

On Galois representations of local fields with imperfect residue fields

By

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Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$. Let V be a p -adic representation of the absolute Galois group $G_K = \text{Gal}(\overline{K}/K)$ where we fix an algebraic closure $\overline{K}$ of K . When the residue field k is perfect (i.e. $e = 0$), Berger has proved a conjecture of Fontaine (Conjecture 1.1. below) which claims that, if V is a de Rham representation of G_K , V becomes a potentially semi-stable representation of G_K . (See Theorem 1.2.) Here, we generalize this result to the case when the residue field k is not necessarily perfect. For this, we prove some results on p -adic representations in the imperfect residue field case (see Theorem 1.3.) which are obtained by using the recent theory of p -adic differential modules and deduce this generalization of the result of Berger as a corollary. (See Theorem 1.4.)

In this survey article, we first state the results in Section 1. In Section 2, we review the property of the p -adic periods ring B_{dR} . Then, in Section 3 and Section 4, we give a sketch of the proof of Theorem 1.3.

§ 1. Results

Let K , k , G_K and V be as above. Fontaine, Hyodo, Kato and Tsuzuki define the p -adic periods rings (associated to K) which are equipped with the continuous action of G_K . (See [F1], [Ka1], [Ka2], [Tz3], [Br2] etc.)

$$(\mathbb{Q}_p \subset) B_{\text{cris}} \subset B_{\text{st}} \subset B_{\text{dR}}.$$

With these rings, we classify the p -adic representation V of G_K as follows. We call the p -adic representation V of G_K

1. a de Rham representation of G_K if and only if we have the equality

$$\dim_{\mathbb{Q}_p} V = \dim_{(B_{\text{dR}})^{G_K}} (B_{\text{dR}} \otimes V)^{G_K} :$$

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2. a semi-stable representation of G_K if and only if we have the equality

$$\dim_{\mathbb{Q}_p} V = \dim_{(B_{\text{st}})^{G_K}} (B_{\text{st}} \otimes V)^{G_K} :$$

3. a crystalline representation of G_K if and only if we have the equality

$$\dim_{\mathbb{Q}_p} V = \dim_{(B_{\text{cris}})^{G_K}} (B_{\text{cris}} \otimes V)^{G_K}.$$

(In general, we have the inequality $\dim_{\mathbb{Q}_p} V \geq \dim_{(B_*)^{G_K}} (B_* \otimes V)^{G_K}$ for $*$ $\in$ {dR, st, cris}.) It is well-known that we have the following implications (see [F1] etc.)

$$\text{cray. rep. of } G_K \implies \text{st. rep. of } G_K \implies \text{dR. rep. of } G_K.$$

Furthermore, we call the p -adic representation V of G_K a potentially de Rham (resp. semi-stable, crystalline) representation of G_K if V is a de Rham (resp. semi-stable, crystalline) representation of G_L where L/K is a finite extension. Then, it is well-known that a potentially de Rham representation of G_K is a de Rham representation of G_K . (See Section 2.) Thus, it is not difficult to see that a potentially semi-stable representation of G_K is a de Rham representation of G_K . Fontaine conjectured the converse.

Conjecture 1.1. *If the p -adic representation V is a de Rham representation of G_K , then V is a potentially semi-stable representation of G_K .*

Then, Berger has proved the following thing.

Theorem 1.2. *The conjecture of Fontaine is true if the residue field k is perfect.*

The aim of this note is to give a sketch of the proof of the generalization of this theorem to the imperfect residue field case. (Theorem 1.5.) For this, we state some results on p -adic representations in the imperfect residue field case. (Theorem 1.3.)

Let us fix some notations. Fix a lifting $(b_i)_{1 \leq i \leq e}$ of a p -basis of k in $\mathcal{O}_K$ (the ring of integers of K), and fix a p^m -th root b_i^{1/p^m} of b_i in $\overline{K}$ for each $m \geq 1$ satisfying $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$. Put

$$K^{(l)} = \cup_{m \geq 1} K(b_i^{1/p^m}, 1 \leq i \leq e) \quad \text{and} \quad K' = \text{the } p\text{-adic completion of } K^{(l)}.$$

which depend on the choice of $\{b_i^{1/p^m}\}$. Then, K' is a complete discrete valuation field with perfect residue field, which is a canonical “**perfectization**” of K . Furthermore, we can regard the Galois group $G_{K'} = \text{Gal}(\overline{K}/K')$ as a subgroup of G_K (see Section 2 for details) and think V as a p -adic representation of $G_{K'}$. Then, we obtain the following theorem ([Mol] and [Mo2]).

Theorem 1.3. *Let K be a complete discrete valuation field of characteristic 0 with residue field of characteristic $p > 0$ such that $[k : k^p] < \infty$ and K' be as above. Let V denote a p -adic representation of G_K . Then, we have the following equivalences.*

1. *V is a de Rham representation of G_K if and only if V is a de Rham representation of $G_{K'}$.*
2. *V is a potentially semi-stable representation of G_K if and only if V is a potentially semi-stable representation of $G_{K'}$.*
3. *V is a potentially crystalline representation of G_K if and only if V is a potentially crystalline representation of $G_{K'}$.*

Remark 1.4. Though we don't introduce the definition of Hodge-Tate representations in this note, we also show that V is a Hodge-Tate representation of G_K if and only if V is a Hodge-Tate representation of $G_{K'}$. (For the definition of Hodge-Tate representations, see [F1] etc.)

With Theorem 1.2. and Theorem 1.3., we have the following equivalences:

$$\begin{array}{ccc} V : \text{dR. rep. of } G_K & \Longleftrightarrow & V : \text{dR. rep. of } G_{K'} \\ \Updownarrow & & \Updownarrow \\ V : \text{pst. rep. of } G_K & \Longleftrightarrow & V : \text{pst. rep. of } G_{K'} \end{array}$$

Thus, we obtain the generalization of Theorem 1.2. to the imperfect residue field case.

Theorem 1.5. *The conjecture of Fontaine is true even if the residue field k is not necessarily perfect.*

For simplicity, in this note, we shall consider only the de Rham representation case of Theorem 1.3..

§ 2. Preliminaries on the p -adic periods ring B_{dR}

§ 2.1. Definitions and properties of the ring B_{dR}

2.1.1. The case $e = 0$ (i.e. k is perfect)

Let K be as in Introduction and assume that the residue field k is perfect. Choose an algebraic closure $\overline{K}$ of K and put $\mathbb{C}_p =$ the p -adic completion of $\overline{K}$. Put

$$\widetilde{E} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, \dots) \mid x^{(i)} \in \mathbb{C}_p, (x^{(i+1)})^p = x^{(i)}\}.$$

Define a valuation v_E on $\tilde{E}$ by $v_E(x) = v_p(x^{(0)})$ where v_p denotes the normalized valuation of $\mathbb{C}_p$ by $v_p(p) = 1$. Let $\epsilon = (\epsilon^{(n)})$ be an element of $\tilde{E}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. The field $\tilde{E}$ is the completion of an algebraic closure of $k((\epsilon - 1))$ for this valuation. Define $\tilde{E}^+$ to be the ring of integers for this valuation. Put $\tilde{A}^+ = W(\tilde{E}^+)$ and

$$\tilde{B}^+ = \tilde{A}^+[1/p] = \left\{ \sum_{k \gg -\infty} p^k [x_k] \mid x_k \in \tilde{E}^+ \right\}$$

where $[*]$ denotes the Teichmüller lift of $*$ in $\tilde{E}^+$. This ring is equipped with a surjective homomorphism

$$\theta : \tilde{B}^+ \rightarrow \mathbb{C}_p : \sum p^k [x_k] \mapsto \sum p^k x_k^{(0)}.$$

The ring B_{dR}^+ is defined to be the completion by the $\text{Ker}(\theta)$ -adic topology of $\tilde{B}^+$:

$$B_{\text{dR}}^+ = \varprojlim_{n \geq 0} \tilde{B}^+ / (\text{Ker}(\theta)^n).$$

This is a discrete valuation ring and $t = \log([\epsilon])$ (which converges in B_{dR}^+) is a generator of the maximal ideal. Put $B_{\text{dR}} = B_{\text{dR}}^+[1/t]$. This is a field and is equipped with an action of the Galois group G_K and a filtration defined by $\text{Fil}^i B_{\text{dR}} = t^i B_{\text{dR}}^+$ ($i \in \mathbb{Z}$). The ring $(B_{\text{dR}})^{G_K}$ is canonically isomorphic to K . If V is a p -adic representation of G_K , then $D_{\text{dR}}(V) = (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K -vector space. We say that a p -adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\text{dR}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\text{dR}}(V)).$$

Furthermore, a potentially de Rham representation V of G_K is a de Rham representation of G_K . (See [F1].)

2.1.2. The case e is general (i.e. k is not necessarily perfect)

Let K be as in Introduction and assume that the residue field k is not necessarily perfect. If we construct B_{dR}^+ , B_{dR} as in the perfect residue case (we denote $B_{\text{dR}}^{+, \text{naiv}}$, $B_{\text{dR}}^{\text{naiv}}$):

1. $\mathbb{C}_p$ = the p -adic completion of $\overline{K}$
2. $\tilde{E} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p$ and $\tilde{E}^+$
3. $\tilde{A}^+ = W(\tilde{E}^+)$, $\tilde{B}^+ = \tilde{A}^+[1/p]$ and $\theta : \tilde{B}^+ \rightarrow \mathbb{C}_p$
4. $B_{\text{dR}}^{+, \text{naiv}} = \varprojlim_{n \geq 0} \tilde{B}^+ / (\text{Ker}(\theta))^n$ and $B_{\text{dR}}^{\text{naiv}} = B_{\text{dR}}^{+, \text{naiv}}[1/t]$

then contrary to the perfect residue field case, we have $(B_{\text{dR}}^{\text{naiv}})^{G_K} \neq K$ in general. Now, we shall recall the imperfect residue field version of B_{dR} .

First, construct the ring $\tilde{A}^+$ for K as above. Let $\alpha : \mathcal{O}_K \otimes_{\mathbb{Z}} \tilde{A}^+ \rightarrow \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ be the natural surjection and define $\tilde{A}_{(K)}^+$ to be

$$\tilde{A}_{(K)}^+ = \varprojlim_{n \geq 0} (\mathcal{O}_K \otimes_{\mathbb{Z}} \tilde{A}^+) / (\text{Ker}(\alpha))^n.$$

Let $\theta_K : \tilde{A}_{(K)}^+ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \rightarrow \mathbb{C}_p$ be the natural extension of $\theta : \tilde{A}^+[1/p] \rightarrow \mathbb{C}_p$. Then, the imperfect residue field version of B_{dR}^+ is defined to be the $\text{Ker}(\theta_K)$ -adic completion of $\tilde{A}_{(K)}^+ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$:

$$B_{\text{dR}}^+ = \varprojlim_{n \geq 0} (\tilde{A}_{(K)}^+ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) / (\text{Ker}(\theta_K))^n.$$

Fix a lifting $(b_i)_{1 \leq i \leq e}$ of a p -basis of k in $\mathcal{O}_K$ as in Introduction. Let $\tilde{b}_i = (b_i^{(n)}) \in \tilde{E}^+$ such that $b_i^{(0)} = b_i$, and then the series which defines $\log([\tilde{b}_i]/b_i)$ converges in B_{dR}^+ to an element t_i . This ring B_{dR}^+ is endowed with an action of the Galois group G_K and a filtration defined by $\text{Fil}^i B_{\text{dR}}^+ = m_{\text{dR}}^i$ where the maximal ideal m_{dR} of B_{dR}^+ is generated by $\{t, t_1, \dots, t_e\}$. Put $B_{\text{dR}} = B_{\text{dR}}^+[1/t]$. Then, K is canonically embedded in B_{dR} and $(B_{\text{dR}})^{G_K} = K$. If V is a p -adic representation of G_K , then $D_{\text{dR}}(V) = (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K -vector space. We say that a p -adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\text{dR}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\text{dR}}(V)).$$

Furthermore, we can show that a potentially de Rham representation V of G_K is a de Rham representation V of G_K in the same way as in the perfect residue field case.

§ 2.2. Comparison of the case 2.1.1 and 2.1.2

Fix the notations as in Section 2.1.2 and let $K^{(\prime)}$ and K' be as in Introduction. First, by the construction, we see that there exists a G_K -equivariant injection

$$(2.1) \quad f : B_{\text{dR}}^{+, \text{naiv}} \hookrightarrow B_{\text{dR}}^+.$$

On the other hand, since K' is a complete discrete valuation field with perfect residue field, we can construct the ring $B_{\text{dR}}^{+, \prime}$ for K' as in Section 2.1.1. We will see that there exists a morphism from the ring B_{dR}^+ to the ring $B_{\text{dR}}^{+, \prime}$. Since $K^{(\prime)}$ is a Henselian discrete valuation field, we have an isomorphism $G_{K'} \simeq G_{K^{(\prime)}} (\subset G_K)$. With this isomorphism, we identify $G_{K'}$ as a subgroup of G_K . Then, there exists a $G_{K'}$ -equivariant surjection

$$(2.2) \quad g : B_{\text{dR}}^+ \twoheadrightarrow B_{\text{dR}}^{+, \prime}.$$

Now, we will show that there exists a morphism between the ring $B_{\text{dR}}^{+, \text{naiv}}$ with the ring $B_{\text{dR}}^{+, \prime}$. We have a bijective map from the set of finite extensions of $K^{(\prime)}$ contained in $\bar{K}$

to the set of finite extensions of K' contained in $\overline{K'}$ defined by $L \mapsto LK'$. Furthermore, LK' is the p -adic completion of L . Hence, we have an isomorphism of rings

$$\mathcal{O}_{\overline{K}}/p^n \mathcal{O}_{\overline{K}} \simeq \mathcal{O}_{\overline{K'}}/p^n \mathcal{O}_{\overline{K'}}$$

where $\mathcal{O}_{\overline{K}}$ and $\mathcal{O}_{\overline{K'}}$ denote rings of integers of $\overline{K}$ and $\overline{K'}$. Thus, the fields $\mathbb{C}_p(K)$ (= the p -adic completion of $\overline{K}$) and $\mathbb{C}_p(K')$ (= the p -adic completion of $\overline{K'}$) are isomorphic (we will simply write $\mathbb{C}_p$). In the end, we have an isomorphism of rings

$$B_{\text{dR}}^{+, \text{naiv}} \simeq B_{\text{dR}}^{+, '}$$

which coincides with the composition ((2.1) and (2.2))

$$g \circ f : B_{\text{dR}}^{+, \text{naiv}} \hookrightarrow B_{\text{dR}}^+ \rightarrow B_{\text{dR}}^{+, '}$$

From now on, we identify the ring $B_{\text{dR}}^{+, \text{naiv}}$ with the ring $B_{\text{dR}}^{+, '}$. Then, it is well-known that the homomorphism

$$(2.3) \quad f : B_{\text{dR}}^{+, '}[t_1, \dots, t_e] \rightarrow B_{\text{dR}}^+$$

is an isomorphism of filtered algebras. (See [Br2] and [Ka1].) From this isomorphism, it follows easily that

$$i : B_{\text{dR}}^{+, '} \hookrightarrow B_{\text{dR}}^+ \quad \text{and} \quad pr : B_{\text{dR}}^+ \twoheadrightarrow B_{\text{dR}}^{+, '} : t_i \mapsto 0$$

are $G_{K'}$ -equivariant homomorphisms and the composition

$$pr \circ i : B_{\text{dR}}^{+, '} \hookrightarrow B_{\text{dR}}^+ \twoheadrightarrow B_{\text{dR}}^{+, '}$$

is identity.

§ 3. Preliminaries on p -adic differential modules

In this section, we will introduce the recent theory of p -adic differential modules which plays an important role in this note. First, let us fix the notations. Put K , $K^{(j)}$ and K' as in Introduction. Put $K_{\infty}^{(j)} = \cup_{m \geq 0} K^{(j)}(\zeta_{p^m})$ and $K'_{\infty} = \cup_{m \geq 0} K'(\zeta_{p^m})$ where ζ_{p^m} denotes a primitive p^m -th root of unity in $\overline{K}$ such that $\zeta_{p^{m+1}}^p = \zeta_{p^m}$. Let $\hat{K}'_{\infty}$ denote the p -adic completion of K'_{∞} . These fields $K_{\infty}^{(j)}$ and $\hat{K}'_{\infty}$ are independent of the choice of $\{b_i^{1/p^m}\}$ (K'_{∞} isn't). Then, we have

$$\hat{K}'_{\infty} \supset K'_{\infty} \supset K_{\infty}^{(j)}.$$

Let H_K denote the kernel of the cyclotomic character $\chi : G_{K'} \rightarrow \mathbb{Z}_p^*$. Note that, since we have $H_K \simeq G_{K_{\infty}^{(j)}}$, the subgroup H_K of G_K is independent of the choice of K' . Define

$\Gamma_K = G_K/H_K$. Let $\Gamma_0 = \text{Gal}(K_\infty^{(l)}/K^{(l)})$ be the subgroup of Γ_K . Let Γ_i ($i \neq 0$) be the subgroup of Γ_K such that actions of $\beta_i \in \Gamma_i$ ($i \neq 0$) are given by

$$\beta_i(\epsilon^{(n)}) = \epsilon^{(n)} \quad \text{and} \quad \beta_i(b_j^{(n)}) = b_j^{(n)} \quad (i \neq j).$$

Define the homomorphism $c_i : \Gamma_i \rightarrow \mathbb{Z}_p$ such that we have

$$\beta_i(b_i^{(n)}) = b_i^{(n)}(\epsilon^{(n)})^{c_i(\beta_i)}.$$

§ 3.1. Definitions of p -adic differential modules

We will give the definitions of p -adic differential modules $D_{\text{Sen}}(V)$, $D_{\text{Bri}}(V)$, $D_{\text{dif}}^+(V)$ and $D_{e\text{-dif}}^+(V)$ which are obtained by Sen, Brinon, Fontaine and Andreatta-Brinon. We will have the following diagram:

$$\begin{array}{ccccc} (B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} V)^{H_K} & \supset & D_{\text{dif}}^+(V) & \supset & D_{e\text{-dif}}^+(V) \\ \downarrow & & \downarrow & & \downarrow \\ (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K} & \supset & D_{\text{Sen}}(V) & \supset & D_{\text{Bri}}(V). \end{array}$$

The following results in Section 3.1.1 and 3.1.3 are obtained when V is a p -adic representation of $G_L = \text{Gal}(\bar{L}/L)$ where L is a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$. However, in Section 3.1.1 and 3.1.3, for simplicity, we will state the results when V is a p -adic representation of $G_{K'}$.

3.1.1. The module $D_{\text{Sen}}(V)$

In the article [S3], Sen shows that the $\hat{K}'_\infty$ -vector space $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ has dimension d ($= \dim_{\mathbb{Q}_p} V$) and the union of the finite dimensional K'_∞ -subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under Γ_0 ($\simeq G_{K'}/H_K$) is a K'_∞ -vector space of dimension d stable under Γ_0 (called $D_{\text{Sen}}(V)$). We have $\mathbb{C}_p \otimes_{K'_\infty} D_{\text{Sen}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}'_\infty \otimes_{K'_\infty} D_{\text{Sen}}(V) \rightarrow (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to 1, then the series of operators on $D_{\text{Sen}}(V)$:

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

converges to an operator $\nabla^{(0)} : D_{\text{Sen}}(V) \rightarrow D_{\text{Sen}}(V)$ and does not depend on the choice of γ .

3.1.2. The module $D_{\text{Bri}}(V)$

In the article [Br1], Brinon generalizes Sen's work above. He shows that the union of the finite dimensional $K_\infty^{(l)}$ -subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under Γ_K is a $K_\infty^{(l)}$ -vector

space of dimension d stable under Γ_K (we call it $D_{\text{Bri}}(V)$). We have $\mathbb{C}_p \otimes_{K_\infty^{(\gamma)}} D_{\text{Bri}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}'_\infty \otimes_{K_\infty^{(\gamma)}} D_{\text{Bri}}(V) \rightarrow (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. As in the case of $D_{\text{Sen}}(V)$, the $K_\infty^{(\gamma)}$ -vector space $D_{\text{Bri}}(V)$ is endowed with the action of the operator

$$\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

if $\gamma \in \Gamma_0$ is close enough to 1. In addition to this operator $\nabla^{(0)}$, if $\beta_i \in \Gamma_i$ is close enough to 1, then the series of operators on $D_{\text{Bri}}(V)$:

$$\frac{\log(\beta_i)}{c_i(\beta_i)} = -\frac{1}{c_i(\beta_i)} \sum_{n \geq 1} \frac{(1-\beta_i)^n}{n}$$

converges to an operator $\nabla^{(i)} : D_{\text{Bri}}(V) \rightarrow D_{\text{Bri}}(V)$ and does not depend on the choice of β_i .

3.1.3. The module $D_{\text{dif}}^+(V)$

Let the ring B_{dR}^+ be as in Section 2.1.2. In the article [F5], by using Sen's theory, Fontaine shows that the union of $K'_\infty[[t, t_1, \dots, t_e]]$ -submodules of finite type of $(B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under Γ_0 ($\simeq G_{K'}/H_K$) is a $K'_\infty[[t, t_1, \dots, t_e]]$ -module of rank $d = \dim_{\mathbb{Q}_p} V$ stable under Γ_0 (called $D_{\text{dif}}^+(V)$). We have $B_{\text{dR}}^+ \otimes_{K'_\infty[[t, t_1, \dots, t_e]]} D_{\text{dif}}^+(V) = B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} V$ and the natural map $(B_{\text{dR}}^+)^{H_K} \otimes_{K'_\infty[[t, t_1, \dots, t_e]]} D_{\text{dif}}^+(V) \rightarrow (B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to 1, then the series of operators on $D_{\text{dif}}^+(V)$:

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

converges to an operator $\nabla^{(0)} : D_{\text{dif}}^+(V) \rightarrow D_{\text{dif}}^+(V)$ and does not depend on the choice of γ .

Remark 3.1. This $D_{\text{dif}}^+(V)$ is a little different from the original one constructed by Fontaine in [F5].

3.1.4. The module $D_{e\text{-dif}}^+(V)$ Let the ring B_{dR}^+ be as in Section 2.1.2. In the article [A-B], Andreatta and Brinon generalize Fontaine's work above. They show that the union of $K_\infty^{(\gamma)}[[t, t_1, \dots, t_e]]$ -submodules of finite type of $(B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under Γ_K is a $K_\infty^{(\gamma)}[[t, t_1, \dots, t_e]]$ -module of rank d stable under Γ_K (we call it $D_{e\text{-dif}}^+(V)$). We have $B_{\text{dR}}^+ \otimes_{K_\infty^{(\gamma)}[[t, t_1, \dots, t_e]]} D_{e\text{-dif}}^+(V) = B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} V$ and the natural map $(B_{\text{dR}}^+)^{H_K} \otimes_{K_\infty^{(\gamma)}[[t, t_1, \dots, t_e]]} D_{e\text{-dif}}^+(V) \rightarrow (B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. As in the case of $D_{\text{dif}}^+(V)$, the $K_\infty^{(\gamma)}[[t, t_1, \dots, t_e]]$ -module $D_{e\text{-dif}}^+(V)$ is endowed with the action of

the operator

$$\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

if $\gamma \in \Gamma_0$ is close enough to 1. In addition to this operator $\nabla^{(0)}$, if $\beta_i \in \Gamma_i$ is close enough to 1, then the series of operators on $D_{e-\text{dif}}^+(V)$:

$$\frac{\log(\beta_i)}{c_i(\beta_i)} = -\frac{1}{c_i(\beta_i)} \sum_{n \geq 1} \frac{(1-\beta_i)^n}{n}$$

converges to an operator $\nabla^{(i)} : D_{e-\text{dif}}^+(V) \rightarrow D_{e-\text{dif}}^+(V)$ and does not depend on the choice of β_i .

§ 3.2. Properties of differential operators

First, we consider the “meaning” of the equation $\nabla^{(j)}(F) = 0$. By definitions of differential operators, it follows easily that F is fixed by actions of an open subgroup of Γ_j . Thus, we can say that

“Find solutions $\{f_k\}_{k=1}^{d=\dim_{\mathbb{Q}_p} V}$ (linearly independent over K) of $\nabla^{(j)}(f_k) = 0$ for $0 \leq j \leq e$ in $D_{e-\text{dif}}^+(V)[1/t]$ ”

$\Downarrow$

“ V is a potentially de Rham rep. of G_K , that is, a de Rham rep. of G_K ”.

Thus, the theory of p -adic differential modules plays an important role in the proof of Theorem 1.3. Now, we will describe actions of operators $\nabla^{(j)}$ ($0 \leq j \leq e$) on the module $D_{e-\text{dif}}^+(V)$. First, by a standard argument, we can show that, if $x \in D_{e-\text{dif}}^+(V)$, we have

$$\nabla^{(0)}(x) = \lim_{\gamma \rightarrow 1} \frac{\gamma(x) - x}{\chi(\gamma) - 1} \quad \text{and} \quad \nabla^{(i)}(x) = \lim_{\beta_i \rightarrow 1} \frac{\beta_i(x) - x}{c_i(\beta_i)}.$$

With this presentation, we can easily describe actions of operators $\nabla^{(j)}$ ($0 \leq j \leq e$) on the ring $K_\infty^{(l)}[[t, t_1, \dots, t_e]]$ as follows.

Lemma 3.2. *We have*

$$\nabla^{(0)} = t \frac{d}{dt} \quad \text{and} \quad \nabla^{(i)} = t \frac{d}{dt_i} \quad (i \neq 0) \quad \text{on } K_\infty^{(l)}[[t, t_1, \dots, t_e]].$$

We extend naturally actions of $K_\infty^{(l)}$ -linear derivations $\nabla^{(0)}$ and $\nabla^{(i)}$ ($i \neq 0$) on $D_{e-\text{dif}}^+(V)$ to $D_{e-\text{dif}}(V) = D_{e-\text{dif}}^+(V)[1/t]$ by putting $\nabla^{(0)}(\frac{1}{t}) = -\frac{1}{t}$ and $\nabla^{(i)}(\frac{1}{t}) = 0$ ($i \neq 0$). Now, compute the bracket $[,]$ of operators $\nabla^{(j)}$ ($0 \leq j \leq e$).

Proposition 3.3. *On the $K_\infty^{(l)}[[t, t_1, \dots, t_e]][1/t]$ -module $D_{e-\text{dif}}(V)$ as above, we have the following relation*

1. $\nabla^{(0)}\nabla^{(i)} - \nabla^{(i)}\nabla^{(0)} = \nabla^{(i)}$ for all $i \neq 0$:
2. $\nabla^{(j)}\nabla^{(i)} - \nabla^{(i)}\nabla^{(j)} = 0$ for all $i, j \neq 0$.

The following proposition describe actions of $\nabla^{(i)}$ ($i \neq 0$) and plays a key role in the proof of Theorem 1.3..

Proposition 3.4. *Let M be a finite generated free $K_\infty^{(l)}[[t, t_1, \dots, t_e]][1/t]$ -module endowed with $K_\infty^{(l)}$ -linear operators $\{\nabla^{(j)}\}_{j=0}^e$ which satisfy Leibniz rule and relations in Proposition 3.3. Assume that M has a basis $\{g_j\}_{j=1}^d$ over $K_\infty^{(l)}[[t, t_1, \dots, t_e]][1/t]$ which satisfies $\nabla^{(0)}(g_j) = 0$. Then, the action of $\nabla^{(i)}$ ($i \neq 0$) is given by*

$$\nabla^{(i)}(g_j) = t \sum_{k=1}^d c_k g_k, \quad c_k \in K_\infty^{(l)}[[t, t_1, \dots, t_e]] \text{ and } \nabla^{(0)}(c_k) = 0.$$

Proof. Since $\{g_j\}_{j=1}^d$ forms a basis of M over $K_\infty^{(l)}[[t, t_1, \dots, t_e]][1/t]$, we have

$$(3.1) \quad \nabla^{(i)}(g_j) = \sum_{k=1}^d a_k g_k \quad (a_k \in K_\infty^{(l)}[[t, t_1, \dots, t_e]][1/t]).$$

Then, by the relation of Proposition 3.3., we have

$$\sum_{k=1}^d \nabla^{(0)}(a_k) g_k = \sum_{k=1}^d a_k g_k$$

(note that we have $\nabla^{(0)}(g_j) = 0$ by hypothesis). Hence, we obtain the differential equation

$$\nabla^{(0)}(a_k) = a_k.$$

Define $c_k = a_k/t$, then it satisfies $\nabla^{(0)}(c_k) = a_k/t - a_k/t = 0$ and we see that c_k is contained in $K_\infty^{(l)}[[t, t_1, \dots, t_e]]$. Thus, the solutions of this differential equation have the following forms

$$(3.2) \quad a_k = c_k t \quad \text{where } c_k \in K_\infty^{(l)}[[t, t_1, \dots, t_e]] \text{ and } \nabla^{(0)}(c_k) = 0.$$

Hence, we have, from (3.1) and (3.2),

$$\nabla^{(i)}(g_j) = t \sum_{k=1}^d c_k g_k \quad \text{where } c_k \in K_\infty^{(l)}[[t, t_1, \dots, t_e]] \text{ and } \nabla^{(0)}(c_k) = 0.$$

□

Corollary 3.5. *With notations as in Proposition 3.4. above, we have*

$$(\nabla^{(1)})^{k_1} \dots (\nabla^{(e)})^{k_e}(g_j) = t^{k_1 + \dots + k_e} \sum_{k=1}^d c_k g_k, \quad c_k \in K_{\infty}^{(j)}[[t, t_1, \dots, t_e]] \quad \text{and} \quad \nabla^{(0)}(c_k) = 0.$$

§ 4. Proof of Theorem 1.3. in the de Rham representation case

Let us recall some notations

- K is a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < \infty$:
- V is a p -adic representation of G_K of dimension d over $\mathbb{Q}_p$:
- K' = the p -adic completion of $\cup_{m \geq 0} K(b_i^{1/p^m}, 1 \leq i \leq e)$ is the complete discrete valuation field of characteristic 0 with perfect residue field:
- there exists a G_K -equivariant isomorphism

$$B_{\text{dR}} = B_{\text{dR}}^+[1/t] \simeq B_{\text{dR}}^{+, '}[t_1, \dots, t_e][1/t].$$

(1) V : de Rham rep. of $G_K \implies V$: de Rham rep. of $G_{K'}$

Proof. Since V is a de Rham representation of G_K , there exists a G_K -equivariant isomorphism of B_{dR} -modules:

$$(4.1) \quad B_{\text{dR}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\text{dR}})^d.$$

Now, by tensoring $B'_{\text{dR}} \otimes_{B_{\text{dR}}}$ (which is induced by the $G_{K'}$ -equivariant surjection $pr : B_{\text{dR}} \rightarrow B'_{\text{dR}}$) over (4.1), we obtain a $G_{K'}$ -equivariant isomorphism of B'_{dR} -modules:

$$B'_{\text{dR}} \otimes_{\mathbb{Q}_p} V \simeq (B'_{\text{dR}})^d.$$

This means that V is a de Rham representation of $G_{K'}$. □

(2) V : de Rham rep. of $G_{K'} \implies V$: de Rham rep. of G_K

This is the difficult part of this note and the theory of p -adic differential modules plays a central role in the following proof. We have to bridge the gap between G_K and $G_{K'}$. Then, roughly speaking, since the differential operators $\{\nabla^{(i)}\}_{i=1}^e$ reflect this difference, it suffices to construct the solutions $\{f_k\}_{k=1}^{d=\dim_{\mathbb{Q}_p} V}$ of $\nabla^{(i)}(f_k) = 0$ for $1 \leq i \leq e$.

Lemma 4.1. *If V is a de Rham representation of $G_{K'}$, there exists a $G_{K'}$ -equivariant isomorphism*

$$B_{\text{dR}} \otimes D_{e-\text{dif}}(V) \simeq (B_{\text{dR}})^d.$$

Proof. Since V is a de Rham representation of $G_{K'}$, there exists a $G_{K'}$ -equivariant isomorphism of B'_{dR} -modules:

$$(4.2) \quad B'_{\text{dR}} \otimes_{\mathbb{Q}_p} V \simeq (B'_{\text{dR}})^d.$$

Now, by tensoring $B_{\text{dR}} \otimes_{B'_{\text{dR}}}$ (which is induced by the $G_{K'}$ -equivariant injection $i : B'_{\text{dR}} \hookrightarrow B_{\text{dR}}$) over (4.2), we obtain a $G_{K'}$ -equivariant isomorphism of B_{dR} -modules:

$$B_{\text{dR}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\text{dR}})^d.$$

On the other hand, we have a G_K -equivariant isomorphism

$$B_{\text{dR}} \otimes D_{e-\text{dif}}(V) \simeq B_{\text{dR}} \otimes_{\mathbb{Q}_p} V.$$

Thus, we obtain the desired isomorphism. $\square$

Finally, we shall give the proof of (2).

Proof. We shall construct the $K_\infty^{(\iota)}$ -linearly independent elements $\{f_j^{(*)}\}_{j=1}^d \in D_{e-\text{dif}}(V)$ such that $\nabla^{(i)}(f_j^{(*)}) = 0$ for $0 \leq i \leq e$ and $1 \leq j \leq d$.

(A) Construction of $\{f_j^{(*)}\}_{j=1}^d \in D_{e-\text{dif}}(V)$

Since V is a de Rham representation of $G_{K'}$, we have a basis $\{f_j\}_{j=1}^d$ of $D_{e-\text{dif}}(V)$ over $K_\infty^{(\iota)}[[t, t_1, \dots, t_e]][1/t]$ such that, from Lemma 4.1.,

$$\nabla^{(0)}(f_j) = 0 \quad \text{for all } 1 \leq j \leq d.$$

Thus, we can apply Corollary 3.5. to the $K_\infty^{(\iota)}[[t, t_1, \dots, t_e]][1/t]$ -module $D_{e-\text{dif}}(V)$ generated by $\{f_j\}_{j=1}^d$ and then we can deduce

$$(\nabla^{(1)})^{k_1} \dots (\nabla^{(e)})^{k_e}(g_j) = t^{k_1 + \dots + k_e} \sum_{k=1}^d c_k g_k, \quad c_k \in K_\infty^{(\iota)}[[t, t_1, \dots, t_e]] \quad \text{and} \quad \nabla^{(0)}(c_k) = 0.$$

Then, if we define $f_j^{(*)} \in D_{e-\text{dif}}(V)$ (converge for $(t, t_1, \dots, t_e)$ -adic topology) by

$$f_j^{(*)} = \sum_{0 \leq k_1, \dots, k_e} (-1)^{k_1 + \dots + k_e} \frac{t_1^{k_1} \dots t_e^{k_e}}{k_1! \dots k_e! t^{k_1 + \dots + k_e}} (\nabla^{(1)})^{k_1} \dots (\nabla^{(e)})^{k_e}(f_j),$$

it follows easily that we have $\nabla^{(i)}(f_j^{(*)}) = 0$ for $0 \leq i \leq e$.

(B) $\{f_j^{(*)}\}_{j=1}^d \in D_{e-\text{dif}}(V)$ is linearly independent over $K_\infty^{(\iota)}$

By the presentation of $f_j^{(*)}$, we have

$$f_j^{(*)} = f_j + g_j \quad \text{where } f_j \notin, g_j \in (t_1, \dots, t_e)D_{e-\text{dif}}(V).$$

Since $\{f_j\}_{j=1}^d$ forms a basis of $D_{e-\text{dif}}(V)$ over $K_\infty^{(')}[[t, t_1, \dots, t_e]][1/t]$, it is, in particular, linearly independent over $K_\infty^{(')} (\subset K_\infty^{(')}[[t, t_1, \dots, t_e]][1/t])$. Then, it follows easily that $\{f_j^{(*)}\}_{j=1}^d$ is linearly independent over $K_\infty^{(')}$ in $D_{e-\text{dif}}(V)$.

(C) Conclusion

Therefore, on the K -vector space generated by $\{f_j^{(*)}\}_{j=1}^d$, $\log(\gamma)$ and $\log(\beta_i)$ act trivially ($\Leftrightarrow \nabla^{(0)}(f_j^{(*)}) = 0$ and $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $1 \leq j \leq d$). Thus, this means that Γ_K acts on this K -vector space via finite quotient and there exists a finite extension L/K such that $\{f_j^{(*)}\}_{j=1}^d$ forms a basis of $D_{\text{dR}}(V_L)$ over $L (\subset K_\infty^{(')})$ where V_L denotes the restriction of V to G_L . Since a potentially de Rham representation of G_K is a de Rham representation of G_K , we complete the proof. $\square$

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