quad \mathbb{E}t(0, N\mathbf{e}_1) + cN^{1-\delta} \leq \mathbb{E}\tilde{t}(0, N\mathbf{e}_1).$$

We introduce the following concentration inequality.

Lemma 9. *Suppose $\mathbb{E}\tau_e^2 < \infty$. There exists $C > 0$ such that for any sufficiently large N ,*

$$(3.3) \quad \mathbb{P}(|t(0, N\mathbf{e}_1) - \mathbb{E}t(0, N\mathbf{e}_1)| > N^{1-2\delta}) \leq N^{-(1-4\delta)},$$

$$(3.4) \quad \mathbb{P}(|\tilde{t}(0, N\mathbf{e}_1) - \mathbb{E}\tilde{t}(0, N\mathbf{e}_1)| > N^{1-2\delta}) \leq N^{-(1-4\delta)}.$$

Proof. The proof of this lemma follows from Theorem 3.1 in [1] which was first proved in [7]. Indeed, since $\mathbb{E}\tau_e^2, \mathbb{E}\tilde{\tau}_e^2 < C$ with some constant $C \in (0, \infty)$ independent of N , Theorem 3.1 in [1] shows that

$$(3.5) \quad \mathbb{E}[(t(0, N\mathbf{e}_1) - \mathbb{E}t(0, N\mathbf{e}_1))^2] \leq CN,$$

$$(3.6) \quad \mathbb{E}[(\tilde{t}(0, N\mathbf{e}_1) - \mathbb{E}\tilde{t}(0, N\mathbf{e}_1))^2] \leq CN,$$

with some constant $C > 0$. Then, we have

$$(3.7) \quad \begin{aligned} \mathbb{P}(|t(0, N\mathbf{e}_1) - \mathbb{E}t(0, N\mathbf{e}_1)| > N^{1-2\delta}) &\leq N^{-2(1-2\delta)} \mathbb{E}[(t(0, N\mathbf{e}_1) - \mathbb{E}t(0, N\mathbf{e}_1))^2] \\ &\leq CN^{-(1-4\delta)}, \end{aligned}$$

which yields (3.3). The same argument proves (3.4). $\square$

If $\delta > 0$ is sufficiently small, by (3.2), this lemma leads us to

$$(3.8) \quad \begin{aligned} &\mathbb{P}(t(0, N\mathbf{e}_1) = \tilde{t}(0, N\mathbf{e}_1)) \\ &\leq \mathbb{P}(|t(0, N\mathbf{e}_1) - \mathbb{E}t(0, N\mathbf{e}_1)| > N^{1-2\delta}) + \mathbb{P}(|\tilde{t}(0, N\mathbf{e}_1) - \mathbb{E}\tilde{t}(0, N\mathbf{e}_1)| > N^{1-2\delta}) \\ &\leq 2CN^{-1/2}. \end{aligned}$$

Since $\min_{\mathbb{O}_N} \mathcal{M}(\pi_N) < cf_{d,r}(N) - 1$ implies $t(0, N\mathbf{e}_1) = \tilde{t}(0, N\mathbf{e}_1)$, we have Theorem 3. Similarly under the condition of Theorem 4, (3.1) holds with $r = 0$ (see subsection 3.3). This yields Theorem 4 by the same argument as above.

Remark 5. Let us comment on how to prove Proposition 1. The proofs of (3.1) can be applied in this case and Theorem 2 in [9] yields the better concentration bounds:

Lemma 10. *Suppose $\mathbb{E}[\tau_e^2] < \infty$. Then for any m and $\delta < 1/4$, there exists N_0 such that for any $N \geq N_0$,*

$$(3.9) \quad \mathbb{P}(|t(D(0), D(N\mathbf{e}_1)) - \mathbb{E}t(D(0), D(N\mathbf{e}_1))| > N^{1-2\delta}) \leq N^{-m},$$

$$(3.10) \quad \mathbb{P}(|\tilde{t}(D(0), D(N\mathbf{e}_1)) - \mathbb{E}\tilde{t}(D(0), D(N\mathbf{e}_1))| > N^{1-2\delta}) \leq N^{-m}.$$

Combining with the previous arguments and the Borel-Cantelli lemma, we have the desired conclusion.

3.2. Lower bound for $0 < r < d-1$ or $r = 1$. Our goal is to prove (3.1). The proof is based on the argument in [3], but the choice of box size and configurations inside of the box are considerably more complicated. The following lemma appears in Lemma 5.5 in [3].

Lemma 11. *There exist $\delta_{11} > 0$ and $K > 0$ such that for any $v, w \in \mathbb{Z}^d$,*

$$\mathbb{P}(t(v, w) < (F^- + \delta_{11})|v - w|_1) \leq e^{-K|v - w|_1}.$$

We fix $\delta_{11} > 0$ which satisfies Lemma 11. Note that Lemma 11 also holds with $\tilde{\tau}$ since $\tilde{\tau}_e \geq \tau_e$. Remark that the usefulness of F is used only in Lemma 11 to prove (3.1). Since $\tilde{\tau}$ satisfies the condition of Theorem 3 with the same r and some new constants instead of $\alpha, \beta, \dots$, which are independent of N , it suffices to show (3.1) for τ , i.e.

$$(3.11) \quad \mathbb{E} \left[\min_{\pi_N \in \mathbb{O}_N} \sharp\{e \in \pi_N \mid \tau_e \geq cf_{d,r}(N)\} \right] \geq cN^{1-\delta}.$$

We take $M > 0$ sufficiently large and sufficiently small positive number $s > 0$ to be chosen later. Set $n = \lfloor f_{d,r}(N) \rfloor$ and $n_1 = \lfloor sf_{d,r}(N) \rfloor$, where $\lfloor \cdot \rfloor$ is a floor function. We define three kinds of box whose notation are the same as in [3] (see Figure 3). First, define the hypercubes $S(l; n)$, for $l \in \mathbb{Z}^d$, by

$$S(l; n) = \{v \in \mathbb{Z}^d : nl_i \leq v_i < n(l_i + 1), 1 \leq i \leq d\}.$$

We call these hypercubes n -cubes. Second, we define the large n -cubes $T(l; n)$, for $l \in \mathbb{Z}^d$, by

$$T(l; n) = \{v \in \mathbb{Z}^d : nl_i - n \leq v_i \leq n(l_i + 2), 1 \leq i \leq d\}.$$

Finally, we define the n -boxes $B^j(l; n)$, for $l \in \mathbb{Z}^d$ and $j \in \{\pm 1, \dots, \pm d\}$, as

$$B^j(l; n) = T(l; n) \cap T(l + 2\operatorname{sgn}(j)\mathbf{e}_{|j|}; n),$$

and its inner boundary $\partial B^j(l; n)$ as

$$\partial B^j(l; n) = \{v \in B^j(l; n) \mid \text{there exists } w \notin B^j(l; n) \text{ such that } |v - w|_1 = 1\}.$$

Note that $S(l; n) \subset T(l; n)$ and $B^j(l; n)$ is a closed box of size $3n \times \dots \times 3n \times n \times 3n \times \dots \times 3n$. Let

$$D^j(l; n) = \{v \in B^j(l; n) \mid d_\infty(v, B^j(l; n)^c) > n_1, v \in n_1\mathbb{Z}^d\},$$

$$C^j(l; n) = \{v + k\mathbf{e}_i \mid v + k\mathbf{e}_i \in B^j(l; n), k \in \mathbb{Z}, i \in \{1, \dots, d\}, v \in D^j(l; n)\}.$$

$$\tilde{C}^j(l; n) = \{\langle v, w \rangle \mid v, w \in C^j(l; n), |v - w|_1 = 1\}.$$

Definition 4. An n -box $B^j(l, n)$ is black if the following two hold:

(Black-1) For any $v, w \in B^j(l; n)$ with $|v - w|_1 \geq n$,

$$t(v, w) \geq (F^- + \delta_{11})|v - w|_1,$$

where $\delta_{11} > 0$ is the constant in Lemma 11.

(Black-2) For any $v \in \partial B^j(l; n)$, there exist $w \in C^j(l, n) \cap \partial B^j(l; n)$ and a path π from v to w such that $|v - w|_1 \leq 2dn_1$, $t(\pi) \leq Mn_1$ and π passes only on $\partial B^j(l; n)$.

An n -cube $S(l, n)$ is said to be black if each of its surrounding n -boxes is black.

We change configurations inside $B^j(l; n)$ so that the optimal path passes through $C^j(l; n)$ with small passage times. Note that $|C^j(l; n)| \leq C(s)f_{d,r}(N)$ with some constant $C(s)$ which depends only on s and d .

Lemma 12. If we take M sufficiently large,

$$\mathbb{P}(B^j(l; n) \text{ is black}) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Proof. By Lemma 11,

$$\begin{aligned} & \mathbb{P}(B^j(l; n) \text{ does not satisfy (Black-1)}) \\ (3.12) \quad & \leq 2d\sharp B^j(l; n)^2 \max\{\mathbb{P}(t(v, w) < (F^- + \delta)|v - w|_1) \mid v, w \in \partial B^j(l; n), |v - w|_1 \geq n\} \\ & \leq 2d(3n)^{2d} \exp(-Kn), \end{aligned}$$

and thus (Black-1) holds with high probability. Since $\mathbb{E}[\tau_e^2] < \infty$, there exists $C > 0$ such that for any $v \in \partial B$, there exists $w \in \partial B^j(l; n) \cap C^j(l; n)$ such that $|v - w|_1 \leq 2dn_1$ and by the same argument of the proof of Theorem 2, $\mathbb{P}(t_{\partial B^j(l; n)}(v, w) > 4dn_1\mathbb{E}\tau_e) \leq Cn_1^{-2(d-1)}$. Note that there exists $C(s) > 0$ such that $\sharp[\partial B^j(l; n) \cap C^j(l; n)] < C(s)$. Therefore, by the union bound, (Black-2) also holds with high probability. $\square$

Combining the previous lemma with a similar argument (Peierls argument) of (5.2) in [3] shows the following lemma. We skip the details.

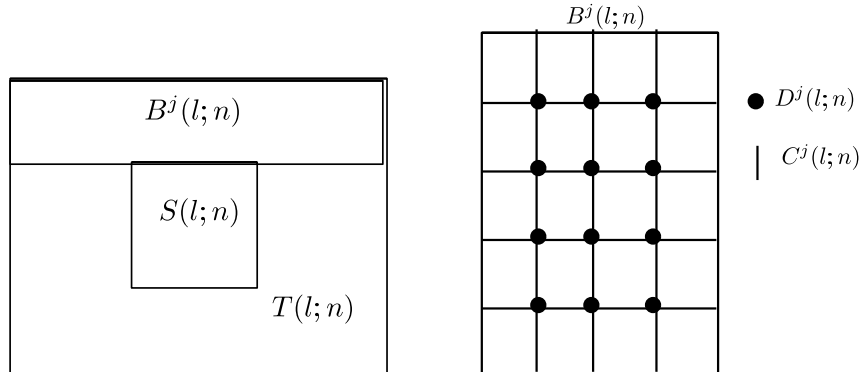


FIGURE 3

Left: The figure of $S(l; n)$, $T^j(l; n)$ and $B^j(l; n)$.
right: The figure of $C(l; n)$ and $D^j(l; n)$

Lemma 13. *There exist $\epsilon, D > 0$ such that for any $N \in \mathbb{N}$,*

$$\mathbb{P} \left(\exists \Gamma \in \mathbb{O}_N \text{ visiting at most } \frac{\epsilon N}{f_{d,r}(N)} \text{ distinct black } n\text{-cubes} \right) \leq \exp \left(-D \frac{N}{f_{d,r}(N)} \right).$$

A path which starts in $S(l; n)$ and ends outside of $T(l; n)$ must have a segment which lies entirely in one of the surrounding n -boxes, and which connects the two opposite large faces of that n -box. This means that this path crosses the n -box in the short direction (see Figure 3). Hereafter “crossing an n -box” means crossing in the short direction. From this and Lemma 13, we have

$$\mathbb{E}[\#\{\text{distinct black } n\text{-box } B^j(l; n) \text{ such that } \exists \Gamma \in \mathbb{O}_N \text{ crossing } B^j(l; n)\}] \geq \frac{\epsilon N}{2f_{d,r}(N)}.$$

For the main step, we set

$$(3.13) \quad \tilde{E}^j(l; n) = \{\langle v, w \rangle \mid v \in C^j(l; n) \setminus \partial B^j(l; n) \text{ and } w \notin C^j(l; n) \setminus \partial B^j(l; n)\}.$$

It is easy to see that there exists $C(s) > 0$ such that

$$(3.14) \quad |\tilde{C}^j(l; n)|, |\tilde{E}^j(l; n)| \leq C(s)f_{d,r}(N).$$

Fix some small constant $c > 0$ depending on s .

Definition 5. *An n -box $B^j(l; n)$ is said to be good if for any $\pi_N \in \mathbb{O}_N$, there exists $e \in \pi_N$ such that both vertices of e are in $B^j(l; n)$ and $\tau_e \geq cf_{d,r}(N)$.*

Definition 6. *We say that the collection $\tau = \{\tau_e\}_{e \in E^d}$ satisfies A_1 -condition if*

$$\begin{cases} cf_{d,r}(N) \leq \tau_e \leq \gamma cf_{d,r}(N) & \text{when } e \in \tilde{E}^j(l; n) \\ \tau_e \leq F^- + c & \text{when } e \in \tilde{C}^j(l; n) \setminus \tilde{E}^j(l; n), \end{cases}$$

where γ is in Theorem 3. If τ satisfies A_1 -condition, we write $\tau \in A_1$.

A resampled configuration $\tau^* = \{\tau_e^*\}_{e \in E^d}$ is taken to be $\tau_e^* = \tau_e$ if $e \notin \tilde{C}^j(l; n) \cup \tilde{E}^j(l; n)$ and independent copy of τ_e if $e \in \tilde{C}^j(l; n) \cup \tilde{E}^j(l; n)$. We enlarge the probability space so that it can measure the event both for τ and τ^* and we still denote the measure by $\mathbb{P}$. Denote by $t^*(\Gamma)$ the passage time of a path Γ for τ^* . Let A be an event defined as

$$(3.15) \quad A = \{\tau^* \in A_1\} \cap \{B^j(l; n) \text{ is black for } \tau\} \cap \{\exists \pi_N \in \mathbb{O}_N \text{ crosses } B^j(l; n)\}.$$

Proposition 5. *If we take s sufficiently small depending on M and c sufficiently small depending on s , except when $0 \in B^j(l; n)$ or $N\mathbf{e}_1 \in B^j(l; n)$, we can estimate of goodness of an n -box from*

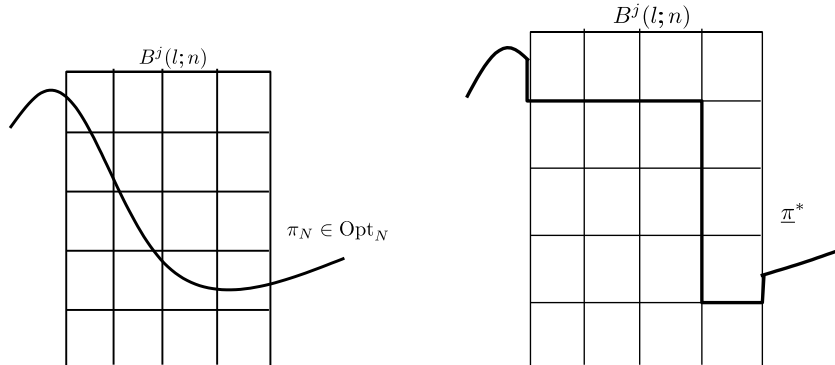


FIGURE 4

Left: $\mathbb{O}_N$ crosses a n -box in the short direction.
Right: How to construct a new path from $\mathbb{O}_N$.

below as

$$\begin{aligned}
 (3.16) \quad & \mathbb{P}(n\text{-box } B^j(l; n) \text{ is good for } \tau) \\
 &= \mathbb{P}(n\text{-box } B^j(l; n) \text{ is good for } \tau^*) \\
 &\geq \mathbb{P}(A).
 \end{aligned}$$

Proof. Since the equality is trivial, we will only give a proof of the inequality. Assume that the event A occurs. We take an arbitrary optimal path $\pi_N \in \mathbb{O}_N$. To prove (3.16), it suffices to show that

$$(3.17) \quad t^*(0, \mathbf{e}_1) < t(0, \mathbf{e}_1).$$

In fact, since we change configurations only on $\tilde{C}^j(l; n)$ and $\tilde{E}^j(l; n)$, π_N^* has to pass through $\tilde{C}^j(l; n) \cup \tilde{E}^j(l; n)$ at least one time. Moreover, in order to enter $\tilde{C}^j(l; n)$, π_N^* has to pass through $\tilde{E}^j(l; n)$. Therefore $B^j(l; n)$ is good for τ^* . This yields (3.16).

We construct a new path π^* from π_N to prove (3.17) as follows (see also Figure 3). Let v and w be the first intersecting point and the last point between π_N and $B^j(l; n)$, respectively. Under the assumption that $B^j(l; n)$ is black, we take $v_1, w_1 \in \partial B^j(l; n) \cap C^j(l; n)$ and a path $\pi_1^* : v \rightarrow v_1$ and $\pi_2^* : w_1 \rightarrow w$ on $\partial B^j(l; n)$ such that $\max\{t(\pi_1^*), t(\pi_2^*)\} \leq Mn_1$ and $\max\{|v - v_1|_1, |w - w_1|_1\} \leq 2dn_1$. Note that τ and τ^* are the same on $\partial B^j(l; n)$. We take a path $\pi_3^* \subset \tilde{C}^j(l; n) \cup \tilde{E}^j(l; n)$ from v_1 to w_1 such that π_3^* has two edges of $\tilde{E}^j(l; n)$ and at most $|v_1 - w_1|_1 + 4n_1$ edges of $\tilde{C}^j(l; n)$. Finally, we connect $\pi_N|_{0 \rightarrow v}$, π_1^* , π_3^* , π_2^* , $\pi_N|_{w \rightarrow N\mathbf{e}_1}$ in this order and we construct π^* . Note that since π_N crosses $B^j(l; n)$, the number of edges of $\max\{|x - y|_1 \mid x, y \in \pi_N \cap B^j(l; n)\} \geq n$. Therefore, on the event A ,

$$\begin{aligned}
 (3.18) \quad & t(v, w) \geq (F^- + \delta_{11})(|v - w|_1 \vee n) \\
 &\geq (F^- + \delta_{11}/2)|v - w|_1 + \delta_{11}n/2,
 \end{aligned}$$

and by $|v_1 - w_1|_1 \leq 3dn$ and $\#\pi_2^* \leq |v_1 - w_1|_1 + 4dn_1 \leq |v - w|_1 + 8dn_1$, we obtain

$$\begin{aligned}
 (3.19) \quad & t^*(\pi_1^*) + t^*(\pi_2^*) + t^*(\pi_3^*) \leq Mn_1 + 2\gamma cf_{d,r}(N) + (F^- + c)(|v - w|_1 + 8dn_1) + Mn_1 \\
 &\leq F^-|v - w|_1 + 4Mn_1.
 \end{aligned}$$

It follows that on the event A , if s is sufficiently small depending on M ,

$$\begin{aligned}
 (3.20) \quad & t(0, N\mathbf{e}_1) - t^*(\pi^*) \geq (F^- + \delta_{11}/2)|v - w|_1 + \delta_{11}n/2 - F^-|v - w|_1 - 4Mn_1 \\
 &\geq \delta_{11}n/4.
 \end{aligned}$$

We take an arbitrary optimal path from 0 to $N\mathbf{e}_1$ for τ^* and write π_N^* . Then, it follows that

$$t(0, N\mathbf{e}_1) = t(\pi_N^*) \geq t^*(\pi^*) + \delta_{11}n/4 \geq t^*(\pi_N^*) + \delta_{11}n/4 = t^*(0, N\mathbf{e}_1) + \delta_{11}n/4.$$

.

□

By (3.14), if we take c to be sufficiently small depending on s again, for sufficiently large N ,

$$\begin{aligned}
 (3.21) \quad & \mathbb{P}(\tau^* \in A_1) \geq \left\{ \beta e^{-\alpha c^r (\log N)^{\frac{r}{r+1}}} \right\}^{|\tilde{E}|} \mathbb{P}(\tau_e^* < F^- + c)^{|\tilde{C}|} \\
 &\geq N^{-\delta}.
 \end{aligned}$$

Thus (3.16) is further bounded from below by

$$(3.22) \quad N^{-\delta} \mathbb{P}(B^j(l; n) \text{ is black for } \tau \text{ and } \exists \pi_N \in \mathbb{O}_N \text{ crosses } B^j(l; n)).$$

From this, we have

$$\begin{aligned}
 (3.23) \quad & \mathbb{E} \left[\inf_{\pi_N \in \mathbb{O}_N} \#\{e \in \pi_N \text{ such that } \tau_e \geq cf_{d,r}(N)\} \right] \\
 &\geq \frac{1}{2d} N^{-\delta} \mathbb{E}[\#\{\text{distinct black } n\text{-box } B^j(l; n) \text{ such that } \exists \pi_N \in \mathbb{O}_N \text{ crosses } B^j(l; n)\}] \\
 &\geq \frac{1}{4d} \frac{\epsilon N^{1-\delta}}{f_{d,r}(N)},
 \end{aligned}$$

where $2d$ appears because of the overlap of n -boxes. Therefore the proof is completed.

3.3. Lower bound for $r = 0$. Let $n = \lfloor sf_{d,0}(N) \rfloor$ and $n_1 = \lfloor s^{1+\frac{1}{2d}} f_{d,0}(N) \rfloor$, and other definitions be the same as before. Then the same arguments as in subsection 3.2 work to prove Lemma 12 and Proposition 5. Moreover since $\#\tilde{E}^j(l; n) \leq 2d(3n/n_1)^{d-1}(3n) \leq s^{1/2} f_{d,0}(N)$, if s is sufficiently small, for any $\delta > 0$ the probability that τ^* satisfies A_1 -condition can be bounded from below by

$$(3.24) \quad \left\{ \beta \frac{\log N}{\log \log N} \right\}^{-\alpha|\tilde{E}|} \mathbb{P}(\tau_e^* < F^- + c)^{|\tilde{C}|} \geq N^{-\delta}.$$

The rest is the same as before.

3.4. Lower bound for $r > d - 1$. We take $M > 0$ sufficiently large and sufficiently small positive number $s > 0$ depending on $M > 0$ specified later and define $s_1 = s^{1+\frac{d}{8r}}$. Set $n = \lfloor sf_{d,r}(N) \rfloor$ and $n_1 = 2\lfloor s_1 f_{d,r}(N) \rfloor$. Here we have defined n_1 to be even for convenience. We use the same definitions of $C^j(l; n)$, $\tilde{C}^j(l; n)$, and $D^j(l; n)$ as above. We change the definitions of $E^j(l; n)$ and $\tilde{E}^j(l; n)$ (See Figure 5) as

$$E^j(l; n) = \{v \in C^j(l; n) \mid \exists w \in C^j(l; n) \text{ such that } |v - w|_1 = n_1/2 = \lfloor s_1 f_{d,r}(N) \rfloor\},$$

$$\tilde{E}^j(l; n) = \{\langle v, w \rangle \mid v \in E^j(l; n), w = v + \mathbf{e}_i \in C^j(l; n), i \in \{1, \dots, d\}\}.$$

Given $a \in C^j(l; n) \setminus E^j(l; n)$, let W_a be the connected component of $C^j(l; n) \setminus E^j(l; n)$ containing a , i.e. $W_a = \text{Conn}(a, C^j(l; n) \setminus E^j(l; n))$. We list the basic properties of W_a and $C^j(l; n)$.

Lemma 14.

- (i) For any $a \in C^j(l; n) \setminus E^j(l; n)$ and $b \in W_a$, there exists a path $\pi = (x_0, \dots, x_l)$ from a to b which lies only on $C^j(l; n) \setminus E^j(l; n)$ and $l = |a - b|_1$.
- (ii) For any $a, b \in C^j(l; n)$ with $|a - b|_1 < n_1/4$, there exists a path $\pi = (x_0, \dots, x_l)$ from a to b which lies only on $C^j(l; n)$ and $l = |a - b|_1$.
- (iii) For any $a \in \partial B^j(l; n)$ and $b \in C^j(l; n)$ with $|a - b|_1 < n_1/4$, there exists a path $\pi = (x_0, \dots, x_l)$ from a to b which lies only on $C^j(l; n) \cup \partial B^j(l; n)$ and $l = |a - b|_1$.
- (iv) If $|a - b|_1 \leq n_1/4$ and $W_a \neq W_b$, there exists $\langle y_1, y_2 \rangle \in \tilde{E}^j(l; n)$ such that $|a - y_1|_1, |b - y_1|_1 \leq n_1/4 + 1$ and a line L including both y_1 and y_2 also includes both a and b .

Proof. (i) It is easy to see that for any connected component, namely W_a , there exists $x \in (D^j(l; n) \cup \partial B^j(l; n)) \cap W_a$ such that

$$W_a \subset \{x + k\mathbf{e}_i \mid i \in \{1, \dots, d\}, k \in \mathbb{Z}^d\}.$$

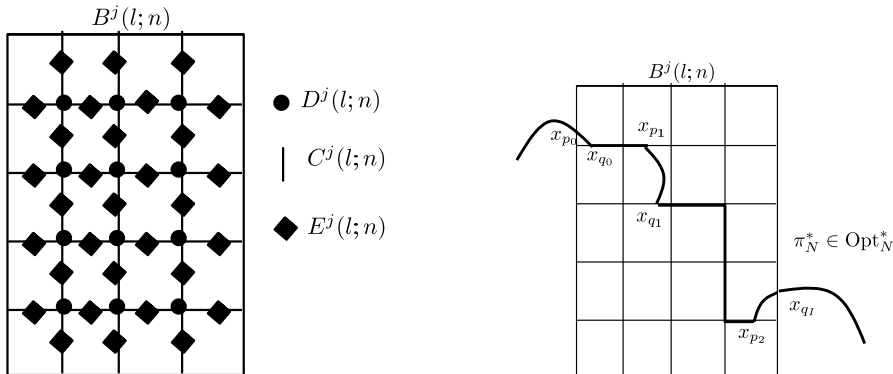


FIGURE 5

Left: The figure of $C^j(l; n)$, $D^j(l; n)$, $E^j(l; n)$.
Right: The figure in the proof of Proposition 10.

This yields (i).

(ii) Fix $a \in C^j(l; n)$. If there exists $x \in D^j(l; n)$ such that $|a - x|_1 < n_1/4$, then since

$$\{y \in C^j(l; n) \mid |a - y|_1 < n_1/4\} \subset \{x + k\mathbf{e}_i \mid i \in \{1, \dots, d\}, k \in \mathbb{Z}^d\},$$

the claim holds. Otherwise, there exists $j \in \{1, \dots, d\}$ such that $d_1(a_j, n_1\mathbb{Z}) \geq n_1/4$ and $a_i \in n_1\mathbb{Z}$ for $i \neq j$. Since $C^j(l; n) \subset \{v \in \mathbb{Z}^d \mid \exists j \in \{1, \dots, d\} \text{ such that if } i \neq j, v_i \in n_1\mathbb{Z}\}$, we have

$$\{y \in C^j(l; n) \mid |a - y|_1 < n_1/4\} \subset \{a + k\mathbf{e}_j \mid k \in \mathbb{Z}\},$$

and since $\{y \in C^j(l; n) \mid |a - y|_1 < n_1/4\}$ is connected, (ii) holds.

(iii),(iv) They follow from the construction of $C^j(l; n)$ directly. $\square$

Given $x \in \mathbb{Z}^d$, we define $\ell(x) = \inf\{|x - y|_1 \mid y \in E^j(l; n)\}$ and given $e = \langle x, y \rangle \in E^d$, $\ell(e) = \inf\{\ell(x), \ell(y)\}$.

Lemma 15. *There exists $C(s) > 0$ such that for any $0 \leq \ell \leq 2dn_1$,*

$$(3.25) \quad \#\{e \in E^d \mid \ell(e) = \ell\} \leq C(s)(\ell + 1)^{d-1}.$$

If $\ell > 2dn_1$,

$$(3.26) \quad \#\{e \in E^d \mid \ell(e) = \ell, e \in B^j(l; n)\} = 0.$$

Proof. Since there exists $C_1(s) > 0$ such that $\#E^j(l; n) \leq C_1(s)$,

$$(3.27) \quad \begin{aligned} \#\{x \in \mathbb{Z}^d \mid \ell(x) = \ell\} &\leq \#\{x \in \mathbb{Z}^d \mid \exists y \in E^j(l; n) \text{ such that } |x - y|_1 = \ell\} \\ &\leq C_1(s)\#\{x \in \mathbb{Z}^d \mid |x|_1 = \ell\}. \end{aligned}$$

By $\#\{x \in \mathbb{Z}^d \mid |x|_1 = \ell\} \leq d^d(\ell + 1)^{d-1}$, we have

$$\#\{x \in \mathbb{Z}^d \mid \ell(x) = \ell\} \leq d^d C_1(s)(\ell + 1)^{d-1}.$$

In particular, we get $\#\{e \in E^d \mid \ell(e) = \ell\} \leq 2d^{d+1}C_1(s)(\ell + 1)^{d-1}$ as desired. If $k > 2dn_1$, (3.26) is trivial due to the way of construction. $\square$

Definition 7. *An n -box $B^j(l, n)$ is black if the following hold:*

(Black-1) *For any $v, w \in B^j(l; n)$ with $|v - w|_1 \geq n$,*

$$t(v, w) \geq (F^- + \delta_{11})|v - w|_1.$$

(Black-2) *For any $v, w \in \partial B^j(l; n)$, there exists a path π from v to w such that $t(\pi) \leq M(|v - w|_1 \vee (\log N)^{1/8dr})$ and π passes only on $\partial B^j(l; n)$.*

(Black-3) *For any edge $e \in \partial B^j(l; n)$, $\tau_e \leq (\log N)^{\frac{1}{8dr}}$.*

Lemma 16.

$$P(B^j(l; n) \text{ is black}) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Proof. The same proof of Lemma 12 yields that (Black-1) holds with high probability. Next, we consider (Black-2). Since $\mathbb{E}\tau_e^{2m} < \infty$ with $m = 2[16d^2r]$, by the same argument of Theorem 2, for any $v, w \in \partial B^j(l; n)$, we have

$$(3.28) \quad \mathbb{P}\left(t_{\partial B^j(l; n)}(v, w) > M(|v - w|_1 \vee (\log N)^{\frac{1}{8dr}})\right) \leq (\log N)^{-\frac{m}{8dr}} \leq (\log N)^{-2d}.$$

By $\#\partial B^j(l; n) \leq C(d)f_{d,r}(N)^{d-1} \leq C(d)(\log N)^{d-1}$ with some constant $C(d) > 0$, (Black-2) holds with high probability. A simple calculation shows that (Black-3) holds with high probability. $\square$

Definition 8. We say that the collection $\tau = \{\tau_e\}_{e \in E^d}$ satisfies A_2 -condition if

$$\begin{cases} c^2 f_{d,r}(N) \leq \tau_e \leq \gamma c^2 f_{d,r}(N), & \text{if } e \in \tilde{E}^j(l; n) \\ \tau_e \leq F^- + c^2, & \text{if } e \in \tilde{C}^j(l; n) \setminus \tilde{E}^j(l; n) \\ \tau_e \geq \frac{c f_{d,r}(N)}{\ell(e)+1} \vee M^2, & \text{if } e \subset \iota(B^j(l; n)) \text{ and } e \notin \tilde{C}^j(l; n) \\ \tau_e \geq (\log N)^{\frac{1}{2dr}}, & \text{if } e \cap \partial B^j(l; n) \neq \emptyset, e \cap (\iota(B^j(l; n))) \neq \emptyset, e \notin \tilde{C}^j(l; n), \end{cases}$$

where recall that $\iota(B) = B \setminus \partial B$. Then we write $\tau \in A_2$.

A resampled configuration $\tau^* = \{\tau_e^*\}_{e \in E^d}$ is taken to be $\tau_e^* = \tau_e$ if $e \cap \iota(B^j(l; n)) = \emptyset$ and independent copy of τ_e if $e \cap \iota(B^j(l; n)) \neq \emptyset$. We define the event A as

$$(3.29) \quad A = \{\tau^* \in A_2\} \cap \{B^j(l; n) \text{ is black for } \tau\} \cap \{\exists \pi_N \in \mathbb{O}_N \text{ crosses } B^j(l; n)\}.$$

Proposition 6. For any $\delta > 0$, if $s > 0$ is sufficiently small depending on M and $c > 0$ is sufficiently small depending on s ,

$$\mathbb{P}(\tau \in A_2) \geq N^{-\delta}.$$

Proof. By the fact $\mathbb{P}(\tau_e > a \vee b) \geq \mathbb{P}(\tau_e > a) \mathbb{P}(\tau_e > b)$ for $a, b \geq 0$,

$$\begin{aligned} \mathbb{P}(\tau \in A_2) &\geq \left(\beta e^{-\alpha c^{2r} f_{d,r}(N)^r} \right)^{|\tilde{E}|} \mathbb{P}(\tau_e \leq F^- + c^2)^{|\tilde{C}|} (1 \wedge \beta)^{2d|B|} \prod_{k=1}^{2dn_1} \left(e^{-\alpha \left(\frac{c f_{d,r}(N)}{k} \right)^r} \right)^{C(s)k^{d-1}} \\ &\quad \cdot \mathbb{P}(\tau_e > M^2)^{2d|B|} \left(\beta e^{-\alpha (\log N)^{1/(2d)}} \right)^{2d|B|} \\ &\geq \exp \left(-\frac{\delta \log N}{30} \right) \exp \left(-\alpha C(s) \sum_{k=1}^{2dn_1+1} c^r \frac{f_{d,r}(N)^r}{k^{r-d+1}} \right) \exp(-4d\alpha M^{2r} n^d) \\ &\geq \exp(-\delta \log N) \geq N^{-\delta}, \end{aligned}$$

where we have used $M^{2r} n^d \leq \frac{\delta}{80\alpha d} \log N$ if s is sufficiently small, because of $r > d - 1$. $\square$

Proposition 7. When $\{0, Ne_1\} \cap B^j(l; n) = \emptyset$, if c is sufficiently small, on the event A , any $\pi_N \in \mathbb{O}_N^*$ does not touch $B^j(l; n) \setminus (\partial B^j(l; n) \cup C^j(l; n))$ and passes on $\tilde{E}^j(l; n)$.

Proof. By the same as in the case $r < d - 1$, we have $t(0, Ne_1) > t^*(0, Ne_1)$, and also any $\pi_N \in \mathbb{O}_N^*$ has to enter the inside of $B^j(l; n)$. We take an arbitrary optimal path $\pi_N^* \in \mathbb{O}_N^*$ and write $\pi_N^* = \{x_1, \dots, x_K\}$. By assuming that there exists $k \in \{1, \dots, K\}$ such that $x_k \in (B^j(l; n) \setminus (\partial B^j(l; n) \cup C^j(l; n)))$, we shall derive a contradiction.

We define $p = \max\{l \leq k | x_l \in \partial B^j(l; n) \cup C^j(l; n)\}$ and $q = \min\{l \geq k | x_l \in \partial B^j(l; n) \cup C^j(l; n)\}$. Note that $q - p \geq 1$. Set $a = x_q$ and $b = x_p$. Define $C(s) > 0$ so that $\sharp E^j(l; n) \leq C(s)$ for any j, l, n .

Step 1 ($a, b \in C^j(l; n)$): First suppose that $a \in \partial B^j(l; n)$. Since $\pi_N^*|_{a \rightarrow b}$ has to pass on an edge whose weight is at least $(\log N)^{1/2dr}$ at least one time and passes only on $B^j(l; n) \setminus (\partial B^j(l; n) \cup C^j(l; n))$ except for the starting and ending points, we have

$$(3.30) \quad t^*(a, b) \geq (\log N)^{1/(2dr)} + (|a - b|_1 - 1)M^2.$$

If $|a - b|_1 \leq n_1/4$, there also exists a path $\pi^* : a \rightarrow b$ on $\partial B^j(l; n) \cup C^j(l; n)$ such that

$$t^*(\pi^*) < M(\log N)^{1/(8dr)} + M|a - b|_1 < t^*(a, b).$$

On the other hand, if $|a - b| > n_1/4$, there also exists a path $\pi^* : a \rightarrow b$ on $\partial B^j(l; n) \cup C^j(l; n)$ such that

$$\begin{aligned} (3.31) \quad t^*(\pi^*) &< M(\log N)^{1/(8dr)} + M|a - b|_1 + C(s)c^2 \gamma f_{d,r}(N) \\ &< (\log N)^{1/(2dr)} + (|a - b|_1 - 1)M^2 \\ &< t^*(a, b). \end{aligned}$$

These are all contradiction if c is sufficiently small depending on s . Thus, we have $a \in C^j(l; n)$. Similarly, we get $b \in C^j(l; n)$.

Step 2 ($|a - b|_1 \leq n_1/4$): Note that by the same reason of (3.30), $t^*(a, b) \geq |a - b|_1 M^2$. Take a path $\pi^* : a \rightarrow b$ on $C^j(l; n)$ such that $t^*(\pi^*) \leq (F^- + c^2)(|a - b|_1 + 4n_1) + C(s)\gamma c^2 f_{d,r}(N)$. It follows that

$$(3.32) \quad \begin{aligned} M^2 |a - b|_1 &\leq t^*(a, b) \\ &= \sum_{i=p}^{q-1} \tau_{\langle x_i, x_{i+1} \rangle} \leq (F^- + c^2)(|a - b|_1 + 4n_1) + C(s)\gamma c^2 f_{d,r}(N), \end{aligned}$$

which leads to

$$|a - b|_1 \leq \frac{4(F^- + c^2)n_1 + C(s)\gamma c^2 f_{d,r}(N)}{M^2 - F^- - c^2} n_1.$$

If we take $M > 0$ sufficiently large, it follows that $|a - b|_1 \leq n_1/4$.

Step 3 ($W_a \neq W_b$): When a and b belong to the same connected component, i.e. $W_a = W_b$, by Lemma 14-(i), we can take a path $\pi^* : a \rightarrow b$ on $C^j(l; n)$ such that

$$(3.33) \quad \begin{aligned} t^*(\pi^*) &\leq (F^- + c^2)|a - b|_1 \\ &< M^2 |a - b|_1 \leq t^*(a, b), \end{aligned}$$

which is also a contradiction. Thus $W_a \neq W_b$.

Step 4 (Conclusion): It follows from Lemma 14-(ii) that we can take a path $\pi^* : a \rightarrow b$ on $C^j(l; n)$ such that

$$t^*(\pi^*) \leq c^2 \gamma f_{d,r}(N) + (F^- + c^2)|a - b|_1.$$

By Lemma 14-(iv), the line between a and b lies on $C^j(l; n)$ and it includes exactly one element of $D^j(l; n)$, which we call x . If there exists $p < k < q$ such that $x_k \in \partial B^j(l; n)$, then $\max\{|a - x_k|_1, |b - x_k|_1\} \geq n_1/4$ and

$$(3.34) \quad \begin{aligned} M^2 n_1/4 &\leq t^*(a, b) \\ &< c^2 \gamma f_{d,r}(N) + (F^- + c^2)|a - b|_1 \\ &\leq c^2 \gamma f_{d,r}(N) + (F^- + c^2)n_1/4, \end{aligned}$$

which is a contradiction. Thus for any $p < k < q$, $x_k \in \iota(B^j(l; n))$. It follows from

$$|x_k - x|_1 \leq |x_k - a|_1 + |x_k - b|_1 \leq q - p \text{ for any } p < k < q$$

that $\ell(\langle x_k, x_{k+1} \rangle) \leq p - q$ and we have

$$(3.35) \quad \begin{aligned} (F^- + c^2)|a - b|_1 + c^2 \gamma f_{d,r}(N) &< M^2 |a - b|_1 \vee \frac{1}{2} c f_{d,r}(N) \\ &\leq M^2 |a - b|_1 \vee \sum_{i=p}^{q-1} \frac{c f_{d,r}(N)}{q - p + 1} \\ &\leq \sum_{i=p}^{q-1} M^2 \vee \frac{c f_{d,r}(N)}{\ell(\langle x_i, x_{i+1} \rangle) + 1} \\ &\leq \sum_{i=p}^{q-1} \tau_{\langle x_i, x_{i+1} \rangle}^* = t^*(a, b), \end{aligned}$$

which is a contradiction.

Finally, we show that any optimal path passes through $\tilde{E}^j(l; n)$. Since any optimal path π_N^* does not touch $B^j(l; n) \setminus (\partial B^j(l; n) \cup C^j(l; n))$, if π_N^* does not touch $\{x \in C^j(l; n) \mid x \not\sim_{C \setminus E} \partial B\}$, since $\pi_N^* \cap \iota(B^j(l; n))^c$ is also a path and we can take an optimal path so that it does not touch the

inside of $B^j(l; n)$, which contradicts what we mentioned at the beginning of the proof. Thus, π_N^* needs to pass on $\tilde{E}^j(l; n)$. $\square$

The rest is the same as before.

3.5. Lower bound for $r = d - 1$ with $d \geq 3$. Let s, s_1, M be as before. We use the same notation as in subsection 3.4 for blackness, $B^j(l; n)$, $C^j(l; n)$, etc.

Let $F_j(l; n) = \{v \in B^j(l; n) \setminus (\partial B^j(l; n) \cup C^j(l; n)) \mid \exists w \in C^j(l; n) \text{ s.t. } |v - w|_1 \leq (\log N)^{1/(8d^2)}\}$. Given $x \in \mathbb{Z}^d$, we define $\ell_1(x)$ and $\ell_2(x)$ as follows:

$$\ell_1(x) = d_1(x, C^j(l; n)), \quad \ell_2(x) = d_1(x, E^j(l; n))_1 - \ell_1(x).$$

Given an edge $e = \langle x, y \rangle$, we define $\ell_1(e)$ and $\ell_2(e)$ as

$$\ell_1(e) = \ell_1(x) \wedge \ell_1(y), \quad \ell_2(e) = \ell_2(x) \wedge \ell_2(y).$$

Lemma 17. *There exists a positive constant $C(s)$ such that if $0 \leq \ell_2 \leq (\log N)^{1/(8d^2)} + 1$ and $0 \leq \ell_2 \leq 2dn_1$, then*

$$(3.36) \quad \#\{e \in E^d \mid e \cap F^j(l; n) \neq \emptyset, \ell_2(e) = k, \ell_1(e) = \ell\} \leq C(s)(\ell + 1)^{d-2}.$$

Otherwise, if $\ell_2 > (\log N)^{1/(8d^2)} + 1$ or $\ell_2 > 2dn_1$, then

$$(3.37) \quad \#\{e \in E^d \mid e \cap F^j(l; n) \neq \emptyset, \ell_2(e) = k, \ell_1(e) = \ell\} = 0.$$

Proof. Let $L = \{ke_1 \mid k \in \mathbb{Z}\}$. (3.36) follows from

$$\#\{e \in E^d \mid \ell_2(e) = k, \ell_1(e) = \ell\} \leq 2d(\#E^j(l; n))\#\{v \in \mathbb{Z}^d \mid |v_1| = k, |v|_1^{(2)} = \ell\},$$

where recall that $|v|_1^{(2)} = |v|_1 - |v_1|$. By the definition of $F^j(l; n)$, (3.37) is trivial. $\square$

Definition 9. *We say that the collection $\tau = \{\tau_e\}_{e \in E^d}$ satisfies A_3 -condition if*

$$(3.38) \quad \begin{cases} c^2 f_{d,d-1}(N) \leq \tau_e \leq \gamma c^2 f_{d,d-1}(N), & \text{if } e \in \tilde{E}^j(l; n) \\ \tau_e \leq F^- + c^2, & \text{if } e \in \tilde{C}^j(l; n) \setminus \tilde{E}^j(l; n) \\ \tau_e \geq \frac{cf_{d,d-1}(N)}{(\ell_1(e)+1)\log(\ell_2(e)+2)} \vee M^2 & \text{if } e \cap F_j(l; n) \neq \emptyset \text{ and } e \cap \partial B_j(l; n) = \emptyset \\ \tau_e \geq (\log N)^{\frac{1}{2d^2}}, & \text{if } e \cap \partial B^j(l; n) \neq \emptyset, e \not\subset \partial B^j(l; n), e \notin \tilde{C}^j(l; n). \end{cases}$$

Then we write $\tau \in A_3$.

A resampled configuration $\tau^* = \{\tau_e^*\}_{e \in E^d}$ is taken to be independent copy if $e \cap (F_j(l; n) \cup C_j(l; n))$ or both $e \cap \partial B_j(l; n)$ and $e \cap \iota(B_j(l; n))$ are not empty and $\tau_e^* = \tau_e$ otherwise.

We define an event A as

$$(3.39) \quad A = \{\tau^* \in A_3\} \cap \{B^j(l; n) \text{ is black for } \tau\} \cap \{\exists \pi_N \in \mathbb{O}_N \text{ crosses } B^j(l; n)\}.$$

Note that on the event A , if $e \notin \tilde{C}^j(l; n)$, then $\tau_e^* \geq \tau_e$. Thus, the following proposition follows.

Proposition 8. *On the event A , the following holds. If for any $v, w \in \iota(B^j(l; n))$ with $|v - w|_1 \geq (\log N)^{\frac{1}{8d^2}}$ and a path $\pi : v \rightarrow w$ satisfying $\pi \cap \tilde{C}^j(l; n) = \emptyset$,*

$$(3.40) \quad t^*(\pi) \geq (F^- + \delta_{11})|v - w|_1.$$

Proposition 9. *For any $\delta > 0$, there exists $s_0 > 0$ such that for any $s < s_0$, there exists $c_0(s) > 0$ such that for any $c < c_0(s)$,*

$$\mathbb{P}(\tau^* \in A_3) \geq N^{-\delta}.$$

Proof. If we take $c > 0$ sufficiently small depending on $s > 0$,

$$\begin{aligned} \mathbb{P}(\tau^* \in A_3) &\geq \left(\beta e^{-\alpha c^{2(d-1)} f_{d,d-1}(N)^{d-1}} \right)^{|\tilde{E}|} \mathbb{P}(\tau_e \leq F^- + c^2)^{|\tilde{C}|} \mathbb{P}(\tau_e > M^2)^{2d|F|} \\ &\quad \cdot \prod_{\ell=1}^{2[(\log N)^{\frac{1}{8d^2}}]} \prod_{k=2}^{4dn_1} \left(\beta e^{-\alpha \left(\frac{c f_{d,d-1}(N)}{\ell \log k} \right)^{d-1}} \right)^{C(s) \ell^{d-2}} \left(\beta e^{-\alpha (\log N)^{1/(2d)}} \right)^{2d|\partial B|} \\ &\geq \exp\left(-\frac{\log N}{30}\right) \exp\left(-\alpha C(s) \sum_{\ell=2}^{4[(\log N)^{\frac{1}{8d^2}}]} \sum_{k=2}^{4dn_1} c^{d-1} \frac{f_{d,d-1}(N)^{d-1}}{\ell (\log k)^{d-1}}\right) \\ &\geq \exp(-\delta \log N) = N^{-\delta}. \end{aligned}$$

where we have used the following fact that for sufficiently small s in the last inequality:

$$\sum_{k=2}^{dn} \frac{1}{(\log k)^{d-1}} \leq \frac{f_{d,d-1}(N)}{(\log \log N)^{d-1}}.$$

□

Proposition 10. *When $\{0, N\mathbf{e}_1\} \cap B^j(l; n) = \emptyset$, if s is sufficiently small depending on M and c is sufficiently small depending on s , on the event A , for any optimal path $\pi_N \in \mathbb{O}_N^*$ needs to pass through $\tilde{E}^j(l; n)$ at least one time.*

Proof. By the same argument as of (3.20), one can check that

$$(3.41) \quad t(0, N\mathbf{e}_1) - \frac{\delta_{11}}{4}n > t^*(0, N\mathbf{e}_1).$$

Thus any optimal path $\pi_N^* \in \mathbb{O}_N^*$ has to enter $B^j(l; n)$. We take $\pi_N^* \in \mathbb{O}_N^*$ which is a self avoiding path and write $\pi_N^* = \{x_1, \dots, x_K\}$. We define the sequence (p_i, q_i) inductively as follows: let

$$p_0 = \inf\{1 \leq l \leq K \mid x_l \in B^j(l; n)\} \text{ and } q_0 = \min\{l > p_0 \mid x_l \in C^j(l; n)\}.$$

Suppose that we could define $(p_i, q_i)_{i=0}^k$, and we define

$$p_{k+1} = \min\{l > q_k \mid x_l \in \mathbb{Z}^d \setminus C^j(l; n)\} - 1 \text{ and } q_{k+1} = \min\{l > p_{k+1} \mid x_l \in C^j(l; n)\}.$$

Let $I = \min\{l \mid q_l = \infty\}$. We redefine $q_I = \max\{1 \leq l \leq K \mid x_l \in B^j(l; n)\}$. (See Figure 5.)

Step 1 ($|x_{p_i} - x_{q_i}|_1 > n_1/4$): We will show that for any $i \in \{1, \dots, I-1\}$, $|x_{p_i} - x_{q_i}|_1 > n_1/4$. Until the Step 1 completed, we assume that $|x_{p_i} - x_{q_i}|_1 \leq n_1/4$ and finally we shall derive a contradiction. By Lemma 14-(i), there exists a path $\pi^* : x_{p_i} \rightarrow x_{q_i}$ on $C^j(l; n)$ such that

$$(3.42) \quad t^*(\pi^*) \leq (F^- + c^2)|x_{p_i} - x_{q_i}|_1 + c^2 f_{d,d-1}(N) \mathbb{I}_{W_{p_i} \neq W_{q_i}}.$$

We divide into three cases:

- (1) $W_{x_{p_i}} = W_{x_{q_i}}$ and $|x_{p_i} - x_{q_i}|_1 < (\log N)^{1/(8d^2)}$,
- (2) $W_{x_{p_i}} = W_{x_{q_i}}$ and $|x_{p_i} - x_{q_i}|_1 \geq (\log N)^{1/(8d^2)}$,
- (3) $W_{x_{p_i}} \neq W_{x_{q_i}}$.

- (1) If there exists $p_i < k < q_i$ such that $|x_{p_i} - x_k|_1 \geq (\log N)^{1/(8d^2)}$, Proposition 8 yields that

$$t^*(x_{p_i}, x_{q_i}) \geq (F^- + \delta_{11})(\log N)^{1/(8d^2)} > (F^- + c^2)|x_{p_i} - x_{q_i}|_1,$$

which is a contradiction. Thus for any $p_i < k < q_i$, $|x_{p_i} - x_k|_1 < (\log N)^{1/(8d^2)}$. If, in addition, there exists $p_i < k < q_i$ such that $x_k \in \partial B^j(l; n)$, by Lemma 14-(iv), we have

$$\max\{d_1(x_{p_i}, \partial B^j(l; n)), d_1(x_{q_i}, \partial B^j(l; n))\} \geq |x_{p_i} - x_{q_i}|_1.$$

Since $x_k \in \iota(B^j(l; n))$ implies $x_k \in F^j(l; n)$ for $p_i < k < q_i$, by A_3 -condition, it yields that

$$t^*(x_{p_i}, x_{q_i}) \geq M^2|x_{p_i} - x_{q_i}|_1 > (F^- + c^2)|x_{p_i} - x_{q_i}|_1,$$

which is also a contradiction as before. On the other hand, if for any $p_i < k < q_i$, $x_k \in F^j(l; n)$, we have

$$t^*(x_{p_i}, x_{q_i}) \geq M^2 |x_{p_i} - x_{q_i}|_1 > (F^- + c^2) |x_{p_i} - x_{q_i}|_1,$$

which is also a contradiction.

(2) By Proposition 8, $t^*(x_{p_i}, x_{q_i}) \geq (F^- + \delta_{11}) |x_{p_i} - x_{q_i}|_1$, which contradicts (3.42).

(3) Suppose that $W_{x_{p_i}} \neq W_{x_{q_i}}$ holds. If there exists $p_i < k < q_i$ such that $x_k \in \partial B^j(l; n)$, by Lemma 14-(iv) and $|x_{p_i} - x_{q_i}|_1 \leq n_1/4$, then we have $\max\{|x_{p_i} - x_k|_1, |x_{q_i} - x_k|_1\} \geq n_1/4$. It follows that

$$t^*(x_{p_i}, x_{q_i}) \geq (F^- + \delta_{11}) n_1/4 > (F^- + c^2) |x_{p_i} - x_{q_i}|_1 + c^2 f_{d,d-1}(N),$$

which contradicts (3.42). Note that since $\ell_1(x_{p_i}) = 0$,

$$\ell_1(x_k) \leq k - p_1.$$

On the other hand, if there exists $p_i < k < q_i$ such that $x_k \in B^j(l; n) \setminus (F^j(l; n) \cup C^j(l; n))$ and for any $p_i < k_1 < q_i$, $x_{k_1} \in \iota(B^j(l; n))$, since $|x_{p_i} - x_{k_1}|_1 \geq (\log N)^{1/(8d^2)}$, then we have

$$\begin{aligned} \sum_{i=p_i}^{k-1} \tau_{\langle x_i, x_{i+1} \rangle} &\geq (F^- + \delta_{11}) |x_{p_i} - x_{q_i}|_1 \vee \sum_{\ell=0}^{[(\log N)^{1/(8d^2)}]} \frac{c f_{d,d-1}(N)}{(\ell+1) \log(2dn_1)} \\ (3.43) \quad &> (F^- + \delta_{11}) |x_{p_i} - x_{q_i}|_1 \vee (c f_{d,d-1}(N)/8d^2) \\ &> (F^- + c^2) |x_{p_i} - x_{q_i}|_1 + c^2 f_{d,d-1}(N) \end{aligned}$$

which also contradicts (3.42). Thus for any $p_i < k < q_i$, $x_k \in C^j(l; n) \cup F^j(l; n)$. Due to the definition of (p_i, q_i) , $x_k \notin C^j(l; n)$, which yields $x_k \in F^j(l; n)$ for any $p_i < k < q_i$. Lemma 14-(iv) yields that there exists $p_i < k_1 < q_i$ such that $\ell_2(x_{k_1}) = 0$. It yields that

$$\max\{\ell_2(x_k) \mid p_i < k < q_i\} \leq q_i - p_i.$$

It follows that

$$\begin{aligned} \sum_{s=p_i}^{q_i-1} \tau_{\langle x_s, x_{s+1} \rangle} &\geq (F^- + \delta_{11}) |x_{p_i} - x_{q_i}|_1 \vee \sum_{s=0}^{q_i-p_i} \frac{c f_{d,d-1}(N)}{(s+1) \log |q_i - p_i + 2|} \\ (3.44) \quad &> (F^- + \delta_{11}) |x_{p_i} - x_{q_i}|_1 \vee c f_{d,d-1}(N)/2, \end{aligned}$$

which is also a contradiction as before. Therefore we have $|x_{q_i} - x_{p_i}|_1 > n_1/4$.

Step 2 ($|x_{q_I} - x_{p_0}|_1 \geq \frac{\delta_{11}n}{4M}$): Since there exists a path $\pi : x_{p_0} \rightarrow x_{q_I}$ on $\partial B^j(l; n)$ such that $t(\pi) \leq M(|x_{p_0} - x_{q_I}|_1 \vee (\log N)^{1/(8d^2)})$, by (Black-3), (3.41) leads to

$$\begin{aligned} \delta_{11}n/4 &\leq t(0, N\mathbf{e}_1) - t^*(0, N\mathbf{e}_1) \leq t(x_{p_0}, x_{q_I}) - t^*(x_{p_0}, x_{q_I}) \\ (3.45) \quad &\leq t(x_{p_0}, x_{q_I}) \\ &\leq M(|x_{q_I} - x_{p_0}|_1 \vee (\log N)^{1/(8d^2)}). \end{aligned}$$

Therefore $\delta_{11}n/4 \leq M|x_{q_I} - x_{p_0}|_1$, which implies

$$|x_{p_0} - x_{q_I}|_1 \geq \frac{\delta_{11}n}{4M}.$$

Step 3 (conclusion): Note that

$$(3.46) \quad |x_{p_0} - x_{q_I}|_1 \leq \sum_{i=0}^I |x_{q_i} - x_{p_i}|_1 + \sum_{i=0}^{I-1} |x_{p_{i+1}} - x_{q_i}|_1.$$

It follows that

$$\begin{aligned}
(3.47) \quad & F^- |x_{p_0} - x_{q_I}|_1 + \delta_{11} \sum_{i=0}^I |x_{q_i} - x_{p_i}|_1 - 2(F^- + \delta)(\log N)^{1/8d^2} \\
& \leq \sum_{i=0}^I (F^- + \delta_{11}) |x_{q_i} - x_{p_i}|_1 + F^- \sum_{i=0}^{I-1} |x_{p_{i+1}} - x_{q_i}|_1 - 2(F^- + \delta)(\log N)^{1/8d^2} \\
& \leq \sum_{i=0}^I t^*(x_{q_i}, x_{p_i}) + \sum_{i=0}^{I-1} t^*(x_{p_{i+1}}, x_{q_i}) = t^*(x_{p_0}, x_{q_I}) \\
& \leq (F^- + c^2) |x_{p_0} - x_{q_I}|_1 + c^2 |E^j(l; n)| f_{d,d-1}(N) + 2dMn_1,
\end{aligned}$$

where we have used Step 1, (Black-2) and Proposition 8 in the second inequality and the same argument as in Step 2 in the third inequality. Note that $|x_{p_0} - x_{q_I}|_1 \leq 3dn$ and

$$2(F^- + \delta)(\log N)^{1/8d^2} + c^2 |x_{p_0} - x_{q_I}|_1 + c^2 |E^j(l; n)| f_{d,d-1}(N) \leq 2Mn_1.$$

It follows from Step 1 that

$$(3.48) \quad (I-1)n_1/4 \leq \delta_{11} \sum_{i=0}^I |x_{q_i} - x_{p_i}|_1 \leq 4Mn_1,$$

which yields that $I \leq 20M$. On the other hand, if π_N^* does not pass through $\tilde{E}^j(l; n)$, $\max_a \#W_a \leq 2dn_1$ implies $|x_{p_{i+1}} - x_{q_i}|_1 \leq 2dn_1$, which yields that

$$(3.49) \quad \sum_{i=0}^{I-1} |x_{p_{i+1}} - x_{q_i}|_1 \leq 40dMn_1.$$

By Step 2, (3.46), (3.48) and (3.49), we have

$$(3.50) \quad \frac{\delta_{11}n}{4M} \leq 40dMn_1 + \frac{4M}{\delta_{11}}n_1,$$

which is a contradiction if s is sufficiently small depending on M . Thus we complete the proof. $\square$

The rest is the same as before.

4. APPENDIX

Proof of Lemma 3. $(\Xi-2)$ follows from $d2^n \leq m/3$ and $(\Xi-3)$ follows from (2.21). $(\Xi-4)$ are trivial from the construction. We shall confirm the first condition. When $j \neq j'$, considering the first coordinate, it is trivial due to the way of construction.

From now on, we suppose that $j = j'$. We take $e \in \Xi^{i_1, \dots, i_j, j} \cap \Xi^{i'_1, \dots, i'_j, j}$ and we shall derive $(i_1, \dots, i_j) = (i'_1, \dots, i'_j)$. Let $v = (d2^{j-1}, \sum_{s=1}^j 2^{j-s}i_s)$, $w = (d2^{j-1}, \sum_{s=1}^j 2^{j-s}i'_s)$. Recall the definition of $\ell^k(v)$:

$$\ell^k(v) = \max\{\ell \in \mathbb{N} \mid v(\ell) \in [0, d2^{k+1}] \times \mathbb{Z}^{d-1}\}.$$

It is easy to see that $\Xi^{v(\ell^k(v)), 0} \cap \Xi^{i'_1, \dots, i'_j, j} = \emptyset$ and $\Xi^{w(\ell^k(w)), 0} \cap \Xi^{i_1, \dots, i_j, j} = \emptyset$ unless $(i_1, \dots, i_j) = (i'_1, \dots, i'_j)$. Thus we can write $e = \langle v(k), v(k+1) \rangle = \langle w(l), w(l+1) \rangle$ with $k, l \in \mathbb{N}$. Note that $|v(k)|_1 = |v|_1 + k$ and $|w(l)|_1 = |w|_1 + l$. Let $|v|_1^{(2)} = \sum_{i=2}^d |v_i|$. Comparing the first-coordinate, $v_1 = w_1$ and

$$\begin{aligned}
(4.1) \quad v(k)_1 &= w(l)_1 = \left\lfloor \frac{k+1}{2} \right\rfloor + \left\lfloor \frac{k}{2} - |v|_1^{(2)} \right\rfloor_+ + v_1 \\
&= \left\lfloor \frac{l+1}{2} \right\rfloor + \left\lfloor \frac{l}{2} - |w|_1^{(2)} \right\rfloor_+ + w_1,
\end{aligned}$$

where $\lfloor \cdot \rfloor$ is a floor function and define $[a]_+ = [a] \vee 0$. We first show that $k = l$. To this end, we suppose that $k > l$ holds. Then since $|v(k)|_1 = |w(l)|_1 = |v|_1 + k = |w|_1 + l$, $|v|_1 < |w|_1$

and $|v|_1^{(2)} < |w|_1^{(2)}$. Thus $\lceil \frac{k+1}{2} \rceil \geq \lceil \frac{l+1}{2} \rceil$ yields that $\left\lceil \frac{k}{2} - |v|_1^{(2)} \right\rceil_+ \geq \left\lceil \frac{l}{2} - |w|_1^{(2)} \right\rceil_+$. By (4.1) and $v_1 = w_1$, all of these are indeed equations:

$$(4.2) \quad \left\lceil \frac{k+1}{2} \right\rceil = \left\lceil \frac{l+1}{2} \right\rceil,$$

$$(4.3) \quad \left\lceil \frac{k}{2} - |v|_1^{(2)} \right\rceil_+ = \left\lceil \frac{l}{2} - |w|_1^{(2)} \right\rceil_+.$$

In particular, $k = l + 1$ by assuming $k > l$. If, in addition, $k \leq 2|v|_1^{(2)}$, then

$$l < l + 1 = k \leq 2|v|_1^{(2)} < 2|w|_1^{(2)}.$$

Recall that when $k \in 2\mathbb{Z}$, the first coordinate changes from $v(k)$ to $v(k+1)$, and when $k \in 2\mathbb{Z} + 1$, other coordinate changes. Thus we have $k \equiv l \pmod{2}$, which leads to a contradiction.

On the other hand, if $k > 2|v|_1^{(2)}$, by (4.3) and $|v|_1^{(2)} + 1 = |w|_1^{(2)}$,

$$(4.4) \quad \begin{aligned} \left\lceil \frac{k}{2} - |v|_1^{(2)} \right\rceil_+ &= \left\lceil \frac{l}{2} - |w|_1^{(2)} \right\rceil_+ \\ &= \left\lceil \frac{k-1}{2} - |v|_1^{(2)} - 1 \right\rceil_+ \\ &= \left\lceil \frac{k}{2} - |v|_1^{(2)} - 3/2 \right\rceil_+, \end{aligned}$$

which leads to $\left\lceil \frac{k}{2} - |v|_1^{(2)} \right\rceil_+ = 0$. Thus we have $k = 2|v|_1^{(2)} + 1$ and $l = 2|v|_1^{(2)}$. However, by (4.2),

$$(4.5) \quad \begin{aligned} \left\lceil \frac{k+1}{2} \right\rceil &= |v|_1^{(2)} + 1 \\ &= \left\lceil \frac{2|v|_1^{(2)} + 1}{2} \right\rceil_+ = |v|_1^{(2)}, \end{aligned}$$

which is a contradiction. As a result, one has $k \leq l$. Similarly, the converse inequality holds and we have $k = l$.

Note that since $|v(k)|_1 = |v|_1 + k$, $|w(k)|_1 = |w|_1 + k$, and $v(k) = w(k)$, $|v|_1 = |w|_1$ and $|v|_1^{(2)} = |w|_1^{(2)}$. Therefore, if $k \geq 2|v|_1^{(2)}$, then $k \geq 2|w|_1^{(2)}$ and $v_q = v(k)/2 = w_q$ for $2 \leq q \leq d$.

Next, suppose that $k < 2|v|_1^{(2)}$. Let $2 \leq p \leq d$ such that if $2 \leq q < p$, $v(k)_p = v(k+2)_p$ and $v(k)_p < v(k+2)_p$. Then for $2 \leq q < p$, $v_q = v(k)_q/2 = w_q$ and for $q > p$, $v_q = v(k) = w_q$. Moreover, since $v_p = v(k)_p - [(k+1)/2] + \sum_{q=2}^{p-1} v_q = w_p$, we have $v = w$. Considering binary number system to each coordinate, we obtain $(i_1, \dots, i_j) = (i'_1, \dots, i'_j)$ as desired. $\square$

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