

The Boltzmann Equation of Gas Mixture of Hard Balls

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The Boltzmann equation of gas of hard balls (spacially homogeneous case) is

$$(0) \begin{cases} \frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \delta^2 N \int_{S^2 \times R^3} |(y-x, l)| \{ u(t, x^*) u(t, y^*) \\ - u(t, x) u(t, y) \} dl dy, \\ u(0, x) = f(x), \end{cases}$$

where $u(t, x)$ is the density of distribution of molecules with speed $x \in R^3$ at time t , dl is the uniform distribution on S^2 , and $(x, y) \rightarrow (x^*, y^*)$ gives the change of speed of molecules by elastic collision :

$$(2) \begin{cases} x^* = x + (y-x, l)l \\ y^* = y - (y-x, l)l \end{cases} \quad l \in S^2$$

and δ, N is the radius and number of molecules, respectively.

When we set $u(t, \Gamma) = \int_{\Gamma} u(t, x) dx$, we can easily extend (0) to the equation of measures,

$$(1) \quad \begin{cases} \frac{\partial u(t, \Gamma)}{\partial t} = \frac{1}{2} \delta^2 N \int_{S^2 \times R^3 \times R^3} |(y-x, l)| \{ \delta(x^*, \Gamma) - \delta(x, \Gamma) \} \underbrace{dl u(t, dx) u(t, dy)} \\ u(0, \Gamma) = f(\Gamma) \end{cases}$$

We assume that $f(\Gamma)$, initial distribution, is the probability measure on R^3 .

We say that $u(t, \cdot)$ preserves mass, momentum or energy if the following condition,

$$\begin{aligned} \int_{R^3} u(t, dx) &= \int_{R^3} f(dx), \\ \int_{R^3} x u(t, dx) &= \int_{R^3} x f(dx), \\ \int_{R^3} |x|^2 u(t, dx) &= \int_{R^3} |x|^2 f(dx), \end{aligned}$$

is satisfied, respectively.

Povzner's results are the following three theorems. ([1], see also [2])

Theorem 1. If $\int_{R^3} |x|^2 f(dx)$ is finite, then

there exists a solution of (1) which preserves mass and momentum.

Theorem 2. If $\int_{\mathbb{R}^3} |x|^\alpha f(dx)$ is finite for some $\alpha \geq 3$, then there exists a solution of (1) which makes $\mu^{(\alpha)}(t)$ locally finite, where $\mu^{(\alpha)}(t) = \int_{\mathbb{R}^3} |x|^\alpha u(t, dx)$, and this solution preserves mass, momentum and energy.

Theorem 3. If $\int_{\mathbb{R}^3} |x|^4 f(dx)$ is finite, then the solution of (1) which makes $\mu^{(q)}(t)$ locally finite is unique, and moreover if f has density, so has $u(t, \cdot)$.

We extend these results to the case of gas mixture. In the case of gas mixture of two species of molecules, (1), (2) and (3) are extended to

$$(4) \left\{ \begin{array}{l} \frac{\partial u_i(t, r)}{\partial t} = \sum_{j=1}^2 a_{ij} N_j \int_{S^2 \times R^3 \times R^3} |(\gamma - x, e)| \{ \delta(x''_j, r) - \delta(x, r) \}, \\ \quad \quad \quad dl \, u_i(t, dx) u_j(t, dy), \\ u_i(0, r) = f_i(r) \end{array} \right.$$

where $a_{11} = \frac{1}{2} \delta_1^2$, $a_{22} = \frac{1}{2} \delta_2^2$ and $a_{12} = a_{21} = \frac{1}{2} \left(\frac{\delta_1 + \delta_2}{2} \right)^2$,

$$(5) \begin{cases} x^{ij} = x + \frac{2m_j}{m_i + m_j} (y - x, l) l \\ y^{ij} = y - \frac{2m_i}{m_i + m_j} (y - x, l) l, \end{cases} \quad (l \in S^2)$$

$$(6) \begin{cases} \int_{R^3} u_i(t, dx) = \int_{R^3} f_i(dx) \text{ and } \int_{R^3} u_2(t, dx) = \int_{R^3} f_2(dx), \\ \sum_{i=1}^2 N_i m_i \int_{R^3} x u_i(t, dx) = \sum_{i=1}^2 N_i m_i \int_{R^3} x f_i(dx), \\ \sum_{i=1}^2 N_i m_i \int_{R^3} |x|^2 u_i(t, dx) = \sum_{i=1}^2 N_i m_i \int_{R^3} |x|^2 f_i(dx), \end{cases}$$

where m_1, m_2 is the mass of two species of molecules, respectively. ([3])

And we can extend Th.1 ~ Th.3 in the following,

Theorem 4. If $\int_{R^3} |x|^2 f_i(dx)$ is finite ($i=1,2$), then there exists a solution of (4) which preserves mass and momentum.

Theorem 5. If $\int_{R^3} |x|^\alpha f_i(dx)$ is finite ($i=1,2$) for some $\alpha \geq 3$, then there exists a solution of (4) which makes $\mu_i^{(\alpha)}(t)$ and $\mu_2^{(\alpha)}(t)$ locally finite, where $\mu_i^{(\alpha)}(t) = \int_{R^3} |x|^\alpha u_i(t, dx)$ ($i=1,2$), and this

solution preserves mass, momentum and energy.

Theorem 6. If $\int_{\mathbb{R}^3} |x|^4 f_i(dx)$ is finite ($i=1,2$), then the solution of (4) which makes $\mu_1^{(4)}(t)$ and $\mu_2^{(4)}(t)$ locally finite is unique, and moreover if f_1 and f_2 have density, so have $u_1(t, \cdot)$ and $u_2(t, \cdot)$.

The proof of these theorems is essentially the same as that of H. Tanaka [2], if we transform (4), (5) and (6) in the following manner.

Let R_1 and R_2 be two copies of R^3 and let Q be the disjoint union of R_1 and R_2 . And let us consider the following measures on Q ,

$$\begin{cases} u(t, P) = \frac{1}{N_1 + N_2} \{ N_1 u_1(t, P \cap R_1) + N_2 u_2(t, P \cap R_2) \} \\ f(P) = \frac{1}{N_1 + N_2} \{ N_1 f_1(P \cap R_1) + N_2 f_2(P \cap R_2) \} \end{cases}$$

then the equation (4), (5) and (6) can easily be transformed to

$$(7) \begin{cases} \frac{\partial u(t, r)}{\partial t} = \int_{S^2 \times Q \times Q} g(x, y, l) \{ \delta(x^*, r) - \delta(x, r) \} dl, \\ u(0, r) = f(r) \end{cases} \quad \underbrace{u(t, dx) u(t, dy),}$$

where $g(x, y, l) = a_{ij} |(y-x, l)|$, if $x \in R_i$ and $y \in R_j$,

$$(8) \begin{cases} x^* = x^{ij} \text{ (as element of } R_i), \\ y^* = y^{ij} \text{ (as element of } R_j), \end{cases} \text{ if } x \in R_i \text{ and } y \in R_j,$$

$$(9) \begin{cases} \int_{R_i} U(t, dx) = \int_{R_i} f(dx), \quad (i=1, 2) \\ \int_Q m_x x U(t, dx) = \int_Q m_x x f(dx), \\ \int_Q m_x |x|^2 U(t, dx) = \int_Q m_x |x|^2 f(dx), \end{cases}$$

where $m_x = m_i$ if $x \in R_i$.

It is easily seen that, we can get the same results for gas mixture of more than two species of molecules in the same manner.

References

- [1] A. Ya. Povzner: On Boltzmann's Equation in the Kinetic Theory of Gases. Mat. Sb., 58 (1962), 63-86.

[2] H. Tanaka : The Boltzmann Equation of Gas of Hard Balls. R.I.M.S. Kôkyûroku 174 (1973), 1-20. (in Japanese)

[3] L. Boltzmann : Vorlesungen über Gas-theorie. Leipzig (1896).