

On weak forking

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Abstract

Weak dividing was defined in [1] and has been characterized in simple theory ([2], [3]). We consider some generalized notion of it.

1. Weak dividing and weak forking

We recall some definitions.

Definition 1 Let $\varphi(x_0, x_1, \dots, x_{n-1})$ be a formula and $p(x)$ be a type. We denote the type $\{\varphi(x_0, x_1, \dots, x_{n-1})\} \cup p(x_0) \cup p(x_1) \cup \dots \cup p(x_{n-1})$ by $[p]^\varphi$.

Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *divides over* A if there are a formula $\varphi(x, b) \in p(x)$ and an infinite sequence $\{b_i : i < \omega\}$ with $b \equiv b_i(A)$ such that $\{\varphi(x, b_i) : i < \omega\}$ is k -inconsistent for some $k < \omega$.

$p(x)$ *weakly divides over* A if there is a formula $\varphi(\bar{x}) \in L_n(A)$ such that $[p \upharpoonright A]^\varphi$ is consistent, while $[p]^\varphi$ is inconsistent.

A collection of types $\{\text{tp}(a_\alpha / A \setminus \{a_\beta \mid \beta < \alpha\}) \mid \alpha < |T|^+\}$ is a *weak dividing left-chain* if for each $\alpha < |T|^+$, $\text{tp}(a_\alpha / A \setminus \{a_\beta \mid \beta < \alpha\})$ weakly divides over $A \cup \{a_\beta \mid \beta < \alpha\}$.

A collection of types $\{\text{tp}(b / A \setminus \{a_\beta \mid \beta < \alpha\}) \mid \alpha < |T|^+\}$ is a *dividing right-chain* if for each $\alpha < |T|^+$, $\text{tp}(b / A \setminus \{a_\beta \mid \beta \leq \alpha\})$ divides over $A \cup \{a_\beta \mid \beta < \alpha\}$.

In this note, we call such formula " $\varphi(\bar{x})$ " in the definition the *witness formula* of weak dividing for the sake of convenience.

I show examples from [3].

Example 2 Let T be the theory of an equivalence relation with two infinite classes of the language $L = \{a \text{ binary relation } E(x, y)\}$. And let $\models \neg E(a, b)$. Then the type $\text{tp}(a/b)$ does not divide over $\emptyset$, while $\text{tp}(a/b)$ weakly divides over $\emptyset$ by the formula $\neg E(x, y)$.

Example 3 Let $(V, <, >)$ be a vector space V over a finite field equipped with an inner product giving orthogonality between two independent vectors. Let a, b, c be independent vectors in V such that $a \perp b$, while $b \not\perp c$ and $a \not\perp c$. Then $\text{tp}(a/bc)$ does not weakly divide over $\emptyset$. But $\text{tp}(a/bc)$ weakly divides over c by the formula $\varphi(x, y) := "x \text{ is a linear combination of } y \text{ and } c"$.

In various characterizations, one of the most important results is the next theorem.

Theorem 4 (Kim [3])

The following are equivalent ;

- (1) T is stable.
- (2) Weak dividing is symmetric in T .
- (3) There is no weak dividing left-chain in T .

I consider some generalization of weak dividing.

Definition 5 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ weakly forks over A if there is a complete type $q(x, y) \in S(A)$ such that $p(x) \cup q(x, y)$ is consistent, and any completion $r(x, y) \in S(B)$ of $p(x) \cup q(x, y)$ weakly divides over A .

We can easily prove the next fact.

Fact 6 Let T be any theory.

If $\text{tp}(a/bA)$ forks over A , then $\text{tp}(b/aA)$ weakly forks over A .

And I consider the definition of weak dividing by the use of formulas. But it can be defined in relation to complete types.

Definition 7 Let $A, B \subset C$. And $\varphi(x)$ is a $L_m(B)$ -formula.

$\varphi(x)$ weakly divides over A if there are a $L_{mn}(A)$ -formula $\phi(\bar{x})$ and a complete type $p(x) \in S_m(A)$ such that $p(x) \cup \{\varphi(x)\}$ is consistent, and $[p]^\phi$ is consistent, while $[\varphi]^\phi$ is inconsistent.

$\varphi(x)$ weakly forks over A if there are $L(B)$ -formulas $\psi_i(x, y)$ ($i < n$) for some $n < \omega$, a complete type $p(x, y) \in S(A)$ and $L(A)$ -formulas $\phi_i(\bar{x}_i, \bar{y}_i)$ ($i < n$) such that $\varphi(x) \vdash \bigvee_{i < n} \exists y \psi_i(x, y)$, $p(x, y) \cup \{\varphi(x)\}$ is consistent, and $\psi_i(x, y)$ weakly divides over A with respect to $p(x, y)$ and $\phi_i(\bar{x}_i, \bar{y}_i)$ for $i < n$.

2. Restricted notions of weak dividing

I considered that we can divide witness formulas into some classes according to properties of stability theory. I told about characterizations of the next restricted weak dividing before.

Definition 8 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *M-weakly divides over A* if there are a formula $\varphi(\bar{x}) \in L_n(A)$ and a Morley sequence $I = \{a_i : i < n+1\}$ of $p \upharpoonright A$ such that $\models \varphi(a_0, a_1, \dots, a_{n-1})$, while the type $[p]^\varphi$ is inconsistent.

$p(x)$ *M-weakly divides over A* if there are a formula $\varphi(\bar{x}) \in L_n(A)$ and a Morley sequence $I = \{a_i : i < n+1\}$ of $p \upharpoonright A$ such that $\models \varphi(a_0, a_1, \dots, a_{n-1})$, while there is no Morley sequence $J = \{b_i : i < n+1\}$ of p over A such that $\models \varphi(b_0, b_1, \dots, b_{n-1})$.

If we set the sequence I indiscernible over A in the definition above, we can define $\mathcal{I}$ -weak dividing and I -weak dividing in the same way.

Another variant of dividing, "thorn"-dividing has been characterized in rosy theory of late years. (see e.g. [6]) I define weak notion of $\mathfrak{p}$ -dividing (thorn-dividing). We recall some definitions.

Definition 9 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *strongly divides over A* if there is a formula $\varphi(x, b) \in p(x)$ such that $b \notin \text{acl}(A)$ and $\{\varphi(x, b_i) : b_i \models \text{tp}(b/A)\}$ is k -inconsistent for some $k < \omega$.

$p(x)$ *$\mathfrak{p}$ -divides over A* if $p(x)$ strongly divides over Ac for some parameter c .

Weak notions of $\mathfrak{p}$ -dividing could be defined in many ways. As $\mathfrak{p}$ -dividing implies dividing, we expect that weak $\mathfrak{p}$ -dividing implies weak dividing.

Definition 10 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *weakly $\mathfrak{p}$ -divides over A* if there is a formula $\varphi(\bar{x}) = \exists y \bigwedge_{i < n} \psi(x_i, y) \in L_n(A)$ such that $[p \upharpoonright A]^\varphi$ is consistent, while $[p]^\varphi$ is inconsistent.

$p(x)$ *weakly $\mathfrak{p}$ -forks over A* if there is a complete type $q(x, y) \in S(A)$ such that $p(x) \cup q(x, y)$ is consistent, and any completion $r(x, y) \in S(B)$ of $p(x) \cup q(x, y)$ weakly $\mathfrak{p}$ -divides over A .

Lastly I raise a question.

Problem

Characterize theories T in which there is no weak $\mathfrak{p}$ -forking left-chain. Are such theories included in rosy theories properly?

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