

Reply: Discontinuous codimension-two bifurcation in a Vlasov equation

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The Comment [1] criticizes the bifurcation analysis performed in [2] on a Vlasov equation. This criticism can be traced back to a discrepancy in the definition of the "paramagnetic phase". Apart from this discrepancy, there is no conflict between [1] and [2].

(1) The definition of the "paramagnetic phase" in [1] is $\langle M_x \rangle = 0$ (Def-1) under the assumption of $M_y = 0$ by symmetry; $\langle M_x \rangle$ denotes the time average of $M_x(t) = \iint \cos(q) F(q, p, t) dq dp$, with $F(q, p, t)$ the time dependent position-velocity distribution function, governed by the Vlasov equation¹. However, the definition in [2] is that the distribution function F is spatially homogeneous (Def-2), namely F does not depend on q . A spatially homogeneous distribution (Def-2) implies $\langle M_x \rangle = 0$ (Def-1), but the converse is not true. Hence Def-2 provides a finer classification of the states. Indeed, if two clusters move in opposite directions [as in Fig.1(h)], $M_x(t)$ oscillates around zero [see Fig.1(b)] and Def-1 is satisfied while Def-2 is not. This is what happens in Fig.7(c) of [2] in the range $K^c < K < K^J$ (K^c : critical point, K^J : jump point), where the order parameter M is defined by $M = ||(M_x, M_y)||$. Increasing K , the two clusters merge when K reaches K^J and the system is in a nonhomogeneous stationary state when $K > K^J$. This scenario is supported by Fig.1(c,f) and by Fig.11 of [2].

(2) Based on Def-2, we find the two-cluster state in the range $K^c < K < K^J$. This third state is missed by adopting Def-1, while the existence of this intermediate oscillatory state is completely consistent with Figs. 2 and 4 of [1]. The authors of [1] insist that we interpreted the two-cluster oscillatory state as indicative of a continuous transition to the ferromagnetic phase. This statement is not true. What we have stated in [2] is a continuous bifurcation between the homogeneous stationary state and the two-cluster oscillatory state, where the bifurcation has been detected using M instead of M_x .

(3) Dynamics in the two-cluster state is illustrated on Fig. 1(b), and is typical of an oscillatory bifurcation: $M_x(t)$ oscillates at a frequency given by the imaginary part of the unstable eigenvalue, and with a slower varying amplitude. Contrary to the claim in [1], the scaling law for this amplitude, which represents the size of the clusters [see Fig.1(h)], is a crucial ingredient to understand this state. Furthermore, we note that this two-cluster state persists for much longer computation times

than shown on Fig.1. These findings on the two-cluster state are actually not new: two cluster solutions are constructed in [3, 4], and the amplitude scaling is analyzed in [5, 6].

All the above points are illustrated by Fig.1, which expands Figs. 7 and 8 of [2]. The initial homogeneous distribution with perturbation is

$$F_\epsilon(q, p) = C e^{-\beta_2 p^2 / 2 - (\beta_4 p^2 / 2)^2} [1 + \epsilon \cos(q)], \quad (1)$$

where C is the normalization factor to satisfy $\iint F_\epsilon(q, p) dq dp = 1$. We used the same parameters as in Fig.7(c) of [2]: $\beta_2 = -0.3$, $\beta_4 = 3$, and $\epsilon = 10^{-6}$. The truncated phase space $(q, p) \in (-\pi, \pi] \times [-4, 4]$ is divided into an $L \times L$ mesh to perform a semi-Lagrangian algorithm, which is a Vlasov solver [7] and is used in [2], where $L = 128$ and the time step is $\Delta t = 0.05$. At $K = 0.96$, the two clusters should be located around $|p| = 0.174$, which is the imaginary part of the most unstable eigenvalues. This prediction is confirmed by the marginal distribution $\int F(q, p, t) dq$ at $t = 5000$ as shown in Fig.2, while no bumps appear at $K = 0.94$.

Summarizing, in the investigated Vlasov dynamics, there are three types of states: a homogeneous stationary state, a nonhomogeneous stationary state, and a two-cluster oscillatory state. The last one can be captured by Def-2, adopted in [2], but not by Def-1, adopted in [1]. The linear stability analysis identifies the continuous bifurcation point K^c between the homogeneous and two-cluster states. Moreover, the nonlinear analysis in [2] approximately identifies the discontinuous bifurcation point K^J between the two-cluster and the nonhomogeneous states.

We conclude that all Molecular Dynamics simulations in [1] are consistent with [2], when the third state (the two-cluster state) is considered. The unique difference is the observation of multistability around the jump point in the former due to finite-size fluctuations, which are absent in the Vlasov simulations of [2]. There is no flaw in [2].

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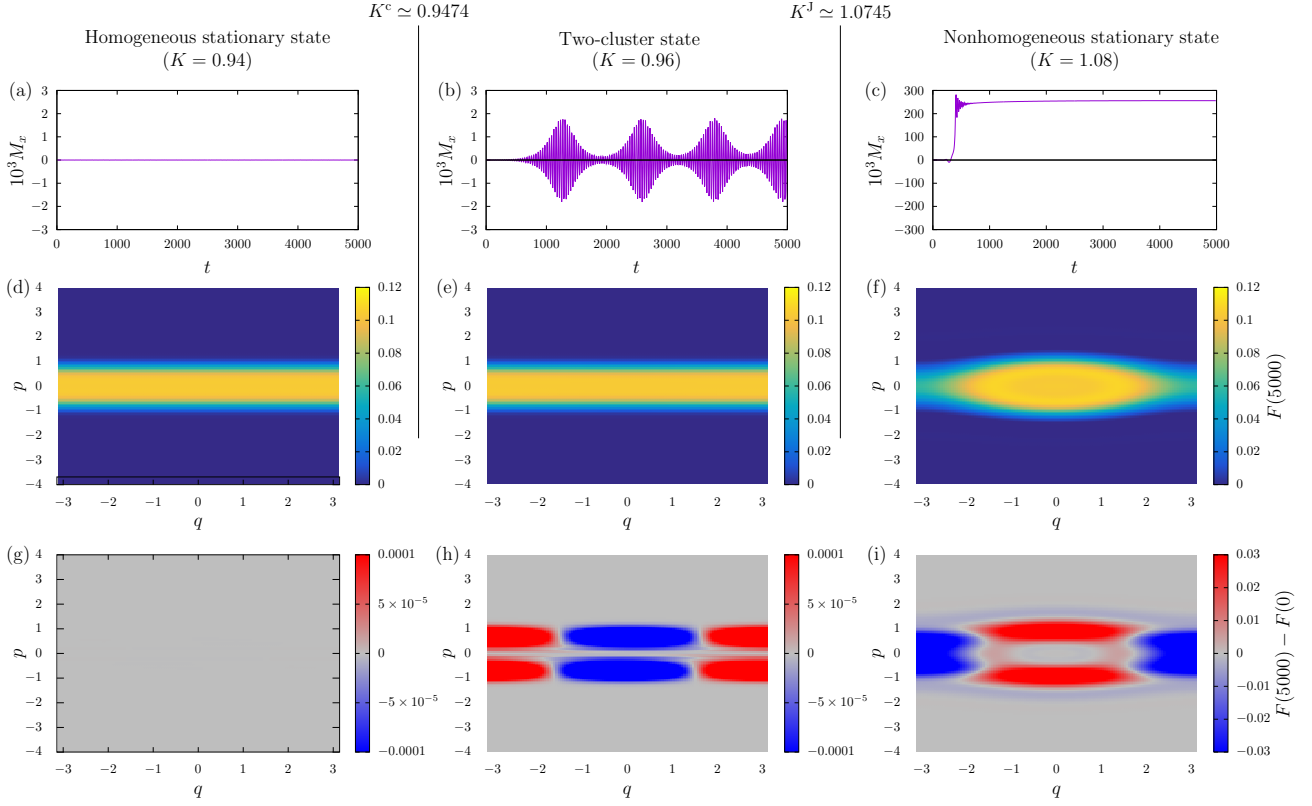


FIG. 1. Two bifurcations in the generalized HMF model. The coupling constant K_1 (simply denoted by K) is the bifurcation parameter, and $K_2 = 0.5$. The two bifurcations are located at $K = K^c$ and $K = K^J$ and separate the homogeneous stationary state, the two-cluster state, and the nonhomogeneous stationary state. $\beta_2 = -0.3, \beta_4 = 3$ and $\epsilon = 10^{-6}$ for the initial perturbed distribution (1). (a,b,c): Temporal evolution of $10^3 M_x(t)$. (d,e,f): Heat maps of $F(q, p, 5000)$. (g,h,i): Heat maps of $F(q, p, 5000) - F(q, p, 0)$. The bifurcation parameter is $K = 0.94$ (a,d,g), $K = 0.96$ (b,e,h), and $K = 1.08$ (c,f,i). Note that two clusters should be located around $|p| \simeq 0.174$ [see Fig.2], which is the imaginary part of the most unstable eigenvalues and which corresponds to the shorter period in the panel (b). Mesh size is 128×128 on a truncated phase space (q, p) which is $(-\pi, \pi] \times [-4, 4]$.

[1] T. N. Teles, R. Pakter, and Y. Levin, Comment on “Discontinuous codimension-two bifurcation in a Vlasov equation”.

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¹ For consistency with [2] we use (q, p) as phase-space coordinates, and denote the magnetization by (M_x, M_y) and M .

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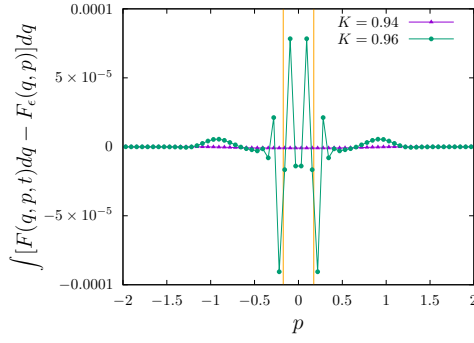


FIG. 2. Modification of marginal distribution at time $t = 5000$ for $K = 0.94$ (purple triangles) and $K = 0.96$ (green circles). The two orange vertical lines mark $|p| = 0.174$ corresponding to the imaginary part of the most unstable eigenvalues at $K = 0.96$. The parameters are the same as Fig. 1.

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