

On Radius Critical Graphs

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Abstracts

The radius $r(G)$ of a connected graph G is defined by:

$$r(G) = \min_{u \in V(G)} \max_{v \in V(G) - u} d(u, v)$$

where $d(u, v)$ is the length of the shortest path in G between u and v . G is radius-critical if deleting any vertex from G reduces its radius by 1. In this paper we relate this notion to the concepts of eccentricity and give characterizations of edge-maximal 3-radius-critical graphs. In particular, we show that every edge-maximal 3-radius-critical graph is edge-maximum.

1. Introduction

In this section we introduce some notions and obtain some preliminary results on radius critical graphs. Let G be a connected graph and let G' be the graph obtained by deleting some given vertex v of G . Let d and d' be the corresponding distance functions. If u and w are vertices in G then $d(u, w)$ is the length of the shortest path from u to w in G . Since G'

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need not be connected, the situation $d'(u,w) = +\infty$ is possible.

It is easily verified that:

$$d'(u,w) \geq d(u,w) \quad \text{for all } u,w \in V(G')$$

The eccentricity $e(u)$ of a vertex u in a graph G is defined by

$$e(u) = \max \{d(u,w) : w \in G\}.$$

Let $e'(u)$ be the eccentricity of vertex u in G' . We denote by $N_G(u)$ the open (nearest) neighborhood of a vertex u , that is the set of vertices adjacent to u . $N_G[u]$ denotes the closed (nearest) neighborhood which is defined by

$$N_G[u] = N_G(u) \cup \{u\}.$$

The furthest neighborhood $FN_G(u)$ of a vertex u is defined by

$$FN_G(u) = \{w \in V(G) \mid d(u,w) = e(u)\}.$$

A vertex v in $FN_G(u)$ is called a furthest neighbor of u . In case v is the unique furthest neighbor of u , we have:

$$e'(u) = e(u) - 1.$$

The furthest neighbor graph $FN(G)$ of a graph G is defined on $V(G)$ where uv is an edge of $FN(G)$ if and only if $u \in FN(v)$ or $v \in FN(u)$. The radius $rad(G)$ of a graph G is defined by

$$rad(G) = \min\{e(u) \mid u \in V(G)\}.$$

For any connected graph G and non cut vertex v we have

$$(1) \quad rad(G') \geq rad(G) - 1.$$

There is, however, no reasonable upper bound on the radius of G' .

If G is the join of the path of size $2n$ and an extra vertex, that is, $p_{2n} + \{v\}$, then

$$rad(G) = 1 \quad \text{and} \quad rad(G') = n.$$

The inequality (1) leads to the definition of the following class of graphs.

Definition. A block G is radius-critical if for each $v \in V(G)$:

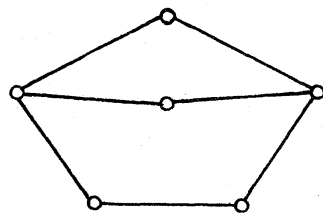
$$\text{rad}(G') = \text{rad}(G) - 1.$$

An r -radius-critical graph is a radius-critical graph with radius r . The above discussion leads to the following characterization of radius-critical graphs.

Lemma 1: G is radius-critical if and only if

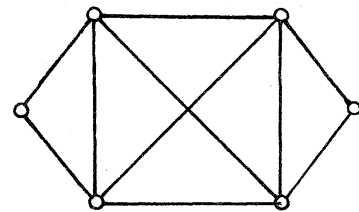
- (i) each vertex in G has a unique furthest neighbor
- and
- (ii) each vertex in G has the same eccentricity, (equi-eccentric).

The graphs (i) and (ii) in figure 1 show the necessity of the two conditions:



equi-eccentric but
non-unique furthest neighbors.

(i)



unique furthest neighbor
but not equi-eccentric.

(ii)

Fig. 1

In fact, condition (i) is not strong enough to make G a block, as paths of even order satisfy (i) but not (ii). Further properties of equi-eccentric graphs may be found in Ando et al [1].

2. Polarities and Radius-critical graphs.

We begin with the following definition:

Definition. ψ is a polarity¹ on a connected graph G if ψ is a fixed point free involution on $V(G)$ such that

$$d(u, v) = d(u, \psi(u)) \Rightarrow v = \psi(u) \quad \text{for all } u \in V(G).$$

Let G be an equi-eccentric block such that each vertex has a unique furthest neighbor. Then it is easily verified that

$$(2) \quad \psi(u) = FN_G(u) \quad \text{for all } u \in V(G)$$

defines a polarity. The next result shows that these are in fact the only polarities on blocks.

Theorem 1. There is a polarity on a block G if and only if G is radius-critical.

Proof: ($\Leftarrow$) By lemma 1, G is radius critical if and only if it is equi-eccentric and each vertex has a unique furthest neighbor. As remarked above, (2) defines a polarity.

($\Rightarrow$) Let G be a block with polarity ψ . First suppose for some vertex u , $e(u) = 1$. Then

$$d(u, \psi(u)) = 1.$$

and G must be K_2 , since

$$d(u, v) = 1 \Rightarrow v = \psi(u).$$

Thus we may assume that $e(u) \geq 2$ for all vertices u . Since G is a block, for any k smaller than $e(u)$, there must be at least two vertices v, w such that

1. The authors are grateful to H. Enomoto for suggesting this term.

$$d(u,v) = d(u,w) = k.$$

Thus, by the definition of polarity

$$d(u, \psi(u)) = e(u) \quad \text{and} \quad FN_G(u) = \{\psi(u)\}.$$

Consider some vertex w in $N_G(\psi(u))$. Clearly

$$e(w) \geq e(u) - 1.$$

The inequality is in fact strict, since otherwise $\psi(w) = u$, contradicting the assumption that ψ is an involution. Therefore,

$$e(w) \geq e(u) \quad \text{for all } w \in N_G(\psi(u)).$$

This implies that G is equi-eccentric. By the lemma, G must be radius-critical.

Corollary. G is radius-critical if and only if

$$FN(G) = nK_2, \quad n \geq 2.$$

Proof: It is easily verified that if $FN(G) = nK_2$ and $n \geq 2$, then G must be a block. The statement then follows from theorem 1.

3. 3-radius-critical graphs

In this section we study edge-maximal 3-radius-critical graphs. We show that a 3-radius-critical graph is edge-maximal if and only if it is edge-maximum. Finally we obtain a characterization of 3-radius-critical graphs.

We first obtain a bound on the maximum degree $\Delta(G)$, of a 3-radius-critical graph G of even order p . Since every vertex of G has eccentricity 3,

$$\psi(N[v]) \cap N[v] = \phi \quad \text{for all } v \in V(G).$$

Otherwise let x be a vertex not only in $N[v]$ but also in $\psi(N[v])$.

Then $d(v, \psi(v)) \leq 2$, a contradiction. This implies that

$$\deg(v) \leq (p-2)/2.$$

Define $H_p = K_{p/2, p/2} - (p/2)K_2$. That is, H_p is the complete $(p/2, p/2)$ bipartite graph minus a one-factor. It is easily verified that H_p is 3-radius-critical, and, by the above remark, also edge-maximum. Thus 3-radius-critical edge-maximum graphs are $(p-2)/2$ -regular.

Lemma 2. Let G be a 3-radius-critical graph. Suppose u and v are non-adjacent vertices satisfying:

$$(i) \ u \in V(G) - \{N[v] \cup [\psi(v)]\}$$

$$(ii) \ \psi(u) \notin N[v]$$

then joining u and v by an edge leaves a 3-radius-critical graph.

Proof: Let H be the graph formed from G by joining u and v , then

$$d_H(x, y) \leq d_G(x, y) \quad \text{for all } x, y \in V(G).$$

So we need only show that

$$(3) \ d_H(x, y) \leq 2 \quad \text{implies that } d_G(x, y) \leq 2.$$

Case 1. $d_H(x, y) = 1$.

Either xy is an edge in G , in which case (3) is trivial, or xy is the edge uv . But if xy is the edge uv , then $d_G(x, y) \leq 2$ because of condition (ii).

Case 2. $d_H(x, y) = 2$

Let w be adjacent to x and y in H . If w is adjacent to x and y in G , (3) is immediate. Thus we may assume that xw is the new edge. Suppose $x = u$, $w = v$. Then condition (ii) implies that $y \neq \psi(u)$. Thus

$$d_G(x, y) = d_G(u, y) = 2.$$

Otherwise $x = v$, $w = u$. But then the condition (i) that $u \in N[\psi(v)]$ implies $y \notin \psi(v)$. Thus

$$d_G(x, y) = d_G(v, y) = 2.$$

Theorem 2. A 3-radius-critical graph G is edge-maximal if and only if it is edge-maximum.

Proof: Suppose G is an edge-maximal 3-radius-critical graph and that x is a vertex of degree less than $(p-2)/2$. Then we will find two vertices satisfying the conditions of lemma 2, yielding a contradiction to the edge maximality of G . Let

$$W = V(G) - \{N_G[x] \cup N_G[\psi(x)]\}.$$

The degree condition on x implies that W is not empty. Suppose $\psi(W) \not\subseteq N[x]$, and choose $y \in W$ such that $\psi(y) \notin N[x]$. Then setting $u = y$ and $v = x$, the conditions of lemma 2 are satisfied.

Otherwise, $\psi(W) \subseteq N[x]$ so $\psi(W) \not\subseteq N[\psi(x)]$. In this case, choose $y \in W$ such that $\psi(y) \in N[\psi(x)]$. Then setting $u = y$ and $v = \psi(x)$, the conditions of lemma 2 are satisfied. Thus the theorem follows.

We remark that edge-minimal 3-radius-critical graphs are not necessarily edge-minimum. Consider, for example, the edge-minimal 3-radius-critical graph $H_5 = K_{5,5} - 5K_2$, which is in fact edge-maximum!

Before giving a final result on edge-maximum 3-radius-critical graphs, we need a new definition.

Definition. The distance two graph G_2 of a graph G is defined on $V(G)$, where uv is an edge of G_2 if and only if $d_G(u,v) = 2$.

Theorem 3. G is an edge-maximal 3-radius-critical graph of order p if and only if G is H_p or G_2 is an edge-maximal 3-radius-critical graph.

Proof: If G is H_p the theorem is immediate. So let G be any edge-maximal 3-radius-critical graph that is not H_p . Let ψ be the polarity of G and set

$$E' = \{uv \in E(\bar{G}) \mid \psi(u) = v\}.$$

Then it may be verified that $G_2 = \bar{G} - E'$. We will verify that G_2 is also 3-radius-critical edge-maximal.

Case 1. $d_G(u,v) = 3$.

In this case $v = \psi(u)$, $uv \in E'$ and thus $d_{G_2}(u,v) \geq 2$. Further, G is also edge maximum, so

$$V(G) = N_G[u] \cup N_G[\psi(u)]$$

since G is $(p-2)/2$ -regular. This implies that

$$N_G[u] \cap N_G[\psi(u)] = \phi,$$

and hence $d_{G_2}(u,v) \geq 3$. On the other hand, G_2 is also $(p-2)/2$ -regular and is not $K_{p/2} \cup K_{p/2}$, since $G \neq H_p$. Thus G_2 is connected and $d_{G_2}(u,v) = 3$.

Case 2. $d_G(u,v) = 2$.

By definition, $d_{G_2}(u,v) = 1$.

Case 3. $d_G(u,v) = 1$.

Since ψ is a polarity,

$$d_G(u, \psi(v)) = d_G(v, \psi(u)) = 2.$$

Hence $u\psi(v)$ and $v\psi(u)$ are edges in G_2 and $d_{G_2}(u,v) = 2$.

We now see that G_2 had radius 3 and every vertex has degree $(p-2)/2$. Further ψ is a polarity on G_2 and G_2 is therefore edge-maximal radius-critical. Under these conditions, we may interchange the roles of G and G_2 in the above case analysis to see that $(G_2)_2 = G$, or in other words, the distance two graph of G_2 is G . This proves the sufficiency of the statement of theorem 3.

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