

N- Fractional Calculus of Some Multi- Powers Functions

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Abstract

In a previous article of the author, N- fractional calculus

$$\left(((z-b)^\beta - c)^\alpha \right)_\nu, \quad ((z-b)^\beta - c \neq 0)$$

are discussed. In this article that of more extended forms

$$\left((((z-b)^\beta - c)^\alpha - d)^\delta \right)_\nu, \quad (((z-b)^\beta - c)^\alpha - d \neq 0)$$

are discussed. Moreover their special cases

$$\left((((z-b)^\beta - c)^\alpha - d)^\delta \right)_n, \quad (n \in \mathbb{Z}_0^+, ((z-b)^\beta - c)^\alpha - d \neq 0)$$

are presented .

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\text{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\text{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_\nu = (f)_\nu = {}_c(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_c \frac{f(\xi)}{(\xi-z)^{\nu+1}} d\xi \quad (\nu \notin \mathbb{Z}), \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (2)$$

where $-\pi \leq \arg(\xi-z) \leq \pi$ for C_- , $0 \leq \arg(\xi-z) \leq 2\pi$ for C_+ ,

$\xi \neq z$, $z \in C$, $\nu \in \mathbb{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\xi}{(\xi-z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}), \quad [\text{Refer to (1)}] \quad (3)$$

with
$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in \mathbb{R}\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in \mathbb{R}\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. "F.O.G. $\{N^\nu\}$ " is an "Action product group which has continuous index ν " for the set of F . (F.O.G.; Fractional calculus operator group) [3]

(III) Lemma. We have [1]

$$(i) \quad ((z-c)^b)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-b)}{\Gamma(-b)} (z-c)^{b-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-b)}{\Gamma(-b)} \right| < \infty \right), \quad (7)$$

$$(ii) \quad (\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty), \quad (8)$$

$$(iii) \quad ((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty), \quad (9)$$

where $z-c \neq 0$ for (i) and $z-c \neq 0, 1$ for (ii), (iii),

$$(iv) \quad (u \cdot v)_\alpha = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad \left(\begin{array}{l} u = u(z), \\ v = v(z) \end{array} \right). \quad (10)$$

§ 1. Preliminary

The Teorem below is reported by the author already (cf. J. Frac. Calc. Vol. 29, May (2006), pp.35 - 44.) . [12]

Theorem D. We have

$$(i) \quad (((z-b)^\beta - c)^\alpha)_\gamma = e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha \beta + \gamma)}{k! \Gamma(\beta k - \alpha \beta)} \left(\frac{c}{(z-b)^\beta} \right)^k \quad (1)$$

$$\left(\left| \frac{\Gamma(\beta k - \alpha \beta + \gamma)}{\Gamma(\beta k - \alpha \beta)} \right| < \infty \right)$$

and

$$(ii) \quad \left(((z-b)^\beta - c)^\alpha \right)_n = (-1)^n (z-b)^{\alpha \beta - n}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha \beta]_n}{k!} \left(\frac{c}{(z-b)^\beta} \right)^k \quad (n \in \mathbb{Z}_0^+) \quad (2)$$

where

$$\left| \frac{c}{(z-b)^\beta} \right| < 1,$$

and

$$[\lambda]_k = \lambda(\lambda+1) \cdots (\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \text{ with } [\lambda]_0 = 1, \\ (\text{Notation of Pochhammer}).$$

§ 2. N-Fractional Calculus of Functions

$$\left(((z-b)^\beta - c)^\alpha - d \right)^\delta$$

Theorem 1. We have

$$(i) \quad \left((((z-b)^\beta - c)^\alpha - d)^\delta \right)_\gamma = e^{-i\pi\gamma} (z-b)^{\alpha \beta \delta - \gamma} \\ \times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k \Gamma(\beta k - \alpha \beta(\delta-m) + \gamma)}{m! \cdot k! \Gamma(\beta k - \alpha \beta(\delta-m))} \left(\frac{c}{(z-b)^\beta} \right)^k \left(\frac{d}{(z-b)^{\alpha \beta}} \right)^m \quad (1)$$

$$\left(\left| \frac{\Gamma(\beta k - \alpha \beta(\delta-m) + \gamma)}{\Gamma(\beta k - \alpha \beta(\delta-m))} \right| < \infty \right)$$

and

$$(ii) \quad \left((((z-b)^\beta - c)^\alpha - d)^\delta \right)_n = (-1)^n (z-b)^{\alpha \beta \delta - n}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k [\beta k - \alpha \beta(\delta-m)]_n}{m! \cdot k!} \left(\frac{c}{(z-b)^\beta} \right)^k \left(\frac{d}{(z-b)^{\alpha \beta}} \right)^m, \quad (2)$$

where

$$((z-b)^\beta - c)^\alpha - d \neq 0, \quad \left| \frac{c}{(z-b)^\beta} \right| < 1, \quad \left| \frac{d}{(z-b)^{\alpha\beta}} \right| < 1, \quad n \in \mathbb{Z}_0^+.$$

Proof of (i). We have

$$(((z-b)^\beta - c)^\alpha - d)^\delta = X^\delta \left(1 - \frac{d}{X}\right)^\delta \quad (X = ((z-b)^\beta - c)^\alpha) \quad (3)$$

$$= X^\delta \sum_{m=0}^{\infty} \frac{[-\delta]_m}{m!} \left(\frac{d}{X}\right)^m \quad (4)$$

$$= \sum_{m=0}^{\infty} \frac{[-\delta]_m (d^m)}{m!} X^{\delta-m} \quad (5)$$

$$= \sum_{m=0}^{\infty} \frac{[-\delta]_m (d^m)}{m!} ((z-b)^\beta - c)^{\alpha(\delta-m)}, \quad (6)$$

hence, operating N-fractional calculus operator N^γ to the both sides of (6), we obtain

$$\begin{aligned} & \left((((z-b)^\beta - c)^\alpha - d)^\delta \right)_\gamma \\ &= \sum_{m=0}^{\infty} \frac{[-\delta]_m (d^m)}{m!} \left(((z-b)^\beta - c)^{\alpha(\delta-m)} \right)_\gamma \end{aligned} \quad (7)$$

$$\begin{aligned} &= \sum_{m=0}^{\infty} \frac{[-\delta]_m (d^m)}{m!} \left[e^{-i\pi\gamma} (z-b)^{\alpha\beta(\delta-m)-\gamma} \right. \\ &\quad \times \sum_{k=0}^{\infty} \frac{[-\alpha(\delta-m)]_k \Gamma(\beta k - \alpha\beta(\delta-m) + \gamma)}{k! \Gamma(\beta k - \alpha\beta(\delta-m))} \left(\frac{c}{(z-b)^\beta} \right)^k \left. \right], \end{aligned} \quad (8)$$

applying Theorem D. (i), under the conditions stated before.

We have then (1) from (8) clearly.

Proof of (ii). Set $\gamma = n$ in (1).

Note 1. We use the notations $\sum_{m,k=0}^{\infty} \dots = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \dots$, for our convenience.

Note.2. When $d = 0$ and $\delta = 1$, Theorem 1 is reduced to Theorem D., clearly.

Corollary 1. We have

$$\begin{aligned}
 (i) \quad & \left(((z^\beta - c)^\alpha - d)^\delta \right)_\gamma = e^{-i\pi\gamma} z^{\alpha\beta\delta - \gamma} \\
 & \times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta - m)]_k \Gamma(\beta k - \alpha\beta(\delta - m) + \gamma)}{m! \cdot k! \Gamma(\beta k - \alpha\beta(\delta - m))} \left(\frac{c}{z^\beta} \right)^k \left(\frac{d}{z^{\alpha\beta}} \right)^m \\
 & \left(\left| \frac{\Gamma(\beta k - \alpha\beta(\delta - m) + \gamma)}{\Gamma(\beta k - \alpha\beta(\delta - m))} \right| < \infty \right)
 \end{aligned} \quad (9)$$

and

$$\begin{aligned}
 (ii) \quad & \left(((z^\beta - c)^\alpha - d)^\delta \right)_n = (-1)^n z^{\alpha\beta\delta - n} \\
 & \times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta - m)]_k [\beta k - \alpha\beta(\delta - m)]_n}{m! \cdot k!} \left(\frac{c}{z^\beta} \right)^k \left(\frac{d}{z^{\alpha\beta}} \right)^m,
 \end{aligned} \quad (10)$$

where

$$(z^\beta - c)^\alpha - d \neq 0, \quad \left| \frac{c}{z^\beta} \right| < 1, \quad \left| \frac{d}{z^{\alpha\beta}} \right| < 1, \quad n \in \mathbb{Z}_0^+.$$

Proof. Set $b = 0$ in Theorem 1.

§ 3. Some Special Cases

[I] When $\beta = \alpha = 1$, we obtain

$$\begin{aligned}
 & \left(((z - b - c - d)^\delta) \right)_\gamma = e^{-i\pi\gamma} (z - b)^{\delta - \gamma} \\
 & \times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [m - \delta]_k \Gamma(k + m - \delta + \gamma)}{m! \cdot k! \Gamma(k + m - \delta)} \left(\frac{c}{z - b} \right)^k \left(\frac{d}{z - b} \right)^m \\
 & \left(\left| \frac{c}{z - b} \right|, \left| \frac{d}{z - b} \right| < 1, \left| \frac{\Gamma(k + m - \delta + \gamma)}{\Gamma(k + m - \delta)} \right| < \infty \right)
 \end{aligned} \quad (1)$$

from Theorem 1. (i).

Now we have the identities

$$\Gamma(m - \delta) = \Gamma(-\delta) [-\delta]_m, \quad (2)$$

$$\Gamma(k + m - \delta) = \Gamma(m - \delta) [m - \delta]_k, \quad (3)$$

$$\Gamma(k + m - \delta + \gamma) = \Gamma(m - \delta + \gamma) [m - \delta + \gamma]_k, \quad (4)$$

then applying (2) ~ (4) into the RHS of (1), we obtain

$$\text{RHS of (1)} = e^{-i\pi\gamma} (z-b)^{\delta-\gamma}$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(m-\delta+\gamma)}{m! \Gamma(-\delta)} \left(\frac{d}{z-b}\right)^m \sum_{k=0}^{\infty} \frac{[m-\delta+\gamma]_k}{k!} \left(\frac{c}{z-b}\right)^k \quad (5)$$

$$= e^{-i\pi\gamma} (z-b)^{\delta-\gamma} \sum_{m=0}^{\infty} \frac{\Gamma(m-\delta+\gamma)}{m! \Gamma(-\delta)} \left(\frac{d}{z-b}\right)^m \left(1 - \frac{c}{z-b}\right)^{\delta-\gamma-m} \quad (6)$$

$$= e^{-i\pi\gamma} (z-b-c)^{\delta-\gamma} \frac{\Gamma(-\delta+\gamma)}{\Gamma(-\delta)} \sum_{m=0}^{\infty} \frac{[-\delta+\gamma]_m}{m!} \left(\frac{d}{z-b-c}\right)^m \quad (7)$$

$$= e^{-i\pi\gamma} \frac{\Gamma(-\delta+\gamma)}{\Gamma(-\delta)} (z-b-c-d)^{\delta-\gamma}, \quad \left(\left| \frac{\Gamma(-\delta+\gamma)}{\Gamma(-\delta)} \right| < \infty \right) \quad (8)$$

using the relationship

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1-z)^{-\lambda}. \quad (9)$$

The result (8) is same as the one obtained by Lemma (i).

[II] When $d=0$ and $\delta=1$, we obtain

$$(((z-b)^{\beta}-c)^{\alpha})_{\gamma} = e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-1]_m [-\alpha(1-m)]_k \Gamma(\beta k - \alpha\beta(1-m) + \gamma)}{m! \cdot k! \Gamma(\beta k - \alpha\beta(1-m))} \left(\frac{c}{(z-b)^{\beta}}\right)^k \left(\frac{0}{(z-b)^{\alpha\beta}}\right)^m \quad (10)$$

$$= e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma} \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^{\beta}}\right)^k \quad (11)$$

$$\left(\left| \frac{c}{(z-b)^{\beta}} \right| < 1, \quad \left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right)$$

from Theorem 1. (i).

That is, in this case we have Theorem D. (i) it self, clearly.

[III] When $n=0$, we have

$$(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta})_0 = (z-b)^{\alpha\beta\delta}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k}{m! \cdot k!} \left(\frac{c}{(z-b)^{\beta}}\right)^k \left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m. \quad (12)$$

$$((z-b)^\beta - c)^\alpha - d \neq 0, \quad \left| \frac{c}{(z-b)^\beta} \right| < 1, \quad \left| \frac{d}{(z-b)^{\alpha\beta}} \right| < 1, .$$

from Theorem 1. (ii).

Indeed we obtain

$$\text{RHS of (12)} = (z-b)^{\alpha\beta\delta}$$

$$\times \sum_{m=0}^{\infty} \frac{[-\delta]_m}{m!} \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^m \sum_{k=0}^{\infty} \frac{[-\alpha(\delta-m)]_k}{k!} \left(\frac{c}{(z-b)^\beta} \right)^k \quad (13)$$

$$= (z-b)^{\alpha\beta\delta} \sum_{m=0}^{\infty} \frac{[-\delta]_m}{m!} \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^m \left(1 - \frac{c}{(z-b)^\beta} \right)^{\alpha\delta - \alpha m} \quad (14)$$

$$= ((z-b)^\beta - c)^{\alpha\delta} \sum_{m=0}^{\infty} \frac{[-\delta]_m}{m!} \left(\frac{d}{((z-b)^\beta - c)^\alpha} \right)^m \quad (15)$$

$$= ((z-b)^\beta - c)^{\alpha\delta} \left(1 - \frac{d}{((z-b)^\beta - c)^\alpha} \right)^\delta \quad (16)$$

$$= (((z-b)^\beta - c)^\alpha - d)^\delta, \quad (17)$$

clearly.

[IV] When $n=1$, we have

$$\begin{aligned} & (((z-b)^\beta - c)^\alpha - d)^\delta \Big|_1 = -(z-b)^{\alpha\beta\delta-1} \\ & \times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k \cdot (\beta k - \alpha\beta\delta + \alpha\beta m)}{m! \cdot k!} \left(\frac{c}{(z-b)^\beta} \right)^k \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^m, \quad (18) \end{aligned}$$

from Theorem 1. (ii).

Then letting

$$R := \left(\frac{u}{(z-b)^\beta} \right)^\alpha \quad (u = (z-b)^\beta - c) \quad (19)$$

and

$$S := \left(1 - \frac{d}{u^\alpha} \right)^\delta \quad (20)$$

we have

$$\sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k \cdot \beta k}{m! \cdot k!} \left(\frac{c}{(z-b)^\beta} \right)^k \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^m$$

$$= \frac{-c\alpha\beta\delta}{u} R^\delta S - \frac{cd\alpha\beta\delta}{u(u^\alpha-d)} R^\delta S, \quad (21)$$

$$- \alpha\beta\delta \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k}{m! \cdot k!} \left(\frac{c}{(z-b)^\beta} \right)^k \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^m$$

$$= -\alpha\beta\delta R^\delta S, \quad (22)$$

and

$$\alpha\beta \sum_{m,k=0}^{\infty} \frac{[-\delta]_m \cdot m [-\alpha(\delta-m)]_k}{m! \cdot k!} \left(\frac{c}{(z-b)^\beta} \right)^k \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^m$$

$$= -\frac{d\alpha\beta\delta}{(u^\alpha-d)} R^\delta S, \quad (23)$$

Therefore, applying (23), (22) and (21) into (18), we obtain

$$(((z-b)^\beta - c)^\alpha - d)^\delta)_1 = (z-b)^{\alpha\beta\delta-1} R^\delta S$$

$$\times \left[\frac{c\alpha\beta\delta}{u} + \frac{cd\alpha\beta\delta}{u(u^\alpha-d)} + \alpha\beta\delta + \frac{d\alpha\beta\delta}{u^\alpha-d} \right] \quad (24)$$

$$= \alpha\beta\delta (z-b)^{\alpha\beta\delta-1} \left(\frac{c}{u} + 1 \right) \left(\frac{d}{u^\alpha-d} + 1 \right) \quad (25)$$

$$= \alpha\beta\delta (z-b)^{-1} (u^\alpha-d)^\delta \left(\frac{(z-b)^\beta}{u} \right) \left(\frac{u^\alpha}{u^\alpha-d} \right) \quad (26)$$

$$= \alpha\beta\delta (u^\alpha-d)^{\delta-1} (u^{\alpha-1}) (z-b)^{\beta-1} \quad (27)$$

$$= \alpha\beta\delta (((z-b)^\beta - c)^\alpha - d)^{\delta-1} ((z-b)^\beta - c)^{\alpha-1} (z-b)^{\beta-1} \quad (28)$$

clearly.

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