

DYER-LASHOF OPERATIONS FOR CERTAIN INFINITE LOOP SPACES

by

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§1. Results (with notation in §2.)

1. Let $X = BU \times Z$. Consider $H_*(X; Z_p) = Z_p[a_0, a_0^{-1}, a_1, \dots]$,

$\deg a_i = 2i$, for an arbitrary odd prime p .

$$Q^m(a_k) = \sum_{n_1, \dots, n_p} c_{n_1, \dots, n_p}^{k, m} a_{n_1} a_{n_2} \dots a_{n_p},$$

where $n_1 + \dots + n_p = k + m(p-1)$, $n_1 \geq n_2 \geq \dots \geq n_p$.

The coefficients $c_{n_1, \dots, n_p}^{k, m}$ are defined as the (unique) solution

of the system of equations:

$$\sum_{\nu=0}^{[k/p]} \binom{k - \nu(p-1)}{\nu} c_{n_1, \dots, n_p}^{k - \nu(p-1), m + \nu} = (-1)^m B_{n_1, \dots, n_p}^{k, m}, \dots (1.1)$$

where

$$B_{n_1, \dots, n_p}^{k, m} = \sum_{\sigma \in \mathfrak{S}_p / H} A_{n_{\sigma(1)}, \dots, n_{\sigma(p)}}^{k, m},$$

(H is the stabilizer of $\{n_1, \dots, n_p\}$)

$A_{n_1, \dots, n_p}^{k, m}$ = the coefficient of $x^{m(p-1)}$ in the polynomial

$$(1+x)^{n_1} (1+2x)^{n_2} \dots (1+(p-1)x)^{n_{p-1}}.$$

The equation can be solved in some special cases. Set

$$k = k'(p - 1) - \alpha, \quad 0 \leq \alpha < p - 1.$$

i) In case $k' \leq p$:
$$C_{n_1, \dots, n_p}^{k, m} = \sum_{j=1}^{k'} (-1)^{m+k'-j} \binom{k'+\alpha-1}{k'-j} B_{n_1, \dots, n_p}^{j(p-1)-\alpha, m+k'-j}.$$

ii) In case $k' = p + 1$:
$$C_{n_1, \dots, n_p}^{k, m} = \sum_{j=2}^{k'} \text{ (same as above).}$$

iii) In case there is a number n such that $\sum_{i=1}^p (n_i - n) < p - 1$,

$$n_i \geq n,$$

$$(-1)^{m} B_{n_1, \dots, n_p}^{k, m} = \#(\mathbb{G}_p/H) \binom{n}{m},$$

and hence $C_{n_1, \dots, n_p}^{k, m} = \#(\mathbb{G}_p/H) \binom{n-k-1}{n-m}$, applying Th. 25.3 of Adem.

iv) In case there is a number n such that $n_1 = n_2 = \dots = n_{p-1} = n$,

$$n_p = n - 1, \quad C_{n_1, \dots, n_p}^{k, m} = \binom{n-k-2}{n-m}.$$

As for the remaining generator a_0^{-1} :

$$Q^m(a_0^{-1}) = \sum (-1)^{\lambda} \binom{\lambda}{\lambda_1, \dots, \lambda_m} (Q^1(a_0))^{\lambda_1} \dots (Q^m(a_0))^{\lambda_m} a_0^{-p(\lambda+1)},$$

where $\sum i \lambda_i = m$, $\lambda = \sum \lambda_i$.

2. Let $X = K(\pi, n)$: the Eilenberg-MacLane space for an arbitrary finitely generated abelian group π , and $n \geq 1$.

For any prime p , the Dyer-Lashof operations on $H_*(K(\pi, n); \mathbb{Z}_p)$ are trivial, except that $Q^0(1) = 1$.

§2. Proofs of 1. : $X = BU \times Z$.

In [1], S. Priddy has computed the Dyer-Lashof operations on $H_*(BU \times Z; \mathbb{Z}_2)$. We shall adapt his methods to the mod p case for an arbitrary odd prime p , and obtain some results.

For the necessary background literature on infinite loop spaces and Dyer-Lashof operations, we refer to J. P. May's papers [2], [4].

It is known that

$$H_*(BU \times Z) = \mathbb{Z}_p[a_0, a_0^{-1}, a_1, a_2, \dots]$$

and that the generator a_i comes from the generator e_{2i} of $H_*(BZ_p)$, ($\mathbb{Z}_p$ is regarded as the subgroup of U_1 , with generator $a = e^{2\pi i/p}$) which is the dual of $y_2^i \in H^*(BZ_p) = \bigwedge(y_1) \otimes \mathbb{Z}_p[y_2]$.

According to [1] the diagram

$$\begin{array}{ccccc} B(\mathbb{Z}_p \wr \mathbb{Z}_p) & \longrightarrow & B(\mathbb{Z}_p \wr U_1) & \xrightarrow{Bj} & BU_p \\ \parallel & & \parallel & & \downarrow \\ W\mathbb{Z}_p \times_{\mathbb{Z}_p} (B\mathbb{Z}_p)^p & \longrightarrow & W\mathbb{Z}_p \times_{\mathbb{Z}_p} (BU_1)^p & & \\ & & \downarrow & & \\ & & W\mathbb{Z}_p \times_{\mathbb{Z}_p} (BU \times Z)^p & \xrightarrow{\theta} & BU \times Z \end{array}$$

is homotopy commutative, where j is the inclusion of the wreath product and θ is the Dyer-Lashof map.

Consider the following commutative diagram of group homomorphisms

$$\begin{array}{ccccccc}
 \mathbb{Z}_p \times \mathbb{Z}_p & \xrightarrow{1 \times \Delta^p} & \mathbb{Z}_p \wr \mathbb{Z}_p & \hookrightarrow & \mathcal{G}_p \wr U_1 & \xrightarrow{j} & U_p \\
 \downarrow 1 \times \Delta^p & & & & & & \downarrow P(\cdot)^{p-1} \\
 \mathbb{Z}_p \times (\underbrace{\mathbb{Z}_p \times \dots \times \mathbb{Z}_p}_p) & \xrightarrow{\varphi} & \underbrace{\mathbb{Z}_p \times \dots \times \mathbb{Z}_p}_p & \hookrightarrow & \underbrace{U_1 \times \dots \times U_1}_p & \xrightarrow{m} & U_p
 \end{array}$$

where

$$\varphi(a; b_1, \dots, b_p) = (ab_1, a^2b_2, \dots, a^pb_p)$$

(group multiplications of the cyclic group $\mathbb{Z}_p$)

m = juxtaposition of matrices

$$P = \frac{1}{\sqrt{p}} \begin{pmatrix} a^{ij} \end{pmatrix}_{\substack{1 \leq i \leq p \\ 1 \leq j \leq p}}, \quad (a = e^{2\pi i/p}),$$

(note that $P^{-1} = \overline{P}$).

We now apply $H_*B(\cdot)$ to the diagram, and compute the images of

the element $e_{2m(p-1)} \otimes e_{2k} \in H_*(B\mathbb{Z}_p \times B\mathbb{Z}_p)$:

$$\begin{array}{l}
 \begin{array}{c}
 e_{2m(p-1)} \otimes e_{2k} \\
 \downarrow B(1 \times \Delta^p)_* \\
 \sum_{i_1 + \dots + i_p = k} e_{2m(p-1)} \otimes e_{2i_1} \otimes \dots \otimes e_{2i_p}
 \end{array} \\
 \begin{array}{c}
 \downarrow m_* \varphi_* \\
 \sum_{i_1 + \dots + i_p = k} \sum_{j_1 + \dots + j_{p-1} = m(p-1)} a_{j_1+i_1} \dots a_{j_{p-1}+i_{p-1}} a_{i_p}
 \end{array} \\
 \begin{array}{c}
 \downarrow B(1 \times \Delta^p)_* \\
 \sum_{i_1 + \dots + i_p = k} e_{2m(p-1)} \otimes e_{2i_1} \otimes \dots \otimes e_{2i_p}
 \end{array} \\
 \begin{array}{c}
 \downarrow m_* \varphi_* \\
 \sum_{i_1 + \dots + i_p = k} \sum_{j_1 + \dots + j_{p-1} = m(p-1)} a_{j_1+i_1} \dots a_{j_{p-1}+i_{p-1}} a_{i_p}
 \end{array}
 \end{array}$$

(1) $\sum_{\nu} (-1)^{\nu+k} \binom{k-\nu(p-1)}{\nu} e_{2(m+p\nu-k)(p-1)} \otimes e_{2k-2\nu(p-1)}^p$

(2) $\sum_{\nu} (-1)^m \binom{k-\nu(p-1)}{\nu} Q^{m+\nu}(a_{k-\nu(p-1)})$

(3) $\sum_{i_1 + \dots + i_p = k} e_{2m(p-1)} \otimes e_{2i_1} \otimes \dots \otimes e_{2i_p}$

(4) $\sum_{i_1 + \dots + i_p = k} \sum_{j_1 + \dots + j_{p-1} = m(p-1)} a_{j_1+i_1} \dots a_{j_{p-1}+i_{p-1}} a_{i_p}$

id.

(proofs): (1) is referred to Lemma 4.6 of [3].

(2) is obtained from the definition of Q^m . (see [3].)

(3) and (4) are proved by investigating the duality. In fact,

$$\langle y_1^{\varepsilon_1} y_2^{i_1} \otimes y_1^{\varepsilon_2} y_2^{i_2}, \Delta_*(e_{2k}) \rangle = \langle y_1^{\varepsilon_1 + \varepsilon_2} y_2^{i_1 + i_2}, e_{2k} \rangle$$

$$= 1 \text{ iff } \varepsilon_1 = \varepsilon_2 = 0, \quad i_1 + i_2 = k,$$

$$\text{and hence } \Delta_*(e_{2k}) = \sum_{i_1 + i_2 = k} e_{2i_1} \otimes e_{2i_2};$$

$$\langle y_2^{k_1 + k_2}, \mu_*(e_{2k_1} \otimes e_{2k_2}) \rangle$$

$$= \langle (y_2 \otimes 1 + \lambda y_1 \otimes y_1 + 1 \otimes y_2)^{k_1 + k_2}, e_{2k_1} \otimes e_{2k_2} \rangle = \binom{k_1 + k_2}{k_1},$$

$$\text{and hence } \mu_*(e_{2k_1} \otimes e_{2k_2}) = \binom{k_1 + k_2}{k_1} e_{2k_1 + 2k_2}.$$

$$\text{Note that } \varphi = (\varphi_1 \times \dots \times \varphi_p) \circ T \circ (\Delta^p \times 1),$$

where T is the shuffling map and $\varphi_\nu = \mu^\nu(s_\nu \times 1)$ with $s_\nu = \mu^\nu \circ \Delta^\nu: a \mapsto a^\nu$.

We see that

$$\begin{aligned} s_\nu^*(e_{2j}) &= (\mu^\nu)_* \sum_{k_1 + \dots + k_\nu = j} e_{2k_1} \otimes \dots \otimes e_{2k_\nu} \\ &= \sum_{k_1 + \dots + k_\nu = j} \binom{k_1 + \dots + k_\nu}{k_1, \dots, k_\nu} e_{2j} \\ &= \underbrace{(1 + \dots + 1)}_\nu^j e_{2j} = \nu^j e_{2j}, \end{aligned}$$

$$\begin{aligned} \text{and hence } \varphi_*(e_{2n} \otimes e_{2i_1} \otimes \dots \otimes e_{2i_p}) \\ = \sum_{j_1 + \dots + j_p = n} 1^{j_1} 2^{j_2} \dots p^{j_p} \binom{j_1 + i_1}{i_1} \dots \binom{j_p + i_p}{i_p} e_{2j_1 + 2i_1} \otimes \dots \otimes e_{2j_p + 2i_p}. \end{aligned}$$

From the preceding diagram we have

$$\sum_{\nu} \binom{k-\nu(p-1)}{\nu} Q^{m+\nu}(a_{k-\nu(p-1)}) = (-1)^m \sum_{n_1, \dots, n_p} B_{n_1, \dots, n_p}^{k, m} a_{n_1} \cdots a_{n_p},$$

and (1.1) follows.

Note that the solution of the equation (1.1) is determined uniquely, since the left hand side is of the form $C^{k, m} + (k: \text{lower})$, and the right hand side is known.

(1.1) can be solved only in some special cases.

First, we get i) and ii), using

Lemma. If $i < p$ and $0 \leq \alpha < p-1$, then

$$\binom{j(p-1)-\alpha}{i} = (-1)^i \binom{i}{i+1} H_{j+\alpha-1} = (-1)^i \binom{i+j+\alpha-1}{j+\alpha-1} \pmod{p}.$$

(this is proved by considering the power series $(1+X)^{jp-(j+\alpha)}$)

and the fact that

$$c_k = \sum_{j=1}^k \binom{k+\alpha-1}{k-j} a_j \text{ is the solution of } \sum_{i=0}^{k-1} (-1)^i \binom{k+\alpha-1}{i} c_{k-i} = a_k.$$

Secondly, in view of $(1+X) \cdots (1+(p-1)X) = 1 - X^{p-1} \pmod{p}$,

$A^{k, m}$ are expressed by some binomial coefficients in cases iii) and iv),

and in these cases $C^{k, m}$ are computed by using some formulas of binomial

coefficients. Th. 25.3 of Adem is referred to [1].

Computation of $Q^m(a_0^{-1})$ is same as in [1]. In fact, we get from the Cartan formula in $H_*(QS^0)$,

$$Q^n([-1]) = \sum_{\sum i \lambda_i = n} (-1)^{\lambda} (\lambda_1, \dots, \lambda_n) (Q^1[1])^{\lambda_1} \dots (Q^n[1])^{\lambda_n} [-1]^{p(\lambda+1)}$$

in $H_*(QS^0)$, and the result follows.

Note that

$$Q^m(a_0) = (-1)^m \sum_{n_1 + \dots + n_{p-1} = m(p-1)} 2^{n_2} \dots (p-1)^{n_{p-1}} a_{n_1} \dots a_{n_{p-1}} a_0,$$

by i).

§3. Proof of 2. : $X = K(\pi, n)$.

Remark 3.1. The Dyer-Lashof map

$$\theta: WZ_p \times_{Z_p} (K(\pi, n))^P \longrightarrow K(\pi, n)$$

always represents an element of $H^n(WZ_p \times_{Z_p} (K(\pi, n))^P; \pi)$.

On the other hand, the properties of θ :

i) $\theta|_{(WZ_p)^0 \times_{Z_p} X^P}: X^P \longrightarrow X$ is equal to the p-ple product map of the

H-space X (see [6]),

ii) $\theta_*(e_m \circ \underbrace{1 \otimes \dots \otimes 1}_p) = 0$ in $H_m(X; Z_p)$, for any $m \geq 1$ (see [2]),

show that

$$\theta_*: H_i(WZ_p \times_{Z_p} (K(\pi, n))^P; Z) \longrightarrow H_i(K(\pi, n); Z)$$

is uniquely determined for $i = n$.

Hence the universal coefficient theorem shows that θ is unique as an element of $H^n(WZ_p \times_{Z_p} (K(\pi, n))^P; \pi)$.

Since θ is unique, the Dyer-Lashof operations on the homology of $K(\pi + \pi', n) = K(\pi, n) \times K(\pi', n)$ can be regarded as the tensor product of those on the direct summands $H_*(K(\pi, n))$, $H_*(K(\pi', n))$.

So it suffices to consider the Dyer-Lashof operations on $H_*(K(Z_p^h, n); Z_p)$ and $H_*(K(Z, n); Z_p)$ (p : prime, $h, n \geq 1$).

Lemma 3.2. Let $H^* = \bigwedge (x_\alpha; \alpha \in A) \otimes Z_p[y_\beta; \beta \in B]$ be an algebra over Z_p , of locally finite type, and with product map Δ^* . Then in its dual coalgebra (H_*, Δ_*) , an element is primitive (i.e., $\Delta_*(z) = z \otimes 1 + 1 \otimes z$) if and only if it is a linear combination of elements $\{(x_\alpha)^*, (y_\beta)^*\}$.

Proof. Let $s(\alpha_1^{i_1} \alpha_2^{i_2} \dots \beta_1^{j_1} \beta_2^{j_2} \dots)$ be the dual basis of $x_{\alpha_1}^{i_1} x_{\alpha_2}^{i_2} \dots y_{\beta_1}^{j_1} y_{\beta_2}^{j_2} \dots$. Then

$$\Delta_*(s(\alpha_1^{i_1} \dots \beta_1^{j_1} \dots)) = \sum_{\substack{i_1' + i_1'' = i_1 \\ j_1' + j_1'' = j_1 \\ \vdots}} s(\alpha_1^{i_1'} \dots \beta_1^{j_1'} \dots) \otimes s(\alpha_1^{i_1''} \dots \beta_1^{j_1''} \dots),$$

and the primitive elements are determined as above.

We shall use the Cartan formula for the coproduct Δ_* :

$$\Delta_* Q^r(x) = \sum_i \sum_j Q^i(x') \otimes Q^{r-i}(x'') \quad (\text{if } \Delta_*(x) = \sum x' \otimes x'') \quad \dots (3.3)$$

and the Nishida relations:

$$Sq_*^s \circ Q^r = \sum_i \binom{r-s}{s-2i} Q^{r-s+i} \circ Sq_*^i \quad (p = 2) \quad \dots (3.4)$$

$$\left. \begin{aligned} \rho_*^s \circ Q^r &= \sum_i (-1)^{i+s} \binom{r-s}{s-pi} Q^{r-s+i} \circ \rho_*^i \\ \rho_*^s \circ \beta_* \circ Q^r &= \sum_i (-1)^{i+s} \binom{r-s}{s-pi} \beta_* \circ Q^{r-s+i} \circ \rho_*^i \\ &\quad + \sum_i (-1)^{i+s} \binom{r-s}{s-pi-1} Q^{r-s+i} \circ \rho_*^i \circ \beta_* \end{aligned} \right\} (p: \text{ odd}) \dots (3.5)$$

(see [7]).

As for the structure of the cohomology of $K(\pi, n)$, we refer to [8]

and [9]. In any case, Lemma 3.2 always applies to the cohomology

group of our consideration.

Proposition 3.6. In $H_*(K(Z_{2^h}, n); Z_2)$, $Q^r(x) = 0$, except $Q^0(1) = 1$.

Proof. We shall prove this by induction on the degree.

Take a homogeneous element x . By the induction hypothesis,

$$\|y\| < \|x\| \Rightarrow Q^r(y) = 0 \quad \text{for any } r, \text{ except that } Q^0(1) = 1.$$

Then the Cartan formula (3.3) implies that $\Delta_* Q^r(x) = Q^r(x) \otimes 1 + 1 \otimes Q^r(x)$,

i.e., $Q^r(x)$ is primitive.

Hence by Lemma 3.2, it suffices to show that $\langle \text{Sq}_h^I(\mathbb{Z}_n), Q^r(x) \rangle = 0$,

where $d(I) + n = r + \|x\|$. (see [8].)

Write $I = (i_1, \dots, i_t)$. If $i_t > 1$, put $I' = I$ and $v = \mathbb{Z}_n$,
 then $\text{Sq}_h^I = \text{Sq}_h^{I'}$. If $i_t = 1$, put $I' = (i_1, \dots, i_{t-1})$ and $v = \beta_h(\tilde{\mathbb{Z}}_n)$,
 then $\text{Sq}_h^I(\mathbb{Z}_n) = \text{Sq}_h^{I'}(v)$. ($\tilde{\mathbb{Z}}_n$ is the fundamental class in $H^n(K(\mathbb{Z}_{2^h}, n); \mathbb{Z}_{2^h})$
 and β_h is the Bockstein homomorphism.)

Note that $\langle \text{Sq}_h^I(\mathbb{Z}_n), Q^r(x) \rangle = \langle v, \text{Sq}_*^{I'} \circ Q^r(x) \rangle$.

Now we use the Nishida relation (3.4). By the induction hypothesis,

$Q^{r-s+i}(\text{Sq}_*^i(x)) = 0$ if $i > 0$. Hence

$$\text{Sq}_*^{I'} \circ Q^r(x) = \binom{r-i_1}{i_1} \binom{r-i_1-i_2}{i_2} \dots \binom{r-d(I')}{i_t \text{ or } i_{t-1}} Q^{r-d(I')}(x).$$

Recall $r-d(I') = n - \|x\| + (d(I) - d(I')) \leq n+1 - \|x\|$.

Since $\|x\| \geq n$ and $Q^r(x) = 0$ if $r < \|x\|$,

$$\text{Sq}_*^{I'} \circ Q^r(x) = 0 \text{ unless } \|x\| = n = 1, r-d(I') = i_t = 1.$$

Hence it suffices to show that

$$\langle \beta_h(\tilde{\mathbb{Z}}_1), Q^1(u_1) \rangle = 0. \quad (u_1 \text{ is the fundamental homology class.})$$

In fact, $\langle \beta_h(\tilde{\mathbb{Z}}_1), Q^1(u_1) \rangle = \langle \beta_h(\tilde{\mathbb{Z}}_1), u_1^2 \rangle = \langle \mu^*(\beta_h(\tilde{\mathbb{Z}}_1)), u_1 \otimes u_1 \rangle$

$$= \langle (\beta_h \otimes \rho + \rho \otimes \beta_h) \circ \mu^*(\tilde{\mathbb{Z}}_1), u_1 \otimes u_1 \rangle = \langle \beta_h(\tilde{\mathbb{Z}}_1) \otimes 1 + 1 \otimes \beta_h(\tilde{\mathbb{Z}}_1), u_1 \otimes u_1 \rangle$$

$$= 0, \quad \text{by virtue of Lemma 3.7 below.}$$

Thus we have shown $Q^r(x) = 0$, and the proposition is proved by induction.

Lemma 3.7. The diagram

$$\begin{array}{ccc}
 H^1(X; Z_{2^h}) & \xrightarrow{\beta_h} & H^2(X; Z_2) \\
 \mu^* \downarrow & & \downarrow \mu^* \\
 H^1(X \times X; Z_{2^h}) & \xrightarrow{\beta_h} & H^2(X \times X; Z_2) \\
 \uparrow \kappa & & \uparrow \kappa \\
 \sum_{i_1+i_2=1} H^{i_1}(X; Z_{2^h}) \otimes H^{i_2}(X; Z_{2^h}) & \xrightarrow{\beta_h \otimes \rho + \rho \otimes \beta_h} & \sum_{j_1+j_2=2} H^{j_1}(X; Z_2) \otimes H^{j_2}(X; Z_2)
 \end{array}$$

is commutative, where ρ = reduction mod 2, β_h = Bockstein homomorphism,

κ = the cross product.

Proof. This follows from the commutativity of the following diagrams:

$$\begin{array}{ccc}
 C^*(X; Z_2) \otimes C^*(X; Z_{2^{h+1}}) & \xrightarrow{1 \otimes \rho} & C^*(X; Z_2) \otimes C^*(X; Z_2) \xrightarrow{\kappa} C^*(X; Z_2) \\
 i_* \otimes 1 \downarrow & & \downarrow i_* \\
 C^*(X; Z_{2^{h+1}}) \otimes C^*(X; Z_{2^{h+1}}) & \xrightarrow{\kappa} & C^*(X; Z_{2^{h+1}})
 \end{array}$$

$$\begin{array}{ccc}
 C^*(X; Z_{2^{h+1}}) \otimes C^*(X; Z_{2^{h+1}}) & \xrightarrow{\kappa} & C^*(X; Z_{2^{h+1}}) \\
 \delta \otimes 1 + 1 \otimes \delta \uparrow & & \uparrow \delta \\
 C^*(X; Z_{2^{h+1}}) \otimes C^*(X; Z_{2^{h+1}}) & \xrightarrow{\kappa} & C^*(X; Z_{2^{h+1}}) \\
 j_* \otimes j_* \downarrow & & \downarrow j_* \\
 C^*(X; Z_{2^h}) \otimes C^*(X; Z_{2^h}) & \xrightarrow{\kappa} & C^*(X; Z_{2^h})
 \end{array}$$

Proposition 3.8. In $H_*(K(Z_{p^h}^n); Z_p)$, p : odd, $Q^r(x) = 0$, except

$$Q^0(1) = 1.$$

Proof. This is entirely same as 3.6. In fact:

it suffices to prove $\langle \rho_h^I(z_n), Q^r(x) \rangle = 0$, with $d(I) + n = 2r(p-1) + \|x\|$.

Write $I = (\varepsilon_0, i_1, \varepsilon_1, \dots, i_t, \varepsilon_t)$ and put $I' = I - (\varepsilon_t)$.

The Nishida relation (3.5) and the induction hypothesis show that

$$\rho_*^{I'} \circ Q^r(x) = \lambda \beta_*^{\varepsilon_{t-1}} \dots \beta_*^{\varepsilon_0} \circ Q^{r-i_1-\dots-i_t}(x), \quad \lambda \in \mathbb{Z}_p.$$

Note that $r-i_1-\dots-i_t = \frac{n-\|x\|+\sum \varepsilon_j}{2(p-1)}$. If $\varepsilon_0+\dots+\varepsilon_{t-1} > 1$, then

$$\beta_*^{\varepsilon_{t-1}} \dots \beta_*^{\varepsilon_0} = 0. \quad \text{If } \varepsilon_0+\dots+\varepsilon_{t-1} \leq 1, \text{ then } r-i_1-\dots-i_t \leq 2/2(p-1),$$

and hence ≤ 0 , thus $Q^{r-i_1-\dots-i_t}(x) = 0$.

Thus $\langle \rho_h^I(\mathbb{Z}_n), Q^r(x) \rangle$ is always shown to be zero.

Proposition 3.9. In $H_*(K(\mathbb{Z}, n); \mathbb{Z}_p)$, p : any prime, $Q^r(x) = 0$,

except $Q^0(1) = 1$.

Proof. This is now obvious, by the proof of 3.6 and 3.8.

Now we complete the proof 2., combining 3.1 and 3.6 ~ 3.9.

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