

The rectangular fractional integral operators

By

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Abstract

With rectangular doubling weight, a generalized Hardy-Littlewood-Sobolev inequality for rectangular fractional integral operators is verified. The result is a nice application of M -linear embedding theorem for dyadic rectangles.

§ 1. Introduction

The purpose of this paper is to demonstrate a nice application of M -linear embedding theorem for dyadic rectangles obtained in [21] and [20], and to study a generalized Hardy-Littlewood-Sobolev inequality for rectangular fractional integral operators.

For a positive integer N , let $0 < \alpha < N$. For the rectangular doubling weight μ^1 on $\mathbb{R}^N$, define the rectangular the fractional integral operator R_α^μ by

$$R_\alpha^\mu f(x) := \int_{\mathbb{R}^N} \mu(R(x, y))^{\frac{\alpha}{N}-1} f(y) \, d\mu(y), \quad x \in \mathbb{R}^N,$$

where $R(x, y)$ stands for the minimal rectangle, with respect to inclusion, which contains two different points, up to coordinates, x and y and has their sides parallel to the coordinate axes. We have the following theorem the proof of which is our goal.

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¹For precise definition, see Subsection 3.1

Theorem 1.1. *For $1 < p < q < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}$, the generalized Hardy-Littlewood-Sobolev inequality*

$$\|R_\alpha^\mu f\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\mu)}$$

holds for all $f \in L^p(\mu)$.

In the case $\mu \equiv 1$ and when we restrict rectangles to cubes, Theorem 1.1 is just the Hardy-Littlewood-Sobolev inequality which is one of the most fundamental norm inequality of real variable harmonic analysis. In the case μ is a doubling weight and when we restrict rectangles to cubes, Theorem 1.1 was studied in Stein's book [16].

Thanks to the fact that μ is rectangular doubling weight, the operator R_α^μ almost commutes with a multi-parameter family of dilation. The weight theory of such product operators, commuting with a multi-parameter family of dilation, is not developed so much up to now despite a number of pioneering works in the 1980's due to Robert Fefferman and Elias Stein (see [3, 4, 5]). The area remains largely open for product fractional integrals (see also [1, 9, 15, 23]).

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2 , do not change in different occurrences. By $A \approx B$ we mean that $c^{-1}B \leq A \leq cB$ with some positive finite constant c independent of appropriate quantities. We write $X \lesssim Y, Y \gtrsim X$ if there is a independent constant c such that $X \leq cY$.

§ 2. M -linear embedding theorem for product dyadic cubes

In what follows we recall M -linear embedding theorem for dyadic rectangles obtained in [21] and [20].

Let $(N_1, N_2, \dots, N_n)$ be an n -tuple of positive integers and let $N = \sum_{i=1}^n N_i$. We decompose the N dimensional Euclidean space $\mathbb{R}^N$ by

$$\mathbb{R}^N = \prod_{i=1}^n \mathbb{R}^{N_i}.$$

By $x \in \mathbb{R}^N$ we will denote

$$x = (x_1, x_2, \dots, x_n), \quad x_i \in \mathbb{R}^{N_i}.$$

By a *product cube* we will always mean a rectangle $R \subset \mathbb{R}^N$ of the form

$$R = \prod_{i=1}^n Q_i, \quad Q_i \in \mathcal{Q}(\mathbb{R}^{N_i}),$$

where $\mathcal{Q}(\mathbb{R}^d)$ stands for the set of all cubes in $\mathbb{R}^d$ with sides parallel to the coordinate axes. We will denote by $\mathcal{R}(\mathbb{R}^N)$ the family of all such product cubes. That is,

$$\mathcal{R}(\mathbb{R}^N) := \prod_{i=1}^n \mathcal{Q}(\mathbb{R}^{N_i}).$$

We denote by $\mathcal{DQ}(\mathbb{R}^d)$ the family of all dyadic cubes $Q = 2^{-k}(m + [0, 1)^d)$, $k \in \mathbb{Z}$, $m \in \mathbb{Z}^d$. We denote by $\mathcal{DR}(\mathbb{R}^N)$ the family of all product dyadic cubes on the product space $\mathbb{R}^N$, that is,

$$\mathcal{DR}(\mathbb{R}^N) := \prod_{i=1}^n \mathcal{DQ}(\mathbb{R}^{N_i}).$$

By a *weight* we will always mean a nonnegative, locally integrable function on the product space $\mathbb{R}^N$ which is positive on a set of positive measure. Given a measurable set E and a weight ω , we will use the following notation:

- Denote by $|E|$ the volume of E ;
- The symbol $\omega(E)$ denotes the quantity $\int_E \omega(x) dx$;
- The symbol $\mathbf{1}_E$ stands for the characteristic function of E ;
- The measure of weight $d\omega$ is defined by $d\omega := \omega(x) dx$.

Let $1 \leq p < \infty$ and ω be a weight. We define the weighted Lebesgue space $L^p(\omega)$ to be a Banach space equipped with the norm

$$\|f\|_{L^p(\omega)} := \left(\int_{\mathbb{R}^N} |f|^p d\omega \right)^{\frac{1}{p}}.$$

Given $1 < p < \infty$, $p' = \frac{p}{p-1}$ denotes the conjugate exponent of p .

The condition (D): We denote by P_i , $i = 1, 2, \dots, n$, the projection onto the coordinate subspace $\mathbb{R}^{N_i}$. For a product cube $R \in \mathcal{R}(\mathbb{R}^N)$, an integer $j = 1, 2, \dots, n$ and a cube $Q \in \mathcal{Q}(\mathbb{R}^{N_j})$, we define the product cube

$$\langle R; Q, j \rangle := \left(\prod_{i=1}^{j-1} P_i(R) \right) \times Q \times \left(\prod_{i=j+1}^n P_i(R) \right),$$

which simply replaces $P_j(R)$ by Q . For a cube $Q \in \mathcal{Q}(\mathbb{R}^d)$, let $\mathcal{D}(Q)$ be the collection of all dyadic subcubes of Q , that is, all those cubes obtained by dividing Q into 2^d congruent cubes of half its length, dividing each of those into 2^d congruent cubes, and so on. By convention, Q itself belongs to $\mathcal{D}(Q)$.

For all positive number $\varepsilon > 0$, we assume that the weight σ satisfies *the condition (D)*:

$$(D) \quad \sum_{Q \in \mathcal{D}(P_j(R))} \sigma(\langle R; Q, j \rangle)^{1+\varepsilon} \leq C_{\sigma, \varepsilon} \sigma(R)^{1+\varepsilon}$$

holds for all product cubes $R \in \mathcal{R}(\mathbb{R}^N)$ and all integers $j = 1, 2, \dots, n$, where the constant $C_{\sigma, \varepsilon}$ depends only on the weight σ and the parameter ε .

The following M -linear embedding theorem for product dyadic cubes is our crucial tool. For the case of dyadic cubes, we refer to a series of works [7, 8, 10, 11, 17, 18, 19, 22].

Theorem 2.1 ([20, Theorem 3.1]). *Let $\mathfrak{K} : \mathcal{DR}(\mathbb{R}^N) \rightarrow [0, \infty)$ be a map. For $k = 1, 2, \dots, M$, let σ_k be a weight on $\mathbb{R}^N$ that satisfies the condition (D) and let $1 < p_k < \infty$ with $\sum_{k=1}^M \frac{1}{p_k} > 1$. The following statements are equivalent:*

(a) *The M -linear embedding inequality for product dyadic cubes*

$$\sum_{R \in \mathcal{DR}(\mathbb{R}^N)} \mathfrak{K}(R) \prod_{k=1}^M \left| \int_R f_k \, d\sigma_k \right| \leq c_1 \prod_{k=1}^M \|f_k\|_{L^{p_k}(\sigma_k)}$$

holds for all $f_k \in L^{p_k}(\sigma_k)$, $k = 1, 2, \dots, M$;

(b) *The Fefferman–Phong-type condition*

$$\mathfrak{K}(R) \prod_{k=1}^M \sigma_k(R)^{\frac{1}{p_k}} \leq c_2$$

holds for all product dyadic cubes $R \in \mathcal{DR}(\mathbb{R}^N)$.

Moreover, the least possible constants c_1 and c_2 are equivalent.

By a simple duality argument for the case $M = 2$ we have the following corollary.

Corollary 2.2. *Let $\mathfrak{K} : \mathcal{DR}(\mathbb{R}^N) \rightarrow [0, \infty)$ be a map. Let ω and σ be the weights on $\mathbb{R}^N$ that satisfy the condition (D) and let $1 < p < q < \infty$. The following statements are equivalent:*

(a) *The weighted norm inequality for rectangular dyadic positive operator $T_{\mathfrak{K}}^\sigma$*

$$\|T_{\mathfrak{K}}^\sigma f\|_{L^q(\omega)} \leq c_1 \|f\|_{L^p(\sigma)}$$

holds for all $f \in L^p(\sigma)$. Here,

$$T_{\mathfrak{K}}^\sigma f(x) := \sum_{R \in \mathcal{DR}(\mathbb{R}^N)} \mathfrak{K}(R) \mathbf{1}_R(x) \int_R f \, d\sigma, \quad x \in \mathbb{R}^N.$$

(b) *The Fefferman–Phong-type condition*

$$\mathfrak{K}(R)\omega(R)^{\frac{1}{q}}\sigma(R)^{\frac{1}{p'}} \leq c_2$$

holds for all product dyadic cubes $R \in \mathcal{DR}(\mathbb{R}^N)$.

Moreover, the least possible constants c_1 and c_2 are equivalent.

We have the following corollary too.

Corollary 2.3. *Let $0 < \alpha < N$. Let μ be a weight on $\mathbb{R}^N$ that satisfies the condition (D) and let $1 < p < q < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}$. Then, the Hardy-Littlewood-Sobolev inequality for the rectangular dyadic positive operator T_α^μ*

$$\|T_\alpha^\mu f\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\mu)}$$

holds for all $f \in L^p(\mu)^2$. Here,

$$T_\alpha^\mu f(x) := \sum_{R \in \mathcal{DR}(\mathbb{R}^N)} \mu(R)^{\frac{\alpha}{N}-1} \mathbf{1}_R(x) \int_R f \, d\mu, \quad x \in \mathbb{R}^N.$$

§ 3. Proof of Theorem 1.1

In what follows we will prove Theorem 1.1.

§ 3.1. Rectangular doubling weight

We need some definitions and remarks.

Definition 3.1. For the cube $Q \in \mathcal{Q}(\mathbb{R}^d)$, let

$$\mathcal{D}^{(1)}(Q) := \{Q' \in \mathcal{D}(Q) : 2\ell(Q') = \ell(Q)\},$$

where, by $\ell(Q)$, we denote the side-lengths of cube Q .

- We say that a weight σ on the product space $\mathbb{R}^N$ is a *doubling weight* if there is a constant $\delta > 0$ such that

$$\sigma(R) \leq \delta \sigma(\langle R; Q, j \rangle)$$

holds for all product cubes $R \in \mathcal{R}(\mathbb{R}^N)$, all integers $j = 1, 2, \dots, n$ and all dyadic cubes $Q \in \mathcal{D}^{(1)}(P_j(R))$.

²We merely check that $\frac{\alpha}{N} - 1 + \frac{1}{q} + 1 - \frac{1}{p} = \frac{\alpha}{N} + \frac{1}{q} - \frac{1}{p} = 0$.

- We say that a weight σ on the product space $\mathbb{R}^N$ is a *reverse doubling weight* if there is a constant $\gamma > 1$ such that

$$\gamma \sigma(\langle R; Q, j \rangle) \leq \sigma(R)$$

holds for all product cubes $R \in \mathcal{R}(\mathbb{R}^N)$, all integers $j = 1, 2, \dots, n$ and all dyadic cubes $Q \in \mathcal{D}^{(1)}(P_j(R))$.

Remark 1. If the weight σ is doubling, then it is reverse doubling. Indeed, for any $R \in \mathcal{R}(\mathbb{R}^N)$, any $j = 1, 2, \dots, n$ and any $Q \in \mathcal{D}^{(1)}(P_j(R))$,

$$\sigma(R) = \sum_{Q' \in \mathcal{D}^{(1)}(P_j(R))} \sigma(\langle R; Q', j \rangle) \geq \left(1 + \frac{2^{N_j} - 1}{\delta}\right) \sigma(\langle R; Q, j \rangle) \geq \left(1 + \frac{1}{\delta}\right) \sigma(\langle R; Q, j \rangle),$$

where we have used

$$\sigma(\langle R; Q', j \rangle) \geq \frac{\sigma(R)}{\delta} \geq \frac{\sigma(\langle R; Q, j \rangle)}{\delta}.$$

Conversely, if the weight σ is reverse doubling and $\gamma > 2^{\max_i N_i} - 1$, then it is doubling. Indeed, for any $R \in \mathcal{R}(\mathbb{R}^N)$, any $j = 1, 2, \dots, n$ and any $Q \in \mathcal{D}^{(1)}(P_j(R))$,

$$\begin{aligned} \sigma(R) &= \sum_{Q' \in \mathcal{D}^{(1)}(P_j(R))} \sigma(\langle R; Q', j \rangle) \leq \frac{2^{N_j} - 1}{\gamma} \sigma(R) + \sigma(\langle R; Q, j \rangle) \\ &\leq \frac{2^{\max_i N_i} - 1}{\gamma} \sigma(R) + \sigma(\langle R; Q, j \rangle). \end{aligned}$$

Hence,

$$\sigma(R) \leq \frac{\gamma}{\gamma + 1 - 2^{\max_i N_i}} \sigma(\langle R; Q, j \rangle).$$

Remark 2. If σ is a reverse doubling weight on the product space $\mathbb{R}^N$, then it satisfies condition (D). Indeed, for any product cube $R \in \mathcal{DR}(\mathbb{R}^N)$, any integer

$j = 1, 2, \dots, n$ and any positive number $\varepsilon > 0$,

$$\begin{aligned}
\sum_{Q \in \mathcal{D}(P_j(R))} \sigma(\langle R; Q, j \rangle)^{1+\varepsilon} &= \sum_{k=0}^{\infty} \sum_{\substack{Q \in \mathcal{D}(P_j(R)) \\ \ell(Q)=2^{-k}\ell(P_j(R))}} \sigma(\langle R; Q, j \rangle)^{1+\varepsilon} \\
&= \sum_{k=0}^{\infty} \sum_{\substack{Q \in \mathcal{D}(P_j(R)) \\ \ell(Q)=2^{-k}\ell(P_j(R))}} \sigma(\langle R; Q, j \rangle)^{\varepsilon} \sigma(\langle R; Q, j \rangle) \\
&\leq \sum_{k=0}^{\infty} \left(\frac{1}{\gamma^k} \right)^{\varepsilon} \sigma(R)^{\varepsilon} \sum_{\substack{Q \in \mathcal{D}(P_j(R)) \\ \ell(Q)=2^{-k}\ell(P_j(R))}} \sigma(\langle R; Q, j \rangle) \\
&= \sigma(R)^{1+\varepsilon} \sum_{k=0}^{\infty} \left(\frac{1}{\gamma^k} \right)^{\varepsilon} \\
&= C_{\gamma, \varepsilon} \sigma(R)^{1+\varepsilon}.
\end{aligned}$$

§ 3.2. The Pérez representation

For a number $c > 0$ and a product cube $R \in \mathcal{R}(\mathbb{R}^N)$, we will use cR to denote the product cube with the same center as R but with c times the side-lengths of R .

We define a C. Pérez type representation of fractional integrals (see [12]) by

$$\bar{R}_{\alpha}^{\mu} f(x) := \sum_{R \in \mathcal{DR}(\mathbb{R}^N)} \mu(R)^{\frac{\alpha}{N}-1} \mathbf{1}_R(x) \int_{3R} f d\mu, \quad x \in \mathbb{R}^N.$$

We now verify the point-wise equivalence

$$(3.1) \quad R_{\alpha}^{\mu} f(x) \approx \bar{R}_{\alpha}^{\mu} f(x).$$

We first observe that, for $u, v \in \mathbb{R}^d$ with $u \neq v$, the minimal dyadic cube $Q \in \mathcal{DQ}(\mathbb{R}^d)$ such that $Q \ni u$ and $3Q \ni v$ satisfies

$$\frac{\ell(Q)}{2} < |u - v| < 2\sqrt{d}\ell(Q).$$

We will refer to such a dyadic cube as $Q(u, v)$.

This observation, together with the fact that μ is a doubling weight, enable us to deduce that

$$\mu(R(x, y)) \approx \mu(R_0(x, y)),$$

where, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^N$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^N$, $x_i \neq y_i$, and

$$R_0(x, y) := \prod_{i=1}^n Q(x_i, y_i).$$

The fact that μ is a reverse doubling weight and a calculus of geometric series also enable us to deduce that

$$\sum_{R \in \mathcal{DR}(\mathbb{R}^N)} \mu(R)^{\frac{\alpha}{N}-1} \mathbf{1}_R(x) \mathbf{1}_{3R}(y) \approx \mu(R_0(x, y))^{\frac{\alpha}{N}-1} \approx \mu(R(x, y))^{\frac{\alpha}{N}-1}.$$

This equation and Fubini's theorem yield (3.1).

§ 3.3. The dyadic grid argument

For $\tau \in \{0, \pm \frac{1}{3}\}^d$, the dyadic grid $\mathcal{DQ}^\tau(\mathbb{R}^d)$ is defined by

$$\mathcal{DQ}^\tau(\mathbb{R}^d) := \{2^{-k}(m + \tau + [0, 1)^d) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}.$$

Claim 3.2. *We claim that, for any dyadic cube $Q \in \mathcal{DQ}(\mathbb{R}^d)$, there exist $\tau \in \{0, \pm \frac{1}{3}\}^d$ and τ -shifted dyadic cube $P \in \mathcal{DQ}^\tau(\mathbb{R}^d)$ such that $3Q \subset P$ and $\ell(P) = 8\ell(Q)$.*

Proof. We need only verify the one-dimensional case $d = 1$. (The claim for $d > 1$ holds after d steps.) We may assume further $k = 0$. Let $Q = [m, m + 1)$, $m \in \mathbb{Z}$. Then $3Q = [m - 1, m + 2)$. We cover $3Q$ by disjoint dyadic intervals of $\mathcal{D}(\mathbb{R})$ with the same length 8. If $3Q$ is covered by such an interval P , then we choose $\tau = 0$ and have $P \in \mathcal{D}^\tau(\mathbb{R})$. We assume that $3Q$ is covered by such two intervals as $P_1 \ni (m - 1)$ and $P_2 \ni (m + 2)$. If $|Q \cap P_1| \geq 1.5$, then we choose $\tau = \frac{1}{3}$ and let $P = \frac{8}{3} + P_1$. If $|Q \cap P_2| > 1.5$, then we choose $\tau = -\frac{1}{3}$ and let $P = -\frac{8}{3} + P_2$. This proves the claim. $\square$

§ 3.4. Proof of Theorem 1.1

Using Claim 3.2, we see that for any product dyadic cube $R \in \mathcal{DR}(\mathbb{R}^N)$ there exist $\tau \in \{0, \pm \frac{1}{3}\}^N$ and $R' \in \mathcal{DR}^\tau(\mathbb{R}^N)$, which is defined by

$$\mathcal{DR}^\tau(\mathbb{R}^N) := \prod_{i=1}^n \mathcal{DQ}^{P_i(\tau)}(\mathbb{R}^{N_i}),$$

such that $3R \subset R'$ and $\ell(P_i(R')) = 8\ell(P_i(R))$, $i = 1, 2, \dots, n$. Moreover, we notice that this correspondence $R \mapsto R'$ becomes *almost bijective*.

By letting

$$T_\alpha^{\mu, \tau} f(x) := \sum_{R \in \mathcal{DR}^\tau(\mathbb{R}^N)} \mu(R)^{\frac{\alpha}{N}-1} \mathbf{1}_R(x) \int_R f \, d\mu, \quad x \in \mathbb{R}^N,$$

since μ is doubling,

$$\begin{aligned} \overline{R}_\alpha^\mu f(x) &= \sum_{R \in \mathcal{DR}(\mathbb{R}^N)} \mu(R)^{\frac{\alpha}{N}-1} \mathbf{1}_R(x) \int_{3R} f \, d\mu \\ &\lesssim \sum_{\tau \in \{0, \pm \frac{1}{3}\}^N} T_\alpha^{\mu, \tau} f(x). \end{aligned}$$

Applying Corollary 2.3 to the right-hand side of this inequality, we obtain Theorem 1.1.

Appendix

As an appendix, we state the Carleson-type embedding theorem for product dyadic cubes. The importance of this theorem is that the Fefferman–Phong-type condition simply links the Carleson-type embedding theorem to the M -linear embedding theorem.

Lemma 3.3 ([20, Lemma 2.2]). *Given a weight σ in the product space $\mathbb{R}^N$ and $1 < p < q < \infty$, the following statements are equivalent:*

(a) *The Carleson-type embedding inequality for product dyadic cubes*

$$\sum_{R \in \mathcal{DR}(\mathbb{R}^N)} \sigma(R)^{\frac{q}{p}} \left(\frac{1}{\sigma(R)} \int_R f \, d\sigma \right)^q \leq c_1 \left(\int_{\mathbb{R}^N} f^p \, d\sigma \right)^{\frac{q}{p}}$$

holds for all nonnegative functions $f \in L^p(\sigma)$;

(b) *The testing condition*

$$\sum_{Q \in \mathcal{D}(P_j(R))} \sigma(\langle R; Q, j \rangle)^{\frac{q}{p}} \leq c_2 \sigma(R)^{\frac{q}{p}}$$

holds for all product dyadic cubes $R \in \mathcal{DR}(\mathbb{R}^N)$ and all integers $j = 1, 2, \dots, n$.

Moreover, the least possible constants c_1 and c_2 enjoy $c_1 \leq Cc_2^n$ and $c_2 \leq c_1$.

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