

UNCONDITIONAL UNIQUENESS FOR THE PERIODIC BENJAMIN-ONO EQUATION BY NORMAL FORM APPROACH

NOBU KISHIMOTO

ABSTRACT. We show unconditional uniqueness of solutions to the Cauchy problem associated with the Benjamin-Ono equation under the periodic boundary condition with initial data given in H^s for $s > 1/6$. This improves the previous unconditional uniqueness result in $H^{1/2}$ by Molinet and Pilod (2012). Our proof is based on a gauge transform and integration by parts in the time variable.

1. INTRODUCTION

In the present article, we study the Cauchy problem associated with the Benjamin-Ono equation with the periodic boundary condition:

$$\partial_t u = -\mathcal{H}\partial_x^2 u + \partial_x(u^2), \quad (t, x) \in (0, T) \times \mathbb{T}, \quad (1.1)$$

with initial condition

$$u|_{t=0} = u_0, \quad (1.2)$$

where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ is the one-dimensional torus, $\mathcal{H}$ denotes the periodic Hilbert transform defined by the Fourier multiplier with symbol $-i \operatorname{sgn}(n) \mathbf{1}_{n \neq 0}$, and u_0 is a given initial datum. We only consider the real-valued solutions in this article. The Benjamin-Ono equation (1.1) is a model for one-dimensional long waves in deep stratified fluids and is known to be completely integrable.

In this article, we show a uniqueness property of rough solutions to (1.1). We begin with some definitions. For $u \in L^2_{\text{loc}}((0, T) \times \mathbb{T})$, we say u is a solution of the equation (1.1) if u satisfies it in the sense of distributions on $(0, T) \times \mathbb{T}$. If in addition a solution u lies in $L^\infty((0, T); H^s)$, where H^s denotes the L^2 -based Sobolev space, then the limit $u|_{t=0} = \lim_{t \rightarrow 0} u(t)$ exists in a weaker topology (see Remark 1.2 below) and the initial condition (1.2) can be interpreted in this sense. We also note that any solution u in $C([0, T]; H^s)$, if $s \geq 0$, automatically belongs to $C^1([0, T]; H^{s-2})$ and satisfies the equation (1.1) in H^{s-2} . Well-posedness of the Cauchy problem (1.1)–(1.2) in H^s means existence of a solution u in $C([0, T]; H^s)$, uniqueness of solutions, and continuity in H^s of the solution map $u_0 \mapsto u$. We say the Cauchy problem is locally well-posed if T is chosen as a finite constant (depending on u_0), and globally well-posed if T can be arbitrarily large (independent of u_0). Depending on the circumstances, uniqueness may be ensured only under some auxiliary conditions. Then, uniqueness in the whole of $C([0, T]; H^s)$ is called *unconditional uniqueness* (in the literature, it sometimes refers to uniqueness in $L^\infty((0, T); H^s)$).

Let us first recall some of the known results on well-posedness of the Cauchy problem (1.1)–(1.2) in H^s . The first result employing the dispersive effect of the linear part in the non-periodic setting (i.e., $x \in \mathbb{R}$) was due to Ponce [16], which was refined by Koch and Tzvetkov [10] and Kenig and Koenig [8]. Tao [18] introduced a technique of a gauge transform in the Benjamin-Ono context, which can transform the original equation containing unfavorable nonlinear interactions

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of low-high type into an equation with milder nonlinearity. The idea of the gauge transform was adapted to the method of the Fourier restriction norm and pushed down the regularity threshold for well-posedness. This was done by Burq and Planchon [3] and by Ionescu and Kenig [7] in the non-periodic case, and by Molinet [13, 14] in the periodic case, which established global well-posedness in L^2 for both $\mathbb{R}$ and $\mathbb{T}$. Moreover, Molinet and Pilod [15] gave a simplified proof of these results in H^s for $s \geq 0$, which even complemented them by showing unconditional uniqueness (uniqueness in $L^\infty((0, T); H^s)$) for $s > 1/4$ in the non-periodic case and for $s \geq 1/2$ in the periodic case. Finally, Gérard et al. [4] recently obtained a sharp global well-posedness result for the periodic problem in the whole range of subcritical regularities $s > -1/2^1$ by exploiting the complete integrability of the equation.

In this article, we study unconditional uniqueness in the periodic case by a different approach: successive applications of integration by parts with respect to t . This simple technique, while it had been used before to derive nonlinear smoothing effect under the presence of dispersion (see, e.g., [1], [17]), came to be recognized as a useful tool to establish unconditional uniqueness for nonlinear dispersive equations after a work of Babin et al. [2] on the periodic Korteweg-de Vries (KdV) equation and a subsequent work of Kwon and Oh [11] on the modified KdV equation. It can also be regarded as a variant of the (Poincaré-Dulac) normal form reduction in the theory of ordinary differential equations, as pointed out by Guo et al. [5] who applied the method to the one-dimensional cubic Schrödinger equation. Recently, this method has also been adapted to the non-periodic setting and to problems of different type; see, e.g., [12], [9] and references therein.

Our main result reads as follows:

Theorem 1.1. *The solution to the Cauchy problem (1.1)–(1.2) is unique in the class $C([0, T]; H^s(\mathbb{T}))$ for $s > 1/6$.*

Remark 1.2. The uniqueness assertion also holds in the class $L^\infty((0, T); H^s(\mathbb{T}))$, $s > 1/6$. To see this, let u be a solution in $L^\infty((0, T); H^s(\mathbb{T}))$, then $u \in W^{1,\infty}((0, T); H^{s-2}) \subset C((0, T); H^{s-2})$ by the equation. In particular, $u(t)$ has limits in H^{s-2} as $t \rightarrow 0, T$ and hence $u \in C([0, T]; H^{s-2})$. By interpolation, for any $s' \in (s-2, s)$ it holds that $\|u(t) - u(t')\|_{H^{s'}} \lesssim \|u(t) - u(t')\|_{H^{s-2}}^{(s-s')/2}$ for almost every $t, t' \in (0, T)$. Therefore, $u \in C([0, T]; H^{s'})$ for any $s' \in (\max\{s-2, 1/6\}, s)$, and Theorem 1.1 can be applied to show uniqueness.

Remark 1.3. It turns out that twice applications of integration by parts in t (normal form reduction) suffice to prove our result, which is similar to the cases of KdV [2] and modified KdV [11] on the torus. It is, however, not likely that further application of normal form reduction can weaken the regularity restriction. In fact, we encounter the requirement $s > 1/6$ in the control of the *resonant* parts just after the first application (see Lemmas 3.7, 3.10, and 3.11 below). It is not clear whether the threshold $s = 1/6$ is optimal or not.

The plan of this article is as follows. The proof begins with applying a suitable gauge transform, which is similar to that of [14] and described in Section 2, to prevent the derivative in the nonlinearity from falling onto high-frequency components. In Section 3 we give multilinear estimates of various terms arising in the normal form reduction steps. The main theorem is proved by use of these estimates in Section 4.

¹For $s < 0$, the quadratic nonlinearity is not well-defined in the distributional framework, and one needs to extend the notion of solutions and well-posedness. See [4] for details.

At the end, we introduce some notation. For a Banach space X , we abbreviate $C([0, T]; X)$ to $C_T X$. In this article, the Fourier coefficients of a 2π -periodic function f are defined by

$$\mathcal{F}f(n) := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z},$$

so that the inverse Fourier transform of a sequence $g = \{g(n)\}_{n \in \mathbb{Z}}$ is given by

$$\mathcal{F}^{-1}g(x) := \sum_{n \in \mathbb{Z}} g(n) e^{inx}, \quad x \in \mathbb{T}.$$

We write $P_{\pm} := \mathcal{F}^{-1} \mathbf{1}_{\pm n > 0} \mathcal{F}$, so that $\mathcal{H} = -iP_+ + iP_-$, and

$$\begin{aligned} P_c f &:= \mathcal{F}^{-1} \mathbf{1}_{n=0} \mathcal{F}f = \mathcal{F}f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx, \\ P_{\neq c} f &:= \mathcal{F}^{-1} \mathbf{1}_{n \neq 0} \mathcal{F}f = f - \mathcal{F}f(0). \end{aligned}$$

For a mean-zero function f on $\mathbb{T}$, define

$$\partial_x^{-1} f(x) := \mathcal{F}^{-1} \left[\frac{1}{in} \mathcal{F}f(n) \right] (x) = \frac{1}{2\pi} \int_0^{2\pi} \int_{\theta}^x f(y) dy d\theta.$$

We also write $\hat{\partial}_x := P_c + \partial_x$. Note that

$$\overline{P_{\pm} f} = P_{\mp} \bar{f}, \quad \partial_x^{-1} \partial_x = \partial_x \partial_x^{-1} P_{\neq c} = P_{\neq c}, \quad P_{\pm} \hat{\partial}_x = P_{\pm} \partial_x,$$

and also that $\hat{\partial}_x$ is a homeomorphism from $H^s(\mathbb{T})$ to $H^{s-1}(\mathbb{T})$ for any $s \in \mathbb{R}$ with the inverse mapping given by $\hat{\partial}_x^{-1} = P_c + \partial_x^{-1} P_{\neq c}$. Finally, for $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ we use the weighted ℓ^p space $\ell_s^p(\mathbb{Z})$ defined by

$$\ell_s^p(\mathbb{Z}) := \left\{ \omega : \mathbb{Z} \rightarrow \mathbb{C} \mid \|\omega\|_{\ell_s^p} := \|\langle \cdot \rangle^s \omega\|_{\ell^p} < \infty \right\}, \quad \langle n \rangle := (1 + n^2)^{1/2}.$$

2. GAUGE TRANSFORM

Let $u \in C_T H^s$ ($s \geq 0$) be a real-valued solution of (1.1) (in the sense of distributions). First, note that the right-hand side of (1.1) is a well-defined distribution in $C_T H^{-2}$. Testing the equation against $\phi(t) e^{inx}$ with $\phi \in C_c^\infty(0, T)$ and $n \in \mathbb{Z}$, we see that $\hat{u}(t, n) = \mathcal{F}[u(t, \cdot)](n) \in C([0, T])$ satisfies

$$\partial_t \hat{u}(t, n) = -in|n| \hat{u}(t, n) + in \sum_{n' \in \mathbb{Z}} \hat{u}(t, n') \hat{u}(t, n - n') \quad \text{in } \mathcal{D}'_t(0, T)$$

for each $n \in \mathbb{Z}$, where the sum is absolutely convergent, and thus the right-hand side is continuous in $t \in [0, T]$. Hence, $\hat{u}(\cdot, n) \in C^1([0, T])$ and the above equation is satisfied in the classical sense. In particular, the mean value $P_c u(t, \cdot)$ is independent of t . In view of the change of the unknown function

$$u(t, x) \mapsto u(t, x - 2t P_c u_0) - P_c u_0,$$

we may focus without loss of generality on mean-zero solutions, i.e., $P_c u(t, \cdot) = \hat{u}(t, 0) \equiv 0$.

Definition 2.1. We define the gauge transform v of a real-valued mean-zero solution $u \in C_T L^2(\mathbb{T})$ of (1.1) as follows:

$$V(t) := e^{-i\partial_x^{-1} u(t)}, \quad v(t) := i\hat{\partial}_x V(t) = iP_c(e^{-i\partial_x^{-1} u(t)}) + e^{-i\partial_x^{-1} u(t)} u(t). \quad (2.1)$$

Note that $v \in C_T H^s$ if $u \in C_T H^s$ and $s \in [0, 1]$, by Lemma 2.2 below. Moreover, we have

$$\begin{aligned} u &= P_+ u + P_- u = P_+ u + \overline{P_+ u}, \quad u = e^{i\partial_x^{-1}u} i\partial_x e^{-i\partial_x^{-1}u} = \bar{V} P_{\neq c} v, \\ P_+ u &= P_+ (\bar{V} P_{\neq c} v) = P_+ (\bar{V} P_+ v) + P_+ (\bar{V} P_- v). \end{aligned} \quad (2.2)$$

Now, we derive the equation for v . We first consider the gauge transform of regularized solutions $u_N(t) := P_{\leq N} u(t)$,² where $P_{\leq N} := \mathcal{F}_n^{-1} \mathbf{1}_{|n| \leq N} \mathcal{F}_x$ for $N > 0$. Set

$$V_N(t) := e^{-i\partial_x^{-1}u_N(t)}, \quad v_N(t) := i\hat{\partial}_x V_N(t).$$

Since the equations

$$\begin{aligned} \partial_t u_N &= -\mathcal{H} \partial_x^2 u_N + \partial_x(u_N^2) + G_N, \\ -i\partial_t \partial_x^{-1} u_N &= i\mathcal{H} \partial_x u_N - iP_{\neq c}(u_N^2) - i\partial_x^{-1} G_N \end{aligned}$$

hold in the classical sense with $G_N := P_{\leq N} \partial_x(u^2) - \partial_x(u_N^2)$, we have

$$\begin{aligned} (\partial_t + \mathcal{H} \partial_x^2) V_N &= V_N (i\mathcal{H} \partial_x u_N - iP_{\neq c}(u_N^2) - i\partial_x^{-1} G_N) + \mathcal{H} [V_N (-i\partial_x u_N - u_N^2)] \\ &= P_{\neq c} [V_N (i\mathcal{H} \partial_x u_N) - (i\mathcal{H}) [V_N \partial_x u_N] - i(P_{\neq c} - i\mathcal{H})(V_N u_N^2)] \\ &\quad + iP_c(u_N^2) V_N + P_c [V_N (i\mathcal{H} \partial_x u_N - iu_N^2)] - iV_N \partial_x^{-1} G_N \\ &= -2P_+ [V_N \cdot P_- \partial_x u_N] + 2P_- [V_N \cdot P_+ \partial_x u_N] - 2iP_- [V_N u_N^2] \\ &\quad + iP_c(u_N^2) V_N + P_c [V_N (i\mathcal{H} \partial_x u_N - iu_N^2)] - iV_N \partial_x^{-1} G_N. \end{aligned}$$

Since $\hat{\partial}_x$ commutes with $\partial_t + \mathcal{H} \partial_x^2$, we obtain

$$\begin{aligned} (\partial_t + \mathcal{H} \partial_x^2) v_N &= -2iP_+ \partial_x [V_N \cdot P_- \partial_x u_N] + 2iP_- \partial_x [V_N \cdot P_+ \partial_x u_N] + 2P_- \partial_x [V_N u_N^2] \\ &\quad - P_c(u_N^2) \hat{\partial}_x V_N - P_c [V_N (\mathcal{H} \partial_x u_N - u_N^2)] + \hat{\partial}_x [V_N \partial_x^{-1} G_N]. \end{aligned} \quad (2.3)$$

The following estimates for the gauge part are easily verified.

Lemma 2.2. *For $0 \leq s \leq 1$ and real-valued mean-zero $f, g \in H^s$, we have*

$$\begin{aligned} \|e^{-i\partial_x^{-1}f}\|_{H^{s+1}} &\lesssim 1 + \|f\|_{H^s}^2, \\ \|e^{-i\partial_x^{-1}f} - e^{-i\partial_x^{-1}g}\|_{H^{s+1}} &\lesssim (1 + \|f\|_{H^s}^2 + \|g\|_{H^s}^2) \|f - g\|_{H^s}. \end{aligned}$$

Proof. The first inequality with $s = 0$ follows from $\|e^{-i\partial_x^{-1}f}\|_{L^2} \lesssim 1$ and $\|\partial_x e^{-i\partial_x^{-1}f}\|_{L^2} \lesssim \|f\|_{L^2}$. For $s \in (0, 1]$, we deduce

$$\|\partial_x e^{-i\partial_x^{-1}f}\|_{H^s} = \|f e^{-i\partial_x^{-1}f}\|_{H^s} \lesssim \|f\|_{H^s} \|e^{-i\partial_x^{-1}f}\|_{H^1}$$

from the Sobolev inequality and apply the estimate for $s = 0$. Concerning the second inequality, we notice

$$\begin{aligned} \|e^{-i\partial_x^{-1}f} - e^{-i\partial_x^{-1}g}\|_{L^2} &\lesssim \|\partial_x^{-1}(f - g)\|_{L^\infty} \lesssim \|f - g\|_{L^2}, \\ \|\partial_x [e^{-i\partial_x^{-1}f} - e^{-i\partial_x^{-1}g}]\|_{L^2} &\lesssim \|f\|_{L^2} \|\partial_x^{-1}(f - g)\|_{L^\infty} + \|f - g\|_{L^2} \|e^{-i\partial_x^{-1}g}\|_{L^\infty} \\ &\lesssim (1 + \|f\|_{L^2}) \|f - g\|_{L^2}, \end{aligned}$$

²Here, we consider regularization of a solution (which is no longer a solution of (1.1)) rather than the solution with a regularized initial datum, in order to justify the equation for v . This is because general solutions of (1.1) are not necessarily approximated by smooth solutions.

which verifies the case $s = 0$. For $s \in (0, 1]$, the Sobolev inequality and the estimates for $s = 0$ obtained above imply

$$\begin{aligned} & \left\| \partial_x [e^{-i\partial_x^{-1}} f - e^{-i\partial_x^{-1}} g] \right\|_{H^s} \\ & \lesssim \|f\|_{H^s} \|e^{-i\partial_x^{-1}} f - e^{-i\partial_x^{-1}} g\|_{H^1} + \|f - g\|_{H^s} \|e^{-i\partial_x^{-1}} g\|_{H^1} \\ & \lesssim \|f\|_{H^s} (1 + \|f\|_{L^2}) \|f - g\|_{L^2} + (1 + \|g\|_{L^2}) \|f - g\|_{H^s}, \end{aligned}$$

which yields the claimed estimate. $\square$

By Lemma 2.2, we see that $(u_N, v_N, V_N) \rightarrow (u, v, V)$ in $C_T(H^s \times H^s \times H^{s+1})$ as $N \rightarrow \infty$ if $0 \leq s \leq 1$. Also note that $\partial_x^{-1} G_N \rightarrow 0$ in $C_T H^{-1}$ as $N \rightarrow \infty$. Furthermore, we see

$$\begin{aligned} & \|P_{\pm}(f P_{\mp} \partial_x g)\|_{H^{-1}} = \|P_{\pm}(P_{\pm} f P_{\mp} \partial_x g)\|_{H^{-1}} \\ & \leq \left[\sum_{N_1 \sim N_2 \gtrsim N \geq 1} + \sum_{N \sim N_1 \gg N_2 \geq 1} \right] N^{-1} \|P_N P_{\pm}(P_{N_1} P_{\pm} f \cdot P_{N_2} P_{\mp} \partial_x g)\|_{L^2} \\ & \lesssim \sum_{N_1 \sim N_2 \geq 1} N_2 \|P_{N_1} f\|_{L^2} \|P_{N_2} g\|_{L^2} + \sum_{N_1 \gg N_2 \geq 1} N_1^{-1} N_2 \|P_{N_1} f\|_{L^\infty} \|P_{N_2} g\|_{L^2} \\ & \lesssim \|f\|_{H^1} \|g\|_{L^2}. \end{aligned}$$

Hence, if $0 \leq s \leq 1$ the right-hand side of (2.3) converges, in $C_T H^{-2}$ for instance, to

$$\begin{aligned} & -2iP_+ \partial_x [V \cdot P_- \partial_x u] + 2iP_- \partial_x [V \cdot P_+ \partial_x u] + 2P_- \partial_x [Vu^2] \\ & - P_c(u^2) \hat{\partial}_x V - P_c[V(\mathcal{H} \partial_x u - u^2)]. \end{aligned}$$

After substituting

$$\begin{aligned} P_+ \partial_x u &= P_+ \partial_x (-i \hat{\partial}_x^{-1} v P_{\neq c} v), & P_- \partial_x u &= P_- \partial_x (-i \hat{\partial}_x^{-1} v P_{\neq c} \bar{v}), \\ Vu^2 &= -i \hat{\partial}_x^{-1} v (P_{\neq c} v)^2, \end{aligned}$$

we obtain the equation for v as

$$\begin{aligned} (\partial_t + \mathcal{H} \partial_x^2) v &= 2iP_+ \partial_x [\hat{\partial}_x^{-1} v \cdot P_- \partial_x (\hat{\partial}_x^{-1} v P_{\neq c} \bar{v})] \\ &+ 2iP_- \partial_x [\hat{\partial}_x^{-1} v \cdot P_+ \partial_x (\hat{\partial}_x^{-1} \bar{v} P_{\neq c} v)] \\ &+ 2iP_- \partial_x [\hat{\partial}_x^{-1} \bar{v} (P_{\neq c} v)^2] + R[u], \\ R[u] &:= -P_c(u^2) \hat{\partial}_x V - P_c[V(\mathcal{H} \partial_x u - u^2)], \end{aligned} \tag{2.4}$$

which is satisfied in the sense of distributions. Note that all the terms but $\partial_t v$ are well-defined as distributions in $C_T H^{-2}$. Taking the Fourier transform in x , we obtain the following equations:

$$\begin{aligned} \partial_t \hat{v}(n) + in|n| \hat{v}(n) &= \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z} \\ n = n_{123}}} m_1(n, n_1, n_2, n_3) \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3) \\ &+ \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z} \\ n = n_{123}}} m_2(n, n_1, n_2, n_3) \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3) \\ &+ \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z} \\ n = n_{123}}} m_3(n, n_1, n_2, n_3) \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3) \\ &+ \mathcal{F}[R[u]](n), \end{aligned}$$

where $n_{ijk\dots}$ stands for $n_i + n_j + n_k + \dots$, $\hat{n} := n - i\mathbf{1}_{n=0}$, and

$$\begin{aligned} m_1(n, n_1, n_2, n_3) &:= 2i \frac{nn_{23}}{\hat{n}_1 \hat{n}_2} \mathbf{1}_{n>0} \mathbf{1}_{n_{23}<0} \mathbf{1}_{n_3 \neq 0}, \\ m_2(n, n_1, n_2, n_3) &:= 2i \frac{nn_{23}}{\hat{n}_1 \hat{n}_2} \mathbf{1}_{n<0} \mathbf{1}_{n_{23}>0} \mathbf{1}_{n_3 \neq 0}, \\ m_3(n, n_1, n_2, n_3) &:= 2i \frac{n}{\hat{n}_1} \mathbf{1}_{n<0} \mathbf{1}_{n_2 n_3 \neq 0}. \end{aligned}$$

Notice that

$$\begin{aligned} n = n_{123}, \quad m_1(n, n_1, n_2, n_3) \neq 0 &\implies n_1 > n, -n_{23} > 0, \\ n = n_{123}, \quad m_2(n, n_1, n_2, n_3) \neq 0 &\implies -n_1 > -n, n_{23} > 0. \end{aligned}$$

In particular, the following bound on multipliers is available:

$$n = n_{123} \implies |m_k(n, n_1, n_2, n_3)| \lesssim \frac{\langle n \rangle}{\min_{1 \leq j \leq 3} \langle n_j \rangle} \quad (k = 1, 2, 3). \quad (2.5)$$

We now introduce a new unknown function

$$\omega(t, n) := e^{itn|n|} \widehat{v}(t, n), \quad (2.6)$$

which obeys the following equation:

$$\begin{aligned} \partial_t \omega(n) &= \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z} \\ n = n_{123}}} e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3) \omega(n_1) \omega(n_2) \omega^*(n_3) \\ &+ \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z} \\ n = n_{123}}} e^{it\Phi} m_2(n, n_1, n_2, n_3) \omega(n_1) \omega^*(n_2) \omega(n_3) \\ &+ \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z} \\ n = n_{123}}} e^{it\Phi} m_3(n, n_1, n_2, n_3) \omega^*(n_1) \omega(n_2) \omega(n_3) \\ &+ \mathcal{R}[u, \omega](n) \\ &=: \mathcal{N}[\omega](n) + \mathcal{R}[u, \omega](n), \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} \Phi &:= n|n| - n_1|n_1| - n_2|n_2| - n_3|n_3|, \\ \tilde{m}_1 &:= m_1 \mathbf{1}_{n_{12}n_{13} \neq 0}, \quad \omega^*(t, n) := \overline{\omega(t, -n)}, \\ \mathcal{R}[u, \omega](n) &:= \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z} \\ n = n_{123}}} e^{it\Phi} m_1(n, n_1, n_2, n_3) \mathbf{1}_{n_{12}n_{13}=0} \omega(n_1) \omega(n_2) \omega^*(n_3) \\ &+ e^{itn|n|} \mathcal{F}[R[u]](n). \end{aligned}$$

Observe that all the sums over n_1, n_2, n_3 in the equation (2.7) converge absolutely for each n if $\omega \in \ell^2(\mathbb{Z})$. In fact, by (2.5) and Young's inequality, for $s \geq 0$ we have

$$\begin{aligned} \langle n \rangle^{s-1} |\mathcal{N}[\omega](n)| &\lesssim \sum_{n = n_{123}} \frac{\max_{1 \leq j \leq 3} \langle n_j \rangle^s}{\min_{1 \leq j \leq 3} \langle n_j \rangle} |\omega(n_1)| |\omega(n_2)| |\omega^*(n_3)| \\ &\lesssim \|\omega\|_{\ell_s^2} \|\omega\|_{\ell_{-1}^2} \|\omega\|_{\ell^2} \lesssim \|\omega\|_{\ell_s^2}^3. \end{aligned}$$

In particular, any solution $u \in C_T H^s(\mathbb{T})$ with $0 \leq s \leq 1$ defines $\omega \in C_T \ell_s^2(\mathbb{Z})$ which is $C^1([0, T])$ in t for each $n \in \mathbb{Z}$ and satisfies (2.7) in the classical sense, with a bound

$$\|\partial_t \omega\|_{C_T \ell_{s-2}^2} \lesssim \|\omega\|_{C_T \ell_s^2}^3 + \|R[u]\|_{C_T H^{s-2}}. \quad (2.8)$$

It is easily verified by Lemma 2.2 that $R[u] \in C_T H^s$ as soon as $u \in C_T H^s$ if $0 \leq s \leq 1$.

Note that restricting to positive frequencies reduces the equation (2.7) to a simpler one:

$$\begin{aligned} \partial_t \omega(n) &= \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z} \\ n = n_{123}}} e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3) \omega(n_1) \omega(n_2) \omega^*(n_3) \\ &\quad + \mathcal{R}[u, \omega](n), \quad n > 0. \end{aligned} \quad (2.9)$$

Since we consider real-valued solutions of (1.1), we only have to bound ω in positive frequencies; see Section 4 for details. However, the full equation (2.7) is also needed when we apply normal form reduction.

3. NORMAL FORM REDUCTION AND MULTILINEAR ESTIMATES

3.1. First normal form reduction. Let us start with the equation (2.9), restricting to positive frequencies. Let $M > 0$ be a large constant, which will be chosen at the end of the proof of uniqueness depending on the solutions we consider. Decompose the equation into “resonant” and “non-resonant” parts as

$$\begin{aligned} \partial_t \omega(n) &= \left[\sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z} \\ n = n_{123} \\ |\Phi| \leq M}} + \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z} \\ n = n_{123} \\ |\Phi| > M}} \right] e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3) \omega(n_1) \omega(n_2) \omega^*(n_3) + \mathcal{R}[u, \omega](n) \\ &=: \mathcal{N}_R[\omega](n) + \mathcal{N}_{NR}[\omega](n) + \mathcal{R}[u, \omega](n), \quad n > 0. \end{aligned}$$

The estimate for $\mathcal{R}$ is easy.

Lemma 3.1. *Let $0 \leq s \leq 1$. Then, we have*

$$\begin{aligned} \|\mathcal{R}[u, \omega]\|_{C_T \ell_s^2} &\lesssim 1 + \|u\|_{C_T H^s}^4 + \|\omega\|_{C_T \ell_s^2}^3, \\ \|\mathcal{R}[u, \omega] - \mathcal{R}[\tilde{u}, \tilde{\omega}]\|_{C_T \ell_s^2} &\lesssim \left(1 + \|u\|_{C_T H^s}^4 + \|\tilde{u}\|_{C_T H^s}^4\right) \|u - \tilde{u}\|_{C_T H^s} \\ &\quad + \left(\|\omega\|_{C_T \ell_s^2}^2 + \|\tilde{\omega}\|_{C_T \ell_s^2}^2\right) \|\omega - \tilde{\omega}\|_{C_T \ell_s^2} \end{aligned}$$

for any real-valued mean-zero functions $u, \tilde{u} \in C_T H^s$ and any $\omega, \tilde{\omega} \in C_T \ell_s^2$.

Proof. Observing $|m_1(n_{123}, n_1, n_2, n_3)| \mathbf{1}_{n_{12}n_{13}=0} \lesssim 1$, the $C_T \ell_s^2$ -norm of the first term of $\mathcal{R}[u, \omega]$ is bounded by $\|\omega\|_{C_T \ell_s^2}^2 \|\omega\|_{C_T \ell_s^2}$ for any $s \geq 0$. Estimates on $R[u]$ in $C_T H^s$ can be easily obtained from Lemma 2.2 when $0 \leq s \leq 1$. $\square$

The resonant part $\mathcal{N}_R$ is also controlled in H^s if $s \geq 0$.

Lemma 3.2. *Let $s \geq 0$. Then, we have*

$$\begin{aligned} \|\mathcal{N}_R[\omega]\|_{C_T \ell_s^2} &\lesssim M \|\omega\|_{C_T \ell_s^2}^3, \\ \|\mathcal{N}_R[\omega] - \mathcal{N}_R[\tilde{\omega}]\|_{C_T \ell_s^2} &\lesssim M \left(\|\omega\|_{C_T \ell_s^2}^2 + \|\tilde{\omega}\|_{C_T \ell_s^2}^2 \right) \|\omega - \tilde{\omega}\|_{C_T \ell_s^2} \end{aligned}$$

for any $\omega, \tilde{\omega} \in C_T \ell_s^2$.

Proof. First of all, we claim that

$$|\Phi| \gtrsim \begin{cases} |n_{12}| |n_{23}| & \text{if } |n_2| \geq |n_3|, \\ |n_{13}| |n_{23}| & \text{if } |n_2| < |n_3| \end{cases} \quad (3.1)$$

whenever $n = n_{123}$ and $\tilde{m}_1(n, n_1, n_2, n_3) \neq 0$. In fact, it holds that $n_1 > n > 0$ and $-n_1 < n_{23} < 0$, and we have

$$\begin{aligned} \Phi &= n^2 - n_1^2 - n_2|n_2| - n_3|n_3| \\ &= \begin{cases} n^2 - n_1^2 - n_2^2 + n_3^2 = 2n_{13}n_{23} & \text{if } n_2 \geq 0 \text{ and } n_3 < 0, \\ n^2 - n_1^2 + n_2^2 - n_3^2 = 2n_{12}n_{23} & \text{if } n_2 < 0 \text{ and } n_3 \geq 0, \\ n^2 - n_1^2 + n_2^2 + n_3^2 = 2(nn_{23} - n_2n_3) & \text{if } n_2, n_3 < 0. \end{cases} \end{aligned} \quad (3.2)$$

In the first two cases the claim follows from $n_{23} < 0$. In the third case, we see that both nn_{23} and $-n_2n_3$ are negative and $|\Phi| \gtrsim |n||n_{23}| \vee |n_2||n_3|$. Let us assume $|n_2| \geq |n_3|$ by symmetry. The claim follows if $|n| \gtrsim |n_{12}|$, while if $|n| \ll |n_{12}|$ we have $|n_{12}| \sim |n_3|$ and $|n_{23}| \sim |n_2|$, which implies (3.1).

We observe that (3.1) yields the bound

$$\left| \frac{\langle n \rangle^s \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \right| \lesssim \frac{\mathbf{1}_{n_{12}n_{13} \neq 0} \langle n_1 \rangle^s}{\langle n_2 \rangle (|n_{12}| \wedge |n_{13}|)},$$

and the desired estimates are reduced to showing

$$\left\| \sum_{n=n_{123}} \frac{\langle n_1 \rangle^s \omega_1(n_1) \omega_2(n_2) \omega_3(n_3)}{\langle n_2 \rangle [\langle n_{12} \rangle \wedge \langle n_{13} \rangle]} \right\|_{\ell^2} \lesssim \|\omega_1\|_{\ell_s^2} \|\omega_2\|_{\ell_s^2} \|\omega_3\|_{\ell_s^2} \quad (3.3)$$

for $\omega_1, \omega_2, \omega_3 \in \ell_s^2$. If $|n_{12}| \geq |n_{13}|$, then by Young's inequality we have

$$\begin{aligned} \text{LHS of (3.3)} &\lesssim \left\| \left(\frac{\omega_2}{\langle \cdot \rangle} \right) * \left(\frac{\langle \cdot \rangle^s \omega_1 * \omega_3}{\langle \cdot \rangle} \right) \right\|_{\ell^2} \leq \|\omega_2\|_{\ell_{-1}^1} \left\| \left(\langle \cdot \rangle^s \omega_1 \right) * \omega_3 \right\|_{\ell_{-1}^2} \\ &\leq \|\omega_2\|_{\ell^2} \left\| \left(\langle \cdot \rangle^s \omega_1 \right) * \omega_3 \right\|_{\ell^\infty} \leq \|\omega_1\|_{\ell_s^2} \|\omega_2\|_{\ell^2} \|\omega_3\|_{\ell^2}. \end{aligned}$$

If $|n_{12}| < |n_{13}|$, then we have

$$\begin{aligned} \text{LHS of (3.3)} &\lesssim \left\| \left(\frac{\langle \cdot \rangle^s \omega_1 * \langle \cdot \rangle^{-1} \omega_2}{\langle \cdot \rangle} \right) * \omega_3 \right\|_{\ell^2} \\ &\leq \left\| \left(\langle \cdot \rangle^s \omega_1 \right) * \langle \cdot \rangle^{-1} \omega_2 \right\|_{\ell_{-1}^1} \|\omega_3\|_{\ell^2} \\ &\leq \|\langle \cdot \rangle^s \omega_1\|_{\ell^2} \|\langle \cdot \rangle^{-1} \omega_2\|_{\ell^1} \|\omega_3\|_{\ell^2} \leq \|\omega_1\|_{\ell_s^2} \|\omega_2\|_{\ell^2} \|\omega_3\|_{\ell^2}. \end{aligned} \quad \square$$

The remaining term $\mathcal{N}_{NR}[u]$, which can be controlled in ℓ_s^2 if $s > 3/2$, does not seem to admit an ℓ_s^2 estimate for lower regularities. Therefore, we apply a differentiation by parts, noting that Φ remains non-zero due to (3.1):

$$\begin{aligned} &\mathcal{N}_{NR}[\omega](n) \\ &= -i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \tilde{m}_1(n, n_1, n_2, n_3) \left[\partial_t \left(\frac{e^{it\Phi}}{\Phi} \omega(n_1) \omega(n_2) \omega^*(n_3) \right) - \frac{e^{it\Phi}}{\Phi} \partial_t \left(\omega(n_1) \omega(n_2) \omega^*(n_3) \right) \right] \\ &=: \partial_t \mathcal{N}_0[\omega](n) + \mathcal{N}_1[\omega](n) + \mathcal{N}_2[\omega](n) + \mathcal{N}_3[\omega](n) + \mathcal{R}_1[u, \omega](n), \end{aligned}$$

where we change the order of summation and time differentiation and apply the product rule, and then substitute the equation (2.7) to obtain the expression of each term as

$$\begin{aligned}\mathcal{N}_0[\omega](n) &:= -i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \frac{e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \omega(n_1) \omega(n_2) \omega^*(n_3), \\ \mathcal{N}_1[\omega](n) &:= i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \frac{e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \mathcal{N}[\omega](n_1) \omega(n_2) \omega^*(n_3), \\ \mathcal{N}_2[\omega](n) &:= i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \frac{e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \omega(n_1) \mathcal{N}[\omega](n_2) \omega^*(n_3), \\ \mathcal{N}_3[\omega](n) &:= i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \frac{e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \omega(n_1) \omega(n_2) \overline{\mathcal{N}[\omega](-n_3)},\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_1[u, \omega](n) &:= i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \frac{e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \left[\mathcal{R}[u, \omega](n_1) \omega(n_2) \omega^*(n_3) \right. \\ &\quad \left. + \omega(n_1) \mathcal{R}[u, \omega](n_2) \omega^*(n_3) + \omega(n_1) \omega(n_2) \overline{\mathcal{R}[u, \omega](-n_3)} \right].\end{aligned}$$

We can justify the above formal computations as follows. The change of summation and time differentiation in $\mathcal{N}_0$ is justified in the framework of distributions (see, e.g., [5, Lemma 5.1]), since the sum is absolutely convergent in view of (2.8). Application of the product rule also makes sense because $\omega(\cdot, n) \in C^1$ for each n if $s \geq 0$. Absolute convergence of all the sums in $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$, which justifies substitution of the equation (2.7), will be proved at the end of this subsection.

The estimate for $\mathcal{R}_1$ is immediately obtained as a corollary of Lemma 3.2.

Lemma 3.3. *Let $0 \leq s \leq 1$. Then, we have*

$$\begin{aligned}\|\mathcal{R}_1[u, \omega]\|_{C_T \ell_s^2} &\lesssim \left(1 + \|u\|_{C_T H^s}^4 + \|\omega\|_{C_T \ell_s^2}^3\right) \|\omega\|_{C_T \ell_s^2}^2, \\ \|\mathcal{R}_1[u, \omega] - \mathcal{R}_1[\tilde{u}, \tilde{\omega}]\|_{C_T \ell_s^2} &\lesssim \left(1 + \|u\|_{C_T H^s}^4 + \|\tilde{u}\|_{C_T H^s}^4\right) \left(\|\omega\|_{C_T \ell_s^2}^2 + \|\tilde{\omega}\|_{C_T \ell_s^2}^2\right) \|u - \tilde{u}\|_{C_T H^s} \\ &\quad + \left(1 + \|u\|_{C_T H^s}^4 + \|\tilde{u}\|_{C_T H^s}^4 + \|\omega\|_{C_T \ell_s^2}^3 + \|\tilde{\omega}\|_{C_T \ell_s^2}^3\right) \left(\|\omega\|_{C_T \ell_s^2} + \|\tilde{\omega}\|_{C_T \ell_s^2}\right) \|\omega - \tilde{\omega}\|_{C_T \ell_s^2}\end{aligned}$$

for any real-valued mean-zero functions $u, \tilde{u} \in C_T H^s$ and any $\omega, \tilde{\omega} \in C_T \ell_s^2$.

Proof. This is just a combination of Lemma 3.1 and the proof of Lemma 3.2. $\square$

Due to the time derivative on $\mathcal{N}_0$, the estimate for it requires an extra small factor. We assume $s > 0$ so that a negative power of M can be included in the estimate.

Lemma 3.4. *Let $s > 0$. Then, there exists $\delta > 0$ such that we have*

$$\begin{aligned}\|\mathcal{N}_0[\omega]\|_{C_T \ell_s^2} &\lesssim M^{-\delta} \|\omega\|_{C_T \ell_s^2}^3, \\ \|\mathcal{N}_0[\omega] - \mathcal{N}_0[\tilde{\omega}]\|_{C_T \ell_s^2} &\lesssim M^{-\delta} \left(\|\omega\|_{C_T \ell_s^2}^2 + \|\tilde{\omega}\|_{C_T \ell_s^2}^2\right) \|\omega - \tilde{\omega}\|_{C_T \ell_s^2}\end{aligned}$$

for any $\omega, \tilde{\omega} \in C_T \ell_s^2$.

Proof. The claim is verified by modifying the proof of Lemma 3.2. We divide the sum in (3.3) into two parts as follows.

If $\langle n_2 \rangle [\langle n_{12} \rangle \wedge \langle n_{13} \rangle] \gtrsim \max_{1 \leq j \leq 3} \langle n_j \rangle$, then it suffices to observe that (3.3) still holds if the denominator in the left-hand side is replaced with $\langle n_2 \rangle^{(1/2)+} [\langle n_{12} \rangle \wedge \langle n_{13} \rangle]^{(1/2)+}$, and that $\langle n_2 \rangle [\langle n_{12} \rangle \wedge \langle n_{13} \rangle] \gtrsim |\Phi|^{1/2} > M^{1/2}$.

Suppose that $\langle n_2 \rangle [\langle n_{12} \rangle \wedge \langle n_{13} \rangle] \ll \max_{1 \leq j \leq 3} \langle n_j \rangle$. This in particular implies $\langle n_2 \rangle \ll \langle n_1 \rangle \vee \langle n_3 \rangle$. If $\langle n_1 \rangle \gg \langle n_3 \rangle$, then $\langle n_1 \rangle \sim \langle n_{12} \rangle \sim \langle n_{13} \rangle \ll \max_{1 \leq j \leq 3} \langle n_j \rangle = \langle n_1 \rangle$ — a contradiction. Hence, it must hold $\langle n_3 \rangle \sim \max_{1 \leq j \leq 3} \langle n_j \rangle \gtrsim |\Phi|^{1/2} > M^{1/2}$ and then $\|\omega_3\|_{\ell^2} \lesssim M^{-s/2} \|\omega_3\|_{\ell_s^2}$ if $s > 0$. $\square$

The remaining terms $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ will be treated in the subsequent subsections. Finally, let us see that the sums in these terms are absolutely convergent. This is the first step where the regularity threshold $s = 1/6$ appears.

Lemma 3.5. *Let $1/6 \leq s < 1/2$ and $\alpha < 3s - 1$. Then, we have*

$$\|\mathcal{N}_j[\omega]\|_{C_T \ell_\alpha^2} \lesssim \|\omega\|_{C_T \ell_s^2}^5$$

for $j = 1, 2, 3$.

Proof. We first upgrade (2.8) as follows: for $0 < s < 1/2$, it holds

$$\|\mathcal{N}[\omega]\|_{C_T \ell_\beta^2} \lesssim \|\omega\|_{C_T \ell_s^2}^3, \quad \beta := 2s - \frac{3}{2} \in (-\frac{3}{2}, -\frac{1}{2}). \quad (3.4)$$

By (2.5), the estimate (3.4) is reduced to showing

$$\|fgh\|_{H^{\beta+1}} \lesssim \|f\|_{H^s} \|g\|_{H^s} \|h\|_{H^{s+1}}.$$

This is shown by applying the Sobolev multiplication law (see e.g. [6, Lemma 3.4]) twice:

$$\|fgh\|_{H^{2s-\frac{1}{2}}} \lesssim \|f\|_{H^s} \|gh\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s} \|h\|_{H^{s+1}},$$

where in the first inequality we have used the assumption $0 < s < 1/2$.

Let $0 < s < 1/2$. In view of (3.4) it suffices to show

$$\left\| \sum_{n=n_{123}} \frac{\tilde{m}_1(n, n_1, n_2, n_3) \langle n \rangle^\alpha n_{\max}^{s-\beta}}{\Phi \langle n_1 \rangle^s \langle n_2 \rangle^s \langle n_3 \rangle^s} \omega_1(n_1) \omega_2(n_2) \omega_3(n_3) \right\|_{\ell^2} \lesssim \|\omega_1\|_{\ell^2} \|\omega_2\|_{\ell^2} \|\omega_3\|_{\ell^2},$$

where $n_{\max} := \max_{1 \leq j \leq 3} \langle n_j \rangle$. From the definition of $\tilde{m}_1$ and (3.1) we have

$$\left| \frac{\tilde{m}_1(n, n_1, n_2, n_3) \langle n \rangle^\alpha n_{\max}^{s-\beta}}{\Phi \langle n_1 \rangle^s \langle n_2 \rangle^s \langle n_3 \rangle^s} \right| \lesssim \frac{\langle n \rangle^{1+\alpha} n_{\max}^{s-\beta}}{\langle n_1 \rangle^{1+s} \langle n_2 \rangle^{1+s} \langle n_3 \rangle^s [\langle n_{12} \rangle \wedge \langle n_{13} \rangle]}, \quad (3.5)$$

and the following four cases are possible to occur.

(i) $\langle n_1 \rangle \gg \langle n_2 \rangle, \langle n_3 \rangle$. In this case we have $\langle n \rangle \sim \langle n_1 \rangle \sim [\langle n_{12} \rangle \wedge \langle n_{13} \rangle] \sim n_{\max}$ and

$$(3.5) \sim \frac{\langle n_1 \rangle^{\alpha-\beta-1}}{\langle n_2 \rangle^{1+s} \langle n_3 \rangle^s} \lesssim \begin{cases} \frac{\langle n_1 \rangle^{\alpha-\beta-1}}{\langle n_2 \rangle^{s+1/2} \langle n_3 \rangle^{s+1/2}} & \text{if } \langle n_2 \rangle \gtrsim \langle n_3 \rangle, \\ \frac{\langle n_1 \rangle^{\alpha-\beta-s-(1/2)+}}{\langle n_2 \rangle^{1+s} \langle n_3 \rangle^{(1/2)+}} & \text{if } \langle n_2 \rangle \ll \langle n_3 \rangle. \end{cases}$$

Hence, by Young's inequality, this case is handled whenever

$$\alpha < \begin{cases} \beta + 1 & = 2s - 1/2 (> 3s - 1) & \text{if } \langle n_2 \rangle \gtrsim \langle n_3 \rangle, \\ \beta + s + 1/2 & = 3s - 1 & \text{if } \langle n_2 \rangle \ll \langle n_3 \rangle. \end{cases}$$

(ii) $\langle n_1 \rangle \sim \langle n_2 \rangle \gtrsim \langle n_3 \rangle$. Since $\langle n \rangle \leq \langle n_1 \rangle$ whenever $\tilde{m}_1 \neq 0$, we have $\langle n \rangle = \langle n_3 + n_{12} \rangle \leq \langle n_3 \rangle + \langle n_{12} \rangle$ and $\langle n \rangle \leq \langle n_3 - n_{13} \rangle \leq \langle n_3 \rangle + \langle n_{13} \rangle$, which imply $\langle n \rangle \leq \langle n_3 \rangle + [\langle n_{12} \rangle \wedge \langle n_{13} \rangle]$. Hence, we have

$$(3.5) \sim \frac{\langle n \rangle^{1+\alpha} n_1^{-\beta-s-2}}{\langle n_3 \rangle^s [\langle n_{12} \rangle \wedge \langle n_{13} \rangle]} \lesssim \frac{\langle n \rangle^{1+\alpha} n_1^{-\beta-s-2+}}{\langle n_3 \rangle^s [\langle n_{12} \rangle \wedge \langle n_{13} \rangle]^{1+}} \\ \lesssim \frac{\langle n \rangle^{1+\alpha-s} n_1^{-\beta-s-2+}}{[\langle n_{12} \rangle \wedge \langle n_{13} \rangle]^{1+}} + \frac{\langle n \rangle^{1+\alpha-s} n_1^{-\beta-s-2+}}{\langle n_3 \rangle^s [\langle n_{12} \rangle \wedge \langle n_{13} \rangle]^{1-s+}}.$$

This yields the desired estimate if $-\beta - s - 2 < 0$ and $\alpha - \beta - 2s - 1 < 0$, which is possible if $\alpha < 4s - 1/2$ ($> 3s - 1$).

(iii) $\langle n_1 \rangle \sim \langle n_3 \rangle \gg \langle n_2 \rangle$. The worst interaction occurs in this case. Since $\langle n \rangle \leq \langle n_2 \rangle + \langle n_{13} \rangle$, we see that

$$(3.5) \sim \frac{\langle n \rangle^{1+\alpha} n_1^{-\beta-s-1}}{\langle n_2 \rangle^{1+s} \langle n_{13} \rangle} \lesssim \begin{cases} \frac{\langle n \rangle^{\alpha-s+} n_1^{-\beta-s-1}}{\langle n_2 \rangle^{0+} \langle n_{13} \rangle} & \text{if } \langle n \rangle \lesssim \langle n_2 \rangle, \\ \frac{\langle n \rangle^{(1/2)+\alpha+} n_1^{-\beta-s-1}}{\langle n_2 \rangle^{1+s} \langle n_{13} \rangle^{(1/2)+}} & \text{if } \langle n \rangle \gg \langle n_2 \rangle. \end{cases}$$

This yields the desired estimate if $-\beta - s - 1 \leq 0$ (i.e., $s \geq 1/6$) and

$$\alpha < \begin{cases} \beta + 2s + 1 & = 4s - 1/2 & \text{if } \langle n \rangle \lesssim \langle n_2 \rangle, \\ \beta + s + 1/2 & = 3s - 1 & \text{if } \langle n \rangle \gg \langle n_2 \rangle. \end{cases}$$

(iv) $\langle n_2 \rangle \sim \langle n_3 \rangle \gg \langle n_1 \rangle$. Recalling $\langle n \rangle \leq \langle n_1 \rangle$, we have

$$(3.5) \sim \frac{\langle n \rangle^{1+\alpha} n_2^{-\beta-s-2}}{\langle n_1 \rangle^{1+s}} \lesssim \frac{\langle n \rangle^{1+\alpha-s+} n_2^{-\beta-s-2}}{\langle n \rangle^{(1/2)+} \langle n_1 \rangle^{(1/2)+}}.$$

Therefore, the claim follows if $-\beta - s - 2 \leq 0$ and $\alpha - \beta - 2s - 1 < 0$, which is possible if $\alpha < 4s - 1/2$. $\square$

Remark 3.6. From the above proof, we see that the condition on α is weakened to $\alpha < 4s - 1/2$ if the sum is restricted to frequencies such that $\langle n_2 \rangle \gtrsim \langle n \rangle$. In particular, $\mathcal{N}_j[\omega]$ is controlled in ℓ_s^2 for $1/6 < s < 1/2$ in this frequency range, since we can choose $\alpha = s$ if $s > 1/6$.

3.2. Estimate for $\mathcal{N}_1$. By the positivity of n_1 , $\mathcal{N}_1[\omega]$ can be written as

$$i \sum_{\substack{n=n_{123} \\ |\Phi|>M}} \sum_{n_1=n_{456}} \left[\frac{e^{it(\Phi+\Phi_1)} \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1(n_1, n_4, n_5, n_6)}{\Phi} \omega(n_2) \omega^*(n_3) \omega(n_4) \omega(n_5) \omega^*(n_6) \right],$$

where $\Phi_1 := n_1|n_1| - n_4|n_4| - n_5|n_5| - n_6|n_6|$. We decompose $\mathcal{N}_1$ as

$$\mathcal{N}_1[\omega](n) = i \sum_{\substack{n=n_{123} \\ |\Phi|>M}} \sum_{n_1=n_{456}} \left(\mathbf{1}_{\mathcal{A}_1^c} + \mathbf{1}_{\mathcal{A}_1} \right) [\cdots] =: \mathcal{N}_{1,R}[\omega](n) + \mathcal{N}_{1,NR}[\omega](n),$$

where $\mathcal{A}_1$, a subset of $\{(n, n_1, \dots, n_6) \in \mathbb{Z}^7 \mid n = n_{123}, n_1 = n_{456}\}$, is defined by

$$\mathcal{A}_1 := \{(n, n_j) : \langle n \rangle \gg \langle n_2 \rangle \gtrsim \langle n_3 \rangle \text{ and } \langle n_5 \rangle \ll \langle n_1 \rangle \wedge \langle n_6 \rangle \text{ and } \langle n_2 \rangle \ll \langle n_6 \rangle\} \\ \cup \{(n, n_j) : \langle n_2 \rangle \ll \langle n \rangle, \langle n_3 \rangle \text{ and } \langle n_5 \rangle \ll \langle n_1 \rangle \wedge \langle n_6 \rangle\}.$$

We show that $\mathcal{N}_{1,R}$ is controlled in ℓ_s^2 , $s > 1/6$.

Lemma 3.7. *Let $1/6 < s < 1/2$. Then, we have*

$$\begin{aligned} \|\mathcal{N}_{1,R}[\omega]\|_{C_T \ell_s^2} &\lesssim \|\omega\|_{C_T \ell_s^2}^5, \\ \|\mathcal{N}_{1,R}[\omega] - \mathcal{N}_{1,R}[\tilde{\omega}]\|_{C_T \ell_s^2} &\lesssim \left(\|\omega\|_{C_T \ell_s^2}^4 + \|\tilde{\omega}\|_{C_T \ell_s^2}^4 \right) \|\omega - \tilde{\omega}\|_{C_T \ell_s^2} \end{aligned}$$

for any $\omega, \tilde{\omega} \in C_T \ell_s^2$.

Proof. In view of Remark 3.6, it suffices to deal with the case $\langle n \rangle \gg \langle n_2 \rangle$.³ We note that this restriction implies $\langle n \rangle \langle n_{23} \rangle \lesssim |\Phi|$ by virtue of (3.1). According to the definition of $\mathcal{A}_1$, consider the following two cases.

i) $\langle n \rangle \gg \langle n_2 \rangle$ and $\langle n_5 \rangle \gtrsim \langle n_1 \rangle \wedge \langle n_6 \rangle$. In this case, we have $|\tilde{m}_1(n_1, n_4, n_5, n_6)| \lesssim 1$. Applying the Sobolev multiplication law twice, we obtain that

$$\begin{aligned} &\left\| \sum_{n_1=n_{456}} e^{it\Phi_1} \tilde{m}_1(n_1, n_4, n_5, n_6) \omega_4(n_4) \omega_5(n_5) \omega_6(n_6) \right\|_{\ell_{3s-1}^2} \\ &\lesssim \|\omega_4 * \omega_5 * \omega_6\|_{\ell_{3s-1}^2} \lesssim \|\omega_4\|_{\ell_s^2} \|\omega_5 * \omega_6\|_{\ell_{2s-1/2}^2} \lesssim \|\omega_4\|_{\ell_s^2} \|\omega_5\|_{\ell_s^2} \|\omega_6\|_{\ell_s^2} \end{aligned}$$

for $1/6 < s < 1/2$. Repeating the proof of Lemma 3.5 with the above estimate instead of (3.4), we obtain a weaker condition on α that $\alpha < 4s - 1/2$ in Cases (i) and (iii). This suffices for the claim since $s < 4s - 1/2$.

ii) $\langle n \rangle \gg \langle n_2 \rangle \gtrsim \langle n_3 \rangle \vee \langle n_6 \rangle$. Using (3.1) and $\langle n \rangle \sim [\langle n_{12} \rangle \wedge \langle n_{13} \rangle] \sim \langle n_1 \rangle < \langle n_4 \rangle$, we see that

$$\begin{aligned} &\left| \frac{\langle n \rangle^s \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1(n_1, n_4, n_5, n_6)}{\Phi} \right| \\ &\lesssim \frac{\langle n_4 \rangle^s}{\langle n_2 \rangle \langle n_5 \rangle} \lesssim \frac{\langle n_4 \rangle^s \langle n_2 \rangle^{(1/6)+} \langle n_3 \rangle^{(1/6)+} \langle n_6 \rangle^{(1/6)+}}{\langle n_5 \rangle \langle n_2 \rangle^{(1/2)+} \langle n_3 \rangle^{(1/2)+} \langle n_6 \rangle^{(1/2)+}}, \end{aligned}$$

which yields the claim if $s > 1/6$ by use of Young's inequality. $\square$

For $\mathcal{N}_{1,NR}$, we apply the normal form reduction argument again. Let us begin with observing that we have $|\Phi + \Phi_1| \gtrsim |\Phi_1|$ for frequencies included in $\mathcal{N}_{1,NR}$. In fact, if $\langle n \rangle \gg \langle n_2 \rangle \gtrsim \langle n_3 \rangle$, $\langle n_5 \rangle \ll \langle n_1 \rangle, \langle n_6 \rangle$, and $\langle n_2 \rangle \ll \langle n_6 \rangle$, then we see from (3.2) and (3.1) that $|\Phi| \lesssim |n_1| |n_2| \ll |n_1 - n_5| |n_6| \sim |n_{46}| |n_{56}| \lesssim |\Phi_1|$, whereas if $\langle n_2 \rangle \ll \langle n \rangle, \langle n_3 \rangle$ and $\langle n_5 \rangle \ll \langle n_1 \rangle, \langle n_6 \rangle$, we have $n_3, n_6 < 0$ and $n_{13} = n - n_2 > 0$, $n_{46} = n_1 - n_5 > 0$, which combined with (3.2) imply that $\Phi < 0$ and $\Phi_1 < 0$, and therefore $|\Phi + \Phi_1| > |\Phi_1|$.

Noting $|\Phi + \Phi_1| \gtrsim |\Phi_1| > 0$ by (3.1), we take a differentiation by parts as

$$\begin{aligned} &\mathcal{N}_{1,NR}[\omega](n) \\ &= \partial_t \left[\sum_{\substack{n=n_{123} \\ |\Phi| > M}} \sum_{n_1=n_{456}} \frac{e^{it(\Phi+\Phi_1)} \mathbf{1}_{\mathcal{A}_1} \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1(n_1, n_4, n_5, n_6)}{\Phi(\Phi + \Phi_1)} \right. \\ &\quad \left. \times \omega(n_2) \omega^*(n_3) \omega(n_4) \omega(n_5) \omega^*(n_6) \right] \\ &\quad - \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \sum_{n_1=n_{456}} \frac{e^{it(\Phi+\Phi_1)} \mathbf{1}_{\mathcal{A}_1} \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1(n_1, n_4, n_5, n_6)}{\Phi(\Phi + \Phi_1)} \\ &\quad \times \partial_t \left[\omega(n_2) \omega^*(n_3) \omega(n_4) \omega(n_5) \omega^*(n_6) \right] \\ &=: \partial_t \mathcal{N}_{1,0}[\omega](n) + \mathcal{N}_{1,1}[\omega](n). \end{aligned}$$

The above formal computations are again justified by the absolute convergence of the sum in $\mathcal{N}_{1,0}$, which follows from Lemma 3.5.

³We easily see that Lemma 3.5 and the estimate (3.4) admit the corresponding difference estimates.

Now, we give estimates for $\mathcal{N}_{1,1}$.

Lemma 3.8. *Let $1/6 < s < 1/2$. Then, we have*

$$\begin{aligned} \|\mathcal{N}_{1,1}[\omega]\|_{C_T \ell_s^2} &\lesssim \left(1 + \|u\|_{C_T H^s}^4 + \|\omega\|_{C_T \ell_s^2}^3\right) \|\omega\|_{C_T \ell_s^2}^4, \\ \|\mathcal{N}_{1,1}[\omega] - \mathcal{N}_{1,1}[\tilde{\omega}]\|_{C_T \ell_s^2} &\lesssim \left(1 + \|u\|_{C_T H^s}^4 + \|\tilde{u}\|_{C_T H^s}^4\right) \left(\|\omega\|_{C_T \ell_s^2}^4 + \|\tilde{\omega}\|_{C_T \ell_s^2}^4\right) \|u - \tilde{u}\|_{C_T H^s} \\ &\quad + \left(1 + \|u\|_{C_T H^s}^4 + \|\tilde{u}\|_{C_T H^s}^4 + \|\omega\|_{C_T \ell_s^2}^3 + \|\tilde{\omega}\|_{C_T \ell_s^2}^3\right) \left(\|\omega\|_{C_T \ell_s^2}^3 + \|\tilde{\omega}\|_{C_T \ell_s^2}^3\right) \|\omega - \tilde{\omega}\|_{C_T \ell_s^2} \end{aligned}$$

for any real-valued mean-zero solutions $u, \tilde{u} \in C_T H^s$ of (1.1) and the corresponding solutions $\omega, \tilde{\omega} \in C_T \ell_s^2$ of (2.7) defined by (2.1), (2.6).

Proof. In view of Lemma 3.1 and (3.4) it suffices to show that

$$\begin{aligned} \left\| \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \sum_{n_1=n_{456}} \left| \frac{\mathbf{1}_{\mathcal{A}_1} \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1(n_1, n_4, n_5, n_6) n_{\max}^{s-(2s-3/2)}}{\Phi(\Phi + \Phi_1) \langle n_2 \rangle^s \langle n_3 \rangle^s \langle n_4 \rangle^s \langle n_5 \rangle^s \langle n_6 \rangle^s} \right. \right. \\ \left. \left. \times \omega_2(n_2) \omega_3(n_3) \omega_4(n_4) \omega_5(n_5) \omega_6(n_6) \right\|_{\ell_s^2} \lesssim \prod_{j=2}^6 \|\omega_j\|_{\ell^2}, \end{aligned}$$

where in the proof of this lemma we denote $\max_{2 \leq j \leq 6} \langle n_j \rangle$ by $n_{\max}$.

Note that $\langle n \rangle < \langle n_1 \rangle \sim \langle n_{46} \rangle \lesssim \langle n_4 \rangle \sim n_{\max}$, $\langle n \rangle \langle n_{23} \rangle \lesssim |\Phi|$, $\langle n_1 \rangle \langle n_{56} \rangle \lesssim |\Phi_1| \lesssim |\Phi + \Phi_1|$ in the above summation. Therefore, it holds that

$$\begin{aligned} &\left| \frac{\langle n \rangle^s \mathbf{1}_{\mathcal{A}_1} \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1(n_1, n_4, n_5, n_6) n_{\max}^{-s+3/2}}{\Phi(\Phi + \Phi_1) \langle n_2 \rangle^s \langle n_3 \rangle^s \langle n_4 \rangle^s \langle n_5 \rangle^s \langle n_6 \rangle^s} \right| \\ &\lesssim \frac{\langle n_4 \rangle^{-2s+1/2}}{\langle n_2 \rangle^{s+1} \langle n_3 \rangle^s \langle n_5 \rangle^{s+1} \langle n_6 \rangle^s \langle n_{46} \rangle^{1-s}}. \end{aligned}$$

This bound is sufficient in the case $1/4 \leq s < 1/2$, since Young's inequality and twice applications of the Sobolev multiplication law yield

$$\begin{aligned} &\left\| \frac{\omega_2}{\langle \cdot \rangle^{s+1}} * \frac{\omega_5}{\langle \cdot \rangle^{s+1}} * \frac{\omega_3}{\langle \cdot \rangle^s} * \frac{\omega_4 * (\omega_6 / \langle \cdot \rangle^s)}{\langle \cdot \rangle^{1-s}} \right\|_{\ell^2} \\ &\leq \|\omega_2\|_{\ell_{-s-1}^1} \|\omega_5\|_{\ell_{-s-1}^1} \left\| \frac{\omega_3}{\langle \cdot \rangle^s} * \frac{\omega_4 * (\omega_6 / \langle \cdot \rangle^s)}{\langle \cdot \rangle^{1-s}} \right\|_{\ell^2} \\ &\lesssim \|\omega_2\|_{\ell^2} \|\omega_5\|_{\ell^2} \|\omega_3\|_{\ell^2} \left\| \omega_4 * \frac{\omega_6}{\langle \cdot \rangle^s} \right\|_{\ell_{-1/2}^2} \lesssim \prod_{j=2}^6 \|\omega_j\|_{\ell^2}. \end{aligned}$$

So we focus on $1/6 < s < 1/4$ and evaluate as

$$\begin{aligned} &\frac{\langle n_4 \rangle^{-2s+1/2}}{\langle n_2 \rangle^{s+1} \langle n_3 \rangle^s \langle n_5 \rangle^{s+1} \langle n_6 \rangle^s \langle n_{46} \rangle^{1-s}} \lesssim \frac{\langle n_6 \rangle^{-2s+1/2} + \langle n_{46} \rangle^{-2s+1/2}}{\langle n_2 \rangle^{s+1} \langle n_3 \rangle^s \langle n_5 \rangle^{s+1} \langle n_6 \rangle^s \langle n_{46} \rangle^{1-s}} \\ &\lesssim \frac{1}{\langle n_2 \rangle^{s+1} \langle n_3 \rangle^s \langle n_5 \rangle^{s+1} \langle n_6 \rangle^{3s-1/2} \langle n_{46} \rangle^{1-s}} + \frac{1}{\langle n_2 \rangle^{s+1} \langle n_3 \rangle^s \langle n_5 \rangle^{s+1} \langle n_6 \rangle^s \langle n_{46} \rangle^{s+1/2}}. \end{aligned}$$

The case corresponding to the first term gives the bound:

$$\begin{aligned} &\|\omega_2\|_{\ell_{-s-1}^1} \|\omega_5\|_{\ell_{-s-1}^1} \left\| \frac{\omega_3}{\langle \cdot \rangle^s} * \frac{\omega_4 * (\omega_6 / \langle \cdot \rangle^{3s-1/2})}{\langle \cdot \rangle^{1-s}} \right\|_{\ell^2} \\ &\lesssim \|\omega_2\|_{\ell^2} \|\omega_5\|_{\ell^2} \|\omega_3\|_{\ell^2} \left\| \omega_4 * \frac{\omega_6}{\langle \cdot \rangle^{3s-1/2}} \right\|_{\ell_{-1/2}^2} \lesssim \prod_{j=2}^6 \|\omega_j\|_{\ell^2}, \end{aligned}$$

while for the second term we argue similarly and obtain the bound:

$$\begin{aligned} & \|\omega_2\|_{\ell^1_{-s-1}} \|\omega_5\|_{\ell^1_{-s-1}} \left\| \frac{\omega_3}{\langle \cdot \rangle^s} * \frac{\omega_4 * (\omega_6 / \langle \cdot \rangle^s)}{\langle \cdot \rangle^{s+1/2}} \right\|_{\ell^2} \\ & \lesssim \|\omega_2\|_{\ell^2} \|\omega_5\|_{\ell^2} \|\omega_3\|_{\ell^2} \|\omega_4 * \frac{\omega_6}{\langle \cdot \rangle^s}\|_{\ell^2_{-2s}} \lesssim \prod_{j=2}^6 \|\omega_j\|_{\ell^2}, \end{aligned}$$

where we have used the Sobolev multiplication law twice. $\square$

As a corollary, we also obtain the estimates for $\mathcal{N}_{1,0}$.

Lemma 3.9. *Let $1/6 < s < 1/2$. Then, there exists $\delta > 0$ such that we have*

$$\begin{aligned} \|\mathcal{N}_{1,0}[\omega]\|_{C_T \ell_s^2} & \lesssim M^{-\delta} \|\omega\|_{C_T \ell_s^2}^5, \\ \|\mathcal{N}_{1,0}[\omega] - \mathcal{N}_{1,0}[\tilde{\omega}]\|_{C_T \ell_s^2} & \lesssim M^{-\delta} \left(\|\omega\|_{C_T \ell_s^2}^4 + \|\tilde{\omega}\|_{C_T \ell_s^2}^4 \right) \|\omega - \tilde{\omega}\|_{C_T \ell_s^2} \end{aligned}$$

for any $\omega, \tilde{\omega} \in C_T \ell_s^2$.

Proof. These estimates follow immediately from the proof of Lemma 3.8, since $n_{\max} \gtrsim |\Phi|^{1/2} > M^{1/2}$. $\square$

3.3. Estimate for $\mathcal{N}_2$. This is an easy case, since we can control this term without further applying normal form reduction. Recall that

$$\begin{aligned} \mathcal{N}_2[\omega](n) & = i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \frac{e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \omega(n_1) \left[\sum_{n_2=n_{456}} e^{it\Phi_2} \tilde{m}_1(n_2, n_4, n_5, n_6) \omega(n_4) \omega(n_5) \omega^*(n_6) \right] \omega^*(n_3) \\ & + i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \frac{e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \omega(n_1) \left[\sum_{n_2=n_{456}} e^{it\Phi_2} m_2(n_2, n_4, n_5, n_6) \omega(n_4) \omega^*(n_5) \omega(n_6) \right] \omega^*(n_3) \\ & + i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \frac{e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \omega(n_1) \left[\sum_{n_2=n_{456}} e^{it\Phi_2} m_3(n_2, n_4, n_5, n_6) \omega^*(n_4) \omega(n_5) \omega(n_6) \right] \omega^*(n_3), \end{aligned}$$

where $\Phi_2 := n_2|n_2| - n_4|n_4| - n_5|n_5| - n_6|n_6|$.

Lemma 3.10. *Let $1/6 < s < 1/2$. Then, we have*

$$\begin{aligned} \|\mathcal{N}_2[\omega]\|_{C_T \ell_s^2} & \lesssim \|\omega\|_{C_T \ell_s^2}^5, \\ \|\mathcal{N}_2[\omega] - \mathcal{N}_2[\tilde{\omega}]\|_{C_T \ell_s^2} & \lesssim \left(\|\omega\|_{C_T \ell_s^2}^4 + \|\tilde{\omega}\|_{C_T \ell_s^2}^4 \right) \|\omega - \tilde{\omega}\|_{C_T \ell_s^2} \end{aligned}$$

for any $\omega, \tilde{\omega} \in C_T \ell_s^2$.

Proof. We focus on the estimate of the first term in $\mathcal{N}_2$; noticing (2.5), we can handle the second and the third terms similarly. As in the proof of Lemma 3.7, we may assume $\langle n_2 \rangle \ll \langle n \rangle$, and hence $\langle n \rangle \langle n_{23} \rangle \lesssim |\Phi|$. Since $\langle n \rangle, \langle n_2 \rangle, \langle n_3 \rangle \lesssim \langle n_1 \rangle$, it holds that

$$\begin{aligned} & \left| \frac{\langle n \rangle^s \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1(n_2, n_4, n_5, n_6)}{\Phi} \right| \\ & \lesssim \frac{\langle n \rangle^s}{\langle n_1 \rangle \langle n_5 \rangle} \lesssim \frac{\langle n_1 \rangle^s \langle n_3 \rangle^{(1/6)+}}{\langle n_1 \rangle^{(1/2)+} \langle n_3 \rangle^{(1/2)+}} \frac{1}{\langle n_2 \rangle^{1/6} \langle n_5 \rangle}. \end{aligned} \tag{3.6}$$

The claim then follows from Young's inequality and the Sobolev multiplication law:

$$\begin{aligned} \|(\langle \cdot \rangle^{-1} \omega_5) * (\omega_4 * \omega_6)\|_{\ell^2_{-1/6}} &\lesssim \| \langle \cdot \rangle^{-1} \omega_5 \|_{\ell^2_{(1/2)+}} \| \omega_4 * \omega_6 \|_{\ell^2_{-1/6}} \\ &\lesssim \| \omega_5 \|_{\ell^2} \| \omega_4 \|_{\ell^2_s} \| \omega_6 \|_{\ell^2_s}, \end{aligned}$$

for $s > 1/6$. □

3.4. Estimate for $\mathcal{N}_3$. We recall

$$\begin{aligned} \mathcal{N}_3[\omega](n) &= i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \frac{e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \omega(n_1) \omega(n_2) \left[\sum_{n_3=n_{456}} e^{it\Phi_3} \tilde{m}_1^*(n_3, n_4, n_5, n_6) \omega^*(n_4) \omega^*(n_5) \omega(n_6) \right] \\ &+ i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \frac{e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \omega(n_1) \omega(n_2) \left[\sum_{n_3=n_{456}} e^{it\Phi_3} m_2^*(n_3, n_4, n_5, n_6) \omega^*(n_4) \omega(n_5) \omega^*(n_6) \right] \\ &+ i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \frac{e^{it\Phi} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \omega(n_1) \omega(n_2) \left[\sum_{n_3=n_{456}} e^{it\Phi_3} m_3^*(n_3, n_4, n_5, n_6) \omega(n_4) \omega^*(n_5) \omega^*(n_6) \right], \end{aligned}$$

where

$$\begin{aligned} m_j^*(n_3, n_4, n_5, n_6) &:= \overline{m_j(-n_3, -n_4, -n_5, -n_6)}, \\ \Phi_3 &:= n_3|n_3| - n_4|n_4| - n_5|n_5| - n_6|n_6|. \end{aligned}$$

Similarly to $\mathcal{N}_1$, we will decompose $\mathcal{N}_3$ into two parts $\mathcal{N}_{3,R} + \mathcal{N}_{3,NR}$ and apply normal form reduction to $\mathcal{N}_{3,NR}$. Define the set $\mathcal{A}_3 \subset \{ (n, n_1, \dots, n_6) \in \mathbb{Z}^7 \mid n = n_{123}, n_3 = n_{456} \}$ as

$$\mathcal{A}_3 := \{ (n, n_j) \mid \langle n_2 \rangle \ll \langle n \rangle \wedge \langle n_3 \rangle, \langle n_5 \rangle \ll \langle n_3 \rangle \wedge \langle n_6 \rangle, |n_{25}| \ll |n_{14}|, |nn_{25}| \ll |n_3 n_{14}| \},$$

and define

$$\begin{aligned} \mathcal{N}_{3,R}[\omega](n) &:= i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \mathbf{1}_{\langle n_2 \rangle \gtrsim \langle n_3 \rangle} \sum_{n_3=n_{456}} \mathbf{1}_{\mathcal{A}_3^c} \frac{e^{it(\Phi+\Phi_3)} \tilde{m}_1(n, n_1, n_2, n_3)}{\Phi} \omega(n_1) \omega(n_2) \\ &\quad \times \left[\tilde{m}_1^*(n_3, n_4, n_5, n_6) \omega^*(n_4) \omega^*(n_5) \omega(n_6) + m_2^*(n_3, n_4, n_5, n_6) \omega^*(n_4) \omega(n_5) \omega^*(n_6) \right. \\ &\quad \left. + m_3^*(n_3, n_4, n_5, n_6) \omega(n_4) \omega^*(n_5) \omega^*(n_6) \right] \\ &+ i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \mathbf{1}_{\langle n_2 \rangle \ll \langle n_3 \rangle} \sum_{n_3=n_{456}} \mathbf{1}_{\mathcal{A}_3} \frac{e^{it(\Phi+\Phi_3)} \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1^*(n_3, n_4, n_5, n_6)}{\Phi} \\ &\quad \times \omega(n_1) \omega(n_2) \omega^*(n_4) \omega^*(n_5) \omega(n_6), \\ \mathcal{N}_{3,NR}[\omega](n) &:= i \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \sum_{n_3=n_{456}} \mathbf{1}_{\mathcal{A}_3} \frac{e^{it(\Phi+\Phi_3)} \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1^*(n_3, n_4, n_5, n_6)}{\Phi} \\ &\quad \times \omega(n_1) \omega(n_2) \omega^*(n_4) \omega^*(n_5) \omega(n_6). \end{aligned}$$

Note that $\tilde{m}_1(n, n_1, n_2, n_3) \neq 0$ and $\langle n_2 \rangle \ll \langle n_3 \rangle$ imply $n_3 < 0$ and $m_2^*(n_3, n_4, n_5, n_6) = m_3^*(n_3, n_4, n_5, n_6) = 0$.

We first treat $\mathcal{N}_{3,R}$.

Lemma 3.11. *Let $1/6 < s < 1/2$. Then, we have*

$$\begin{aligned} \|\mathcal{N}_{3,R}[\omega]\|_{C_T \ell_s^2} &\lesssim \|\omega\|_{C_T \ell_s^2}^5, \\ \|\mathcal{N}_{3,R}[\omega] - \mathcal{N}_{3,R}[\tilde{\omega}]\|_{C_T \ell_s^2} &\lesssim \left(\|\omega\|_{C_T \ell_s^2}^4 + \|\tilde{\omega}\|_{C_T \ell_s^2}^4 \right) \|\omega - \tilde{\omega}\|_{C_T \ell_s^2} \end{aligned}$$

for any $\omega, \tilde{\omega} \in C_T \ell_s^2$.

Proof. As before, the proof has been done for the frequency region $\langle n \rangle \lesssim \langle n_2 \rangle$.

In the case $\langle n_3 \rangle \lesssim \langle n_2 \rangle \ll \langle n \rangle$, the proof is exactly the same as Lemma 3.10 since we have a bound similar to (3.6); using (2.5),

$$\begin{aligned} &\left| \frac{\langle n \rangle^s \tilde{m}_1(n, n_1, n_2, n_3) m_j^*(n_3, n_4, n_5, n_6)}{\Phi} \right| \\ &\lesssim \frac{\langle n \rangle^s}{\langle n_1 \rangle \langle n_{\min} \rangle} \lesssim \frac{\langle n_1 \rangle^s \langle n_2 \rangle^{(1/6)+}}{\langle n_1 \rangle^{(1/2)+} \langle n_2 \rangle^{(1/2)+}} \frac{1}{\langle n_3 \rangle^{1/6} \langle n_{\min} \rangle} \end{aligned}$$

for $j = 1, 2, 3$, where $n_{\min} := \min_{k=4,5,6} |n_k|$.

By the definition of $\mathcal{A}_3$, it suffices to prove the estimate in the following three cases.

i) $\langle n_2 \rangle \ll \langle n \rangle \wedge \langle n_3 \rangle$ and $\langle n_5 \rangle \gtrsim \langle n_3 \rangle \wedge \langle n_6 \rangle$. This case is handled similarly to Case i) of Lemma 3.7.

ii) $\langle n_2 \rangle \ll \langle n \rangle \wedge \langle n_3 \rangle$, $\langle n_5 \rangle \ll \langle n_3 \rangle \wedge \langle n_6 \rangle$, and $|n_{25}| \gtrsim |n_{14}|$. In this case we have $\langle n \rangle, \langle n_3 \rangle \lesssim \langle n_1 \rangle$ and $\langle n_2 \rangle, \langle n_5 \rangle, \langle n_6 \rangle \lesssim \langle n_4 \rangle$, as well as $\langle n_{14} \rangle \lesssim \langle n_2 \rangle \vee \langle n_5 \rangle$. Therefore,

$$\begin{aligned} &\left| \frac{\langle n \rangle^s \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1(n_3, n_4, n_5, n_6)}{\Phi} \right| \lesssim \frac{\langle n \rangle^s}{\langle n_2 \rangle \langle n_5 \rangle} \lesssim \frac{\langle n_1 \rangle^s}{\langle n_{14} \rangle [\langle n_2 \rangle \wedge \langle n_5 \rangle]} \\ &\lesssim \frac{\langle n_1 \rangle^s}{\langle n_{14} \rangle [\langle n_2 \rangle \wedge \langle n_5 \rangle]} \frac{\langle n_4 \rangle^{(1/6)+} [\langle n_2 \rangle \vee \langle n_5 \rangle]^{(1/6)+} \langle n_6 \rangle^{(1/6)+}}{\langle n_4 \rangle^{0+} [(\langle n_2 \rangle \vee \langle n_5 \rangle) \wedge \langle n_6 \rangle]^{(1/2)+}}, \end{aligned}$$

which yields the claim if $s > 1/6$ by use of Young's inequality.

iii) $\langle n_2 \rangle \ll \langle n \rangle \wedge \langle n_3 \rangle$, $\langle n_5 \rangle \ll \langle n_3 \rangle \wedge \langle n_6 \rangle$, and $|nn_{25}| \gtrsim |n_3 n_{14}|$. It holds that

$$\begin{aligned} &\left| \frac{\langle n \rangle^s \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1(n_3, n_4, n_5, n_6)}{\Phi} \right| \\ &\lesssim \frac{\langle n \rangle^s \langle n_3 \rangle}{\langle n_1 \rangle \langle n_2 \rangle \langle n_5 \rangle} \lesssim \frac{\langle n_1 \rangle^s \langle n_{25} \rangle}{\langle n_{14} \rangle \langle n_2 \rangle \langle n_5 \rangle} \lesssim \frac{\langle n_1 \rangle^s}{\langle n_{14} \rangle [\langle n_2 \rangle \wedge \langle n_5 \rangle]}, \end{aligned}$$

and we argue in the same manner as ii). □

For $\mathcal{N}_{3,NR}$, it turns out that $|\Phi + \Phi_3|$ has a lower bound. In fact, frequencies in the sum satisfy

$$\begin{aligned} \Phi + \Phi_3 &= n^2 - n_1^2 - n_2 |n_2| + n_4^2 - n_5 |n_5| - n_6^2 \\ &= 2n_{14} n_{46} + 2nn_{25} - n_{25}^2 - n_2 |n_2| - n_5 |n_5|, \end{aligned}$$

while it holds that

$$\begin{aligned} |nn_{25}| &\ll |n_3 n_{14}| \sim |n_{14} n_{46}|, \\ |n_{25}^2 + n_2 |n_2| + n_5 |n_5|| &\lesssim |n_{25}| (|n_2| \vee |n_5|) \ll |n_3 n_{25}| \ll |n_3| |n_{14}|, \end{aligned}$$

which imply $|\Phi + \Phi_3| \gtrsim |n_3| |n_{14}|$. Note that we may assume $|n_3 n_{14}| \neq 0$; otherwise, the argument for the case $|n_{25}| \gtrsim |n_{14}|$ may be applied since we always have $n_3 < 0$.

We apply a differentiation by parts to $\mathcal{N}_{3,NR}$ as

$$\begin{aligned}
& \mathcal{N}_{3,NR}[\omega](n) \\
&= \partial_t \left[\sum_{\substack{n=n_{123} \\ |\Phi| > M}} \sum_{n_3=n_{456}} \frac{e^{it(\Phi+\Phi_3)} \mathbf{1}_{\mathcal{A}_3} \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1^*(n_3, n_4, n_5, n_6)}{\Phi(\Phi + \Phi_3)} \right. \\
&\quad \left. \times \omega(n_1) \omega(n_2) \omega^*(n_4) \omega^*(n_5) \omega(n_6) \right] \\
&\quad - \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \sum_{n_3=n_{456}} \frac{e^{it(\Phi+\Phi_3)} \mathbf{1}_{\mathcal{A}_3} \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1^*(n_3, n_4, n_5, n_6)}{\Phi(\Phi + \Phi_3)} \\
&\quad \times \partial_t \left[\omega(n_1) \omega(n_2) \omega^*(n_4) \omega^*(n_5) \omega(n_6) \right] \\
&=: \partial_t \mathcal{N}_{3,0}[\omega](n) + \mathcal{N}_{3,1}[\omega](n),
\end{aligned}$$

where, as $\mathcal{N}_{1,NR}$, the formal computations are justified by the absolute convergence of the sum in $\mathcal{N}_{3,0}$.

Lemma 3.12. *Let $1/6 < s < 1/2$. Then, we have*

$$\begin{aligned}
& \|\mathcal{N}_{3,1}[\omega]\|_{C_T \ell_s^2} \lesssim \left(1 + \|u\|_{C_T H^s}^4 + \|\omega\|_{C_T \ell_s^2}^3\right) \|\omega\|_{C_T \ell_s^2}^4, \\
& \|\mathcal{N}_{3,1}[\omega] - \mathcal{N}_{3,1}[\tilde{\omega}]\|_{C_T \ell_s^2} \\
& \lesssim \left(1 + \|u\|_{C_T H^s}^4 + \|\tilde{u}\|_{C_T H^s}^4\right) \left(\|\omega\|_{C_T \ell_s^2}^4 + \|\tilde{\omega}\|_{C_T \ell_s^2}^4\right) \|u - \tilde{u}\|_{C_T H^s} \\
& \quad + \left(1 + \|u\|_{C_T H^s}^4 + \|\tilde{u}\|_{C_T H^s}^4 + \|\omega\|_{C_T \ell_s^2}^3 + \|\tilde{\omega}\|_{C_T \ell_s^2}^3\right) \left(\|\omega\|_{C_T \ell_s^2}^3 + \|\tilde{\omega}\|_{C_T \ell_s^2}^3\right) \|\omega - \tilde{\omega}\|_{C_T \ell_s^2}
\end{aligned}$$

for any real-valued mean-zero solutions $u, \tilde{u} \in C_T H^s$ of (1.1) and the corresponding solutions $\omega, \tilde{\omega} \in C_T \ell_s^2$ of (2.7) defined by (2.1), (2.6).

Proof. Similarly to Lemma 3.8, it suffices to show that

$$\begin{aligned}
& \left\| \sum_{\substack{n=n_{123} \\ |\Phi| > M}} \sum_{n_3=n_{456}} \left| \frac{\mathbf{1}_{\mathcal{A}_3} \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1^*(n_3, n_4, n_5, n_6) n_{\max}^{\frac{3}{2}-s}}{\Phi(\Phi + \Phi_3) \langle n_1 \rangle^s \langle n_2 \rangle^s \langle n_4 \rangle^s \langle n_5 \rangle^s \langle n_6 \rangle^s} \omega_1(n_1) \omega_2(n_2) \omega_4(n_4) \omega_5(n_5) \omega_6(n_6) \right| \right\|_{\ell_s^2} \\
& \lesssim \prod_{j=1,2,4,5,6} \|\omega_j\|_{\ell^2},
\end{aligned}$$

where in the proof of this lemma we denote $\max_{j=1,2,4,5,6} \langle n_j \rangle$ by $n_{\max}$.

Now, we have $n_{\max} \sim \langle n_1 \rangle \vee \langle n_4 \rangle$ and, since $1/6 < s < 1/2$,

$$\begin{aligned}
& \left| \frac{\langle n \rangle^s \mathbf{1}_{\mathcal{A}_1} \tilde{m}_1(n, n_1, n_2, n_3) \tilde{m}_1^*(n_3, n_4, n_5, n_6) n_{\max}^{\frac{3}{2}-s}}{\Phi(\Phi + \Phi_3) \langle n_1 \rangle^s \langle n_2 \rangle^s \langle n_4 \rangle^s \langle n_5 \rangle^s \langle n_6 \rangle^s} \right| \lesssim \frac{n_{\max}^{\frac{3}{2}-s}}{\langle n_1 \rangle \langle n_2 \rangle^{s+1} \langle n_4 \rangle^{2s} \langle n_5 \rangle^{s+1} \langle n_{14} \rangle} \\
& \sim \begin{cases} \frac{1}{\langle n_1 \rangle^{s+\frac{1}{2}} \langle n_2 \rangle^{s+1} \langle n_4 \rangle^{2s} \langle n_5 \rangle^{s+1}} \lesssim \frac{1}{\langle n_1 \rangle^{\frac{1}{2}+} \langle n_2 \rangle^{s+1} \langle n_4 \rangle^{\frac{1}{2}+} \langle n_5 \rangle^{s+1}} & \text{if } \langle n_1 \rangle \gg \langle n_4 \rangle, \\ \frac{1}{\langle n_1 \rangle^{3s-\frac{1}{2}} \langle n_2 \rangle^{s+1} \langle n_5 \rangle^{s+1} \langle n_{14} \rangle} \lesssim \frac{1}{\langle n_2 \rangle^{s+1} \langle n_5 \rangle^{s+1} \langle n_{14} \rangle^{1+}} & \text{if } \langle n_1 \rangle \sim \langle n_4 \rangle, \\ \frac{1}{\langle n_1 \rangle \langle n_2 \rangle^{s+1} \langle n_4 \rangle^{3s-\frac{1}{2}} \langle n_5 \rangle^{s+1}} \lesssim \frac{1}{\langle n_1 \rangle^{\frac{1}{2}+} \langle n_2 \rangle^{s+1} \langle n_5 \rangle^{s+1} \langle n_{46} \rangle^{\frac{1}{2}+}} & \text{if } \langle n_1 \rangle \ll \langle n_4 \rangle, \end{cases}
\end{aligned}$$

where in the case $\langle n_1 \rangle \ll \langle n_4 \rangle$ we have used the fact that $\langle n_{46} \rangle \sim \langle n_3 \rangle \lesssim \langle n_1 \rangle$. Then, the claimed estimates are deduced by Young's inequality. $\square$

Finally, the estimates for $\mathcal{N}_{3,0}$ follow from Lemma 3.12 just as Lemma 3.9 follows from Lemma 3.8.

Lemma 3.13. *Let $1/6 < s < 1/2$. Then, there exists $\delta > 0$ such that we have*

$$\begin{aligned} \|\mathcal{N}_{3,0}[\omega]\|_{C_T \ell_s^2} &\lesssim M^{-\delta} \|\omega\|_{C_T \ell_s^2}^5, \\ \|\mathcal{N}_{3,0}[\omega] - \mathcal{N}_{3,0}[\tilde{\omega}]\|_{C_T \ell_s^2} &\lesssim M^{-\delta} \left(\|\omega\|_{C_T \ell_s^2}^4 + \|\tilde{\omega}\|_{C_T \ell_s^2}^4 \right) \|\omega - \tilde{\omega}\|_{C_T \ell_s^2} \end{aligned}$$

for any $\omega, \tilde{\omega} \in C_T \ell_s^2$.

3.5. Conclusion. We have seen that a real-valued mean-zero solution $u \in C_T H^s$ of (1.1) with $1/6 < s < 1/2$ and the corresponding solution $\omega \in C_T \ell_s^2$ of (2.7) defined by (2.1) and (2.6) satisfy the equation

$$\partial_t \omega(t, n) = \partial_t \mathcal{N}^{(0)}[\omega](t, n) + \mathcal{N}^{(1)}[u, \omega](t, n), \quad t \in [0, T]$$

with

$$\begin{aligned} \mathcal{N}^{(0)}[\omega] &:= \mathcal{N}_0[\omega] + \mathcal{N}_{1,0}[\omega] + \mathcal{N}_{3,0}[\omega], \\ \mathcal{N}^{(1)}[u, \omega] &:= \mathcal{R}[u, \omega] + \mathcal{N}_R[\omega] + \mathcal{R}_1[u, \omega] + \mathcal{N}_{1,R}[\omega] \\ &\quad + \mathcal{N}_{1,1}[\omega] + \mathcal{N}_2[\omega] + \mathcal{N}_{3,R}[\omega] + \mathcal{N}_{3,1}[\omega], \end{aligned}$$

or the associated integral equation

$$\begin{aligned} \omega(t, n) &= \omega(0, n) + \left[\mathcal{N}^{(0)}[\omega](t, n) - \mathcal{N}^{(0)}[\omega](0, n) \right] \\ &\quad + \int_0^t \mathcal{N}^{(1)}[u, \omega](\tau, n) d\tau, \quad t \in [0, T], \end{aligned} \tag{3.7}$$

both in the classical sense for each $n > 0$.

Combining Lemmas 3.1–3.4 and 3.7–3.13, we obtain the following estimates.

Proposition 3.14. *Let $1/6 < s < 1/2$ and $u, \tilde{u} \in C_T H^s$ be any real-valued mean-zero solutions of (1.1), with $\omega, \tilde{\omega} \in C_T \ell_s^2$ being the corresponding solutions of (2.7) defined by (2.1) and (2.6). Then, there exist $\delta = \delta(s) > 0$ and $C = C(s, \|u\|_{C_T H^s}, \|\tilde{u}\|_{C_T H^s}) > 0^4$ such that we have*

$$\begin{aligned} \|\mathcal{N}^{(0)}[\omega] - \mathcal{N}^{(0)}[\tilde{\omega}]\|_{C_T \ell_s^2} &\leq C M^{-\delta} \|\omega - \tilde{\omega}\|_{C_T \ell_s^2}, \\ \|\mathcal{N}^{(1)}[u, \omega] - \mathcal{N}^{(1)}[\tilde{u}, \tilde{\omega}]\|_{C_T \ell_s^2} &\leq C M \left(\|u - \tilde{u}\|_{C_T H^s} + \|\omega - \tilde{\omega}\|_{C_T \ell_s^2} \right). \end{aligned}$$

4. PROOF OF THEOREM 1.1

It suffices to verify the claim for $1/6 < s < 1/2$. Let $u, \tilde{u} \in C_T H^s$ be two real-valued mean-zero solutions of (1.1) on a time interval $[0, T]$ with a common initial datum at $t = 0$. Define the corresponding functions $V, \tilde{V} \in C_T H^{s+1}$, $v, \tilde{v} \in C_T H^s$, and $\omega, \tilde{\omega} \in C_T \ell_s^2$ by (2.1) and (2.6). By a continuity argument, it suffices to prove $u(t) = \tilde{u}(t)$ on $[0, T']$ for some $0 < T' \leq T$. In the following, $\tilde{C} > 0$ denotes any constant depending on $s, \|u\|_{C_T H^s}, \|\tilde{u}\|_{C_T H^s}$.

Let $N > 0$ be a large constant to be chosen later. We estimate $P_{\leq N}(u - \tilde{u})$ and $P_{> N}(u - \tilde{u})$ separately. For the low frequency part, which is smooth and satisfies the integral equation

$$P_{\leq N}(u - \tilde{u})(t) = \int_0^t e^{-(t-\tau)\mathcal{H}\partial_x^2} P_{\leq N} \partial_x (u^2 - \tilde{u}^2)(\tau) d\tau,$$

⁴From Lemma 2.2, $\|\omega\|_{C_T \ell_s^2}$ is controlled by $\|u\|_{C_T H^s}$.

we use the Sobolev multiplication law and obtain

$$\begin{aligned} \|P_{\leq N}(u - \tilde{u})\|_{C_{T'}H^s} &\leq \int_0^{T'} \|P_{\leq N}\partial_x(u^2 - \tilde{u}^2)(t)\|_{H^s} dt \\ &\leq \tilde{C}T'N^2\|u - \tilde{u}\|_{C_{T'}H^s} \end{aligned}$$

for any $0 < T' \leq T$.

For the high frequency part, we use (2.2) to have

$$\begin{aligned} \|P_{>N}(u - \tilde{u})\|_{C_{T'}H^s} &= 2\|P_{>N}P_+(u - \tilde{u})\|_{C_{T'}H^s} \\ &\leq 2\|P_{>N}P_+(\bar{V}P_+v - \tilde{V}P_+\tilde{v})\|_{C_{T'}H^s} + 2\|P_{>N}P_+(\bar{V}P_-v - \tilde{V}P_-\tilde{v})\|_{C_{T'}H^s}. \end{aligned}$$

By Lemma 2.2, we estimate the second term as

$$\begin{aligned} \|P_{>N}P_+(\bar{V}P_-v - \tilde{V}P_-\tilde{v})\|_{C_{T'}H^s} &\leq \|P_{>N}P_+\bar{V} \cdot P_-v - P_{>N}P_+\tilde{V} \cdot P_-\tilde{v}\|_{C_{T'}H^s} \\ &\leq \tilde{C}\left(\|P_{>N}(V - \tilde{V})\|_{C_{T'}H^1}\|v\|_{C_{T'}H^s} + \|P_{>N}\tilde{V}\|_{C_{T'}H^1}\|v - \tilde{v}\|_{C_{T'}H^s}\right) \\ &\leq \tilde{C}N^{-s}\left(\|V - \tilde{V}\|_{C_{T'}H^{s+1}}\|v\|_{C_{T'}H^s} + \|\tilde{V}\|_{C_{T'}H^{s+1}}\|v - \tilde{v}\|_{C_{T'}H^s}\right) \\ &\leq \tilde{C}N^{-s}\|u - \tilde{u}\|_{C_{T'}H^s}. \end{aligned}$$

The first term is estimated as

$$\begin{aligned} &\|P_{>N}P_+(\bar{V}P_+v - \tilde{V}P_+\tilde{v})\|_{C_{T'}H^s} \\ &\leq \|P_{>N/2}(V - \tilde{V})\|_{C_{T'}H^1}\|v\|_{C_{T'}H^s} + \|V - \tilde{V}\|_{C_{T'}H^1}\|P_{>N/2}v\|_{C_{T'}H^s} \\ &\quad + \|\tilde{V}\|_{C_{T'}H^1}\|P_+(v - \tilde{v})\|_{C_{T'}H^s} \\ &\leq \tilde{C}\left((N^{-s} + \|P_{>N/2}v\|_{C_{T'}H^s})\|u - \tilde{u}\|_{C_{T'}H^s} + \|\mathbf{1}_{n>0}(\omega - \tilde{\omega})\|_{C_{T'}\ell_s^2}\right). \end{aligned}$$

Using the equation (3.7) and Proposition 3.14, we have

$$\begin{aligned} \|\mathbf{1}_{n>0}(\omega - \tilde{\omega})\|_{C_{T'}\ell_s^2} &\leq \tilde{C}(M^{-\delta} + T'M)\left(\|u - \tilde{u}\|_{C_{T'}H^s} + \|\omega - \tilde{\omega}\|_{C_{T'}\ell_s^2}\right) \\ &\leq \tilde{C}(M^{-\delta} + T'M)\|u - \tilde{u}\|_{C_{T'}H^s}. \end{aligned}$$

Hence, we obtain the estimate

$$\begin{aligned} &\|u - \tilde{u}\|_{C_{T'}H^s} \\ &\leq \tilde{C}\left(T'(N^2 + M) + N^{-s} + \|P_{>N/2}v\|_{C_{T'}H^s} + M^{-\delta}\right)\|u - \tilde{u}\|_{C_{T'}H^s} \end{aligned}$$

for any $0 < T' \leq T$. We take $N, M > 0$ sufficiently large and then take T' small according to N, M so that

$$\tilde{C}\left(T'(N^2 + M) + N^{-s} + \|P_{>N/2}v\|_{C_{T'}H^s} + M^{-\delta}\right) < 1.$$

For such a T' we obtain $\|u - \tilde{u}\|_{C_{T'}H^s} = 0$ as desired. This concludes the proof of Theorem 1.1.

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN
 Email address: nobu@kurims.kyoto-u.ac.jp