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ASYMPTOTIC EXPANSIONS FOR CERTAIN q-SERIES, q-INTEGRALS, q-DIFFERENTIALS AND A FORMULA OF RAMANUJAN FOR SPECIFIC VALUES OF $\zeta(s)$

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ABSTRACT. This is a summarized version of the author's papers [22][24] on asymptotic aspects of the q-series of Lambert type, q-hypergeometric function, q-integrals and q-differentials. Major portions of the results in these papers are rearranged to state in Parts I and II respectively; the first part is devoted to showing intrinsic linkage between asymptotics of certain q-series and a formula of Ramanujan for specific values of the Riemann zeta-function $\zeta(s)$, while several complete asymptotic expansions for multiple q-integrals and q-differentials of Thomae-Jackson type are presented in the second part.

Part I: Asymptotics for q-series and Ranmanujan's formula for $\zeta(s)$

1.1. **Introduction (I).** Throughout the present article, let q be a complex parameter with $|q|<1$, and the substitution $q=e^{-t}$ will be made as it is needed, where the half-plane $\operatorname{Re} t>0$ is transformed to the unit disk $|q|<1$. It is the main aim of Part I to present intrinsic linkage between asymptotic expansions of certain q-series (see (1.1.6)–(1.1.8) below) and a formula of Ramanujan for specific values of the Riemann zeta-function at odd integers (see (1.1.9) below). This linkage is in fact hidden in Ramanujan's original work; however, the introduction of the q-series (1.1.2) or (1.1.3) and its treatment based on a Mellin transform technique give us an insight for connecting these two aspects together.

Let z and s be complex variables, and let α and μ be real parameters with $\alpha>0$. For our later purposes it is convenient to introduce the generalized Lerch zeta-function $\Phi(s,\alpha,z)$ defined by

$$\Phi(s,\alpha,z)=\sum_{n=0}^{\infty}(\alpha+n)^{-s}z^{n} \tag{1.1.1}$$

for all s if $|z|<1$, for $\operatorname{Re} s>0$ if $|z|=1$ and $z\neq 1$, and for $\operatorname{Re} s>1$ if $z=1$, respectively; this continues to a meromorphic function over the whole s-plane and is one-valued in the complex z-plane cut along the real axis from 1 to $+\infty$ (cf. [13]). We use the notation $e(\mu)=e^{2\pi i\mu}$ hereafter. Then $\Phi(s,\alpha,z)$ reduces to the ordinary Lerch zeta-function $\phi(s,\alpha,\mu)$ when $z=e(\mu)$, so that $\Phi(s,\alpha,1)=\zeta(s,\alpha)$ is the Hurwitz zeta-function, $e(\mu)\Phi(s,1,e(\mu))=\zeta_{\mu}(s)$ the exponential zeta-function, and so $\Phi(s,1,1)=\zeta(s)$ the Riemann zeta-function. We remark that the order of the variables in Φ and ϕ above differs from the usual notation, in order to retain notational consistency with other terminology.

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Let β and ν be real parameters with $\beta > 0$. The main object of the present paper is the q-series of the form

$$S_s\left(\begin{smallmatrix}\alpha,\beta\\ \mu,\nu\end{smallmatrix};q\right) = e(\beta\nu)\sum_{m=0}^{\infty} e((\alpha+m)\mu)q^{(\alpha+m)\beta}\Phi(s,\beta,e(\nu)q^{\alpha+m}), \tag{1.1.2}$$

which is rewritten, by changing the order of summations, as a Lambert series form

$$S_s\left(\begin{smallmatrix}\alpha,\beta\\ \mu,\nu\end{smallmatrix};q\right) = e(\alpha\mu)\sum_{n=0}^{\infty}(\beta+n)^{-s}\frac{e((\beta+n)\nu)q^{\alpha(\beta+n)}}{1-e(\mu)q^{\beta+n}}. \tag{1.1.3}$$

We shall prove complete asymptotic expansions of $S_s\left(\begin{smallmatrix}\alpha,\beta\\ \mu,\nu\end{smallmatrix};q\right)$ as $t \to 0$ in the sectorial region $|\arg t| < \pi/2$ (see Theorem 0 below). Let as usual

$$(z;q)_\infty = \prod_{m=0}^{\infty}(1-zq^m), \qquad (z;q)_n = (z;q)_\infty/(zq^n;q)_\infty$$

for any integer n denote q-shifted factorials. Our main formula (1.2.3) in particular implies a complete asymptotic expansion of $\log(q^\alpha;q)_\infty$ as $q \to 1^-$, and it further allows us to treat the q-series

$$F(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2}, \tag{1.1.4}$$

$$G(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} \qquad \text{and} \qquad H(q) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}. \tag{1.1.5}$$

These are typical examples of the theta series (in the transformed Eulerian form) whose asymptotic behaviours near the singularities at the points $q^k = 1$ $(k = 1, 2, \ldots)$ were first considered by Ramanujan in his last letter to Hardy (see [38]). Ramanujan showed

$$F(q) = \left(\frac{t}{2\pi}\right)^{1/2}\exp\left(\frac{\pi^2}{6t}-\frac{t}{24}\right)+o(1), \tag{1.1.6}$$

$$G(q) = \left(\frac{2}{5-\sqrt{5}}\right)^{1/2}\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}\right)+o(1), \tag{1.1.7}$$

$$H(q) = \left(\frac{2}{5+\sqrt{5}}\right)^{1/2}\exp\left(\frac{\pi^2}{15t}+\frac{11t}{60}\right)+o(1), \tag{1.1.8}$$

as $t \to +0$, and similar asymptotic formulae for certain other q-series. In conjunction with this result, (complete) Stirling's formula for the q-gamma function was first established by Moak [31], while Ueno and Nishizawa [37] developed their theory on a q-analogue of the Hurwitz zeta-function and applied it to rederive the same formula, together with asymptotic expansions of $G(q)$ and $H(q)$, similar to (1.1.7) and (1.1.8). The study on asymptotic aspects for more general q-series of the type $\sum_{n=0}^{\infty}a^nq^{bn^2+cn}/(q;q)_n$ was initiated by Ramanujan [35, p. 366] [36, p.359], and was further proceeded by Berndt [7] [8, Chap. 27]. This direction has recently been systematically explored by McIntosh [25][26][27] and Gordon-McIntosh [17][18], in conjunction with transformation properties of the q-series. It is to be remarked that the basic tool applied by these authors is the Euler-Maclaurin summation device. The Mellin transform technique, on the other hand, was applied by Meinardus [29][30] to derive certain asymptotic formulae for fairly general class of partition-type functions. We refer the reader to [2, Chap. 6] for various related works.

Let B_k $(k = 0, 1, 2, \ldots)$ denote the Bernoulli numbers (cf. [13]). Our main theorem also yields Ramanujan's famous formula for specific values of the Riemann zeta-function at odd integers (cf. [5][6]), which asserts, for any integer $n \neq 0$,

$$\begin{aligned}(1.1.9)\qquad &\xi^{-n}\Big\{\frac{1}{2}\zeta(2n+1) + \sum_{l=1}^{\infty}\frac{l^{-2n-1}}{e^{2l\xi}-1}\Big\} + 2^{2n}\sum_{k=0}^{n+1}\frac{B_{2n+2-2k}B_{2k}}{(2n+2-2k)!(2k)!}\xi^{n+1-k}(-\eta)^k \\ &= (-\eta)^{-n}\Big\{\frac{1}{2}\zeta(2n+1) + \sum_{l=1}^{\infty}\frac{l^{-2n-1}}{e^{2l\eta}-1}\Big\},\end{aligned}$$

where ξ and η are positive real numbers satisfying $\xi\eta = \pi^2$ and the finite sum on the left-hand side is to be regarded as null if $n < -1$ (see Theorem 2 in Section 1.4). It will later turn out that the excluded case $n = 0$ of this formula emerges (in a sense) as asymptotic expansions of $F(q)$, $G(q)$ and $H(q)$ (see Corollary 1.4 in Section 1.3).

1.2. **The main theorem (I).** Let x and y be complex variables. Apostol [3] introduced the sequence of rational functions $\mathcal{B}_k(x, y)$ $(k \geq 0)$ defined by the Taylor series expansion

$$(1.2.1)\qquad \frac{ze^{xz}}{ye^z - 1} = \sum_{k=0}^{\infty}\frac{\mathcal{B}_k(x,y)}{k!}z^k$$

with $|\arg y| < \pi$ near $z = 0$. The function $\mathcal{B}_k(x, y)$, which coincides with the usual Bernoulli polynomial $B_k(x)$ if $y = 1$, is a polynomial in x of degree at most k with coefficients in $\mathbb{Q}(y)$. Next let $\Gamma(s)$ be the gamma function, and $U(a; c; z)$ denote the confluent hypergeometric function defined by

$$(1.2.2)\qquad U(a;c;z) = \frac{1}{\Gamma(a)}\int_0^{\infty e^{i\varphi}} e^{-zw}w^{a-1}(1+w)^{c-a-1}dw$$

for $\operatorname{Re} a > 0$ and $|\arg z + \varphi| < \pi/2$ with any fixed angle $\varphi \in (-\pi, \pi)$, where the path of integration is taken as a half-line from the origin to $\infty e^{i\varphi}$ (cf. [13]); the domain of z is extended to the whole sector $|\arg z| < 3\pi/2$ by rotating suitably the path of integration in (1.2.2).

We now state our main result in Part I.

Theorem 0. *Let α, β, μ and ν be real parameters with $\alpha > 0$ and $\beta > 0$, $q = e^{-t}$, and let $S_s(\alpha, \beta; \mu, \nu; q)$ be defined by (1.1.2) or (1.1.3). Then for any integer $K \geq 0$ and any complex t in the sector $|\arg t| < \pi/2$ the formula*

$$\begin{aligned}(1.2.3)\qquad S_s\big({\textstyle\genfrac{}{}{0pt}{}{\alpha,\beta}{\mu,\nu}};q\big) &= e(\alpha\mu + \beta\nu)\mathcal{B}_0(\beta, e(\nu))\Gamma(1-s)\phi(1-s,\alpha,\mu)t^{s-1} \\ &+ e(\alpha\mu+\beta\nu)\sum_{k=-1}^{K-1}\frac{(-1)^{k+1}\mathcal{B}_{k+1}(\alpha, e(\mu))}{(k+1)!}\phi(s-k,\beta,\nu)t^k \\ &+ R_{s,K}\big({\textstyle\genfrac{}{}{0pt}{}{\alpha,\beta}{\mu,\nu}};q\big)\end{aligned}$$

holds in the region $\operatorname{Re} s < K+1$ *except the points $s = k$ $(k = 0, 1, \ldots, K)$, where $\mathcal{B}_k(x, y)$ is defined by (1.2.1), and the empty sum is to be regarded as null. Here $R_{s,K}\big({\textstyle\genfrac{}{}{0pt}{}{\alpha,\beta}{\mu,\nu}};q\big)$ is the remainder term satisfying the estimate*

$$(1.2.4)\qquad R_{s,K}\big({\textstyle\genfrac{}{}{0pt}{}{\alpha,\beta}{\mu,\nu}};q\big) = O(|t|^K)$$

as $t \to 0$ through $|\arg t| \leq \pi/2 - \delta$ with any small $\delta > 0$, in the region $\operatorname{Re} s < K + 1$, *where the implied O-constant depends at most on s, K, α, β, μ, ν and δ. In particular* *when* $K \geq 1$, $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $0 \leq \mu \leq 1$ *and* $0 \leq \nu \leq 1$ *the explicit expression* (1.2.5)

$$\begin{aligned} R_{s,K}\left({\alpha,\beta \atop \mu,\nu};q\right) &= (-1)^K(2\pi)^{-s}t^{s-1}\Gamma(K+1-s) \\ &\times \Bigg\{ e^{\pi is/2}\sum_{m,n=0}^{\infty}{}' e(-\alpha m-\beta n)(\mu+m)^{-s}f_{s,K}(4\pi^2e^{-\pi i}(\mu+m)(\nu+n)/t) \\ &+ e^{-\pi is/2}\sum_{m,n=0}^{\infty}{}' e(\alpha(1+m)+\beta(1+n))(1-\mu+m)^{-s} \\ &\times f_{s,K}(4\pi^2e^{\pi i}(1-\mu+m)(1-\nu+n)/t) \\ &+ e^{\pi is/2}\sum_{m,n=0}^{\infty}{}' e(-\alpha m+\beta(1+n))(\mu+m)^{-s}f_{s,K}(4\pi^2(\mu+m)(1-\nu+n)/t) \\ &+ e^{-\pi is/2}\sum_{m,n=0}^{\infty}{}' e(\alpha(1+m)-\beta n)(1-\mu+m)^{-s}f_{s,K}(4\pi^2(1-\mu+m)(\nu+n)/t)\Bigg\} \end{aligned}$$

holds for $|\arg t| < \pi/2$, *in the region* $\mathrm{Re}\, s < K$, *where*

(1.2.6) $$f_{s,K}(z) = U(K+1-s;K+1-s;z)$$

with the confluent hypergeometric function defined by (1.2.2), *and the primed summation symbols indicate that the terms including* $\mu+m=0$ *or* $1-\mu+m=0$, *and* $\nu+n=0$ *or* $1-\nu+n=0$ *are to be omitted in they occur.*

Remark. Asymptotic expansions similar to (1.2.3) follow also for the exceptional points $s=k$ $(k=0,1,2,\ldots)$ as limiting cases of Theorem 0, whose important applications are included in these exceptional cases (see Theorems 1–5 below).

1.3. **Applications to q-factorials and allied functions.** It is seen from the relation $z\Phi(1,1,z) = -\log(1-z)$ for $|z|<1$ and (1.1.2) that

(1.3.1) $$S_1(\alpha,1;0,\nu;q) = -\log(e(\nu)q^{\alpha};q)_\infty,$$

and hence Theorem 0 yields

Theorem 1. *Let* α *and* ν *be real with* $\alpha > 0$ *and* $0<\nu<1$. *Then the following asymptotic expansions hold for any integer* $K \geq 1$ *and any complex* t *in* $|\arg t| < \pi/2$:

(1.3.2) $$\begin{aligned} \log(q^{\alpha};q)_\infty &= -\frac{\pi^2}{6t} - B_1(\alpha)\log t - \log\frac{\Gamma(\alpha)}{\sqrt{2\pi}} \\ &+ \frac{1}{4}B_2(\alpha)t - \sum_{k=2}^{K-1}\frac{(-1)^kB_kB_{k+1}(\alpha)}{k(k+1)!}t^k - R_{1,K}\left({\alpha,1 \atop 0,0};q\right); \end{aligned}$$

(1.3.3) $$\begin{aligned} \log(e(\nu)q^{\alpha};q)_\infty &= -\zeta_\nu(2)t^{-1} - B_1(\alpha)\{\log(2\sin\pi\nu)+\pi iB_1(\nu)\} \\ &+ \frac{1}{4}B_2(\alpha)(1-i\cot\pi\nu)t - \sum_{k=2}^{K-1}\frac{(-1)^k\mathcal{B}_k(0,e(\nu))B_{k+1}(\alpha)}{k(k+1)!}t^k \\ &- R_{1,K}\left({\alpha,1 \atop 0,\nu};q\right), \end{aligned}$$

where the remainder terms $R_{1,K}\left({\alpha,1 \atop 0,0};q\right)$ *and* $R_{1,K}\left({\alpha,1 \atop 0,\nu};q\right)$ *satisfy the same estimate as* (1.2.4) *when* $t \to 0$ *through the sector* $|\arg t| \leq \pi/2-\delta$ *with any small* $\delta>0$. *In* *particular if $K \geq 2$ and $0 < \alpha \leq 1$, the explicit expressions as in (1.2.5) follow for the remainder terms.*

Remark A complete asymptotic expansion of $(q^\alpha;q)_\infty$ as $q \to 1^-$ was first established by Moak [31] and later rederived by Ueno-Nishizawa [37] in a slightly different form from that of (1.3.2). McIntosh [25][27] proved (1.3.2) for real $t > 0$ with the error term $R_{1,K} = O(t^K)$ in a more general situation.

Corollary 1.1. *For any real $\alpha > 0$ and any integer $K \geq 1$ the formula*

$$\log(-q^\alpha;q)_\infty = \frac{\pi^2}{12t} - B_1(\alpha)\log 2 + \frac{1}{4}B_2(\alpha)t - \sum_{k=2}^{K-1} \frac{(-1)^k(2^k-1)B_k B_{k+1}(\alpha)}{k(k+1)!}t^k - R_{1,K}\left(\begin{smallmatrix}\alpha,1\\0,1/2\end{smallmatrix};q\right) \tag{1.3.4}$$

holds in $|\arg t| < \pi/2$, where the remainder term $R_{1,K}\left(\begin{smallmatrix}\alpha,1\\0,1/2\end{smallmatrix};q\right)$ satisfies the same estimate as (1.2.4). In particular if $0 < \alpha \leq 1$ and $K \geq 2$ the explicit expression as in (1.2.5) follows for the remainder term.

To describe the subsequent results, the change of the base

$$q = e^{-t} \longmapsto e^{-4\pi^2/t} = \widehat{q} \tag{1.3.5}$$

is frequently applied. Noting the facts

$$B_{2h+1} = 0, \qquad h = 1,2,\dots, \tag{1.3.6}$$

$$B_k(1-\alpha) = (-1)^k B_k(\alpha), \qquad k = 0,1,2\dots \tag{1.3.7}$$

(cf. [13]), we find that every term (with $k \geq 2$) of the series in (1.3.2) and (1.3.4) vanishes when $\alpha = 1$, and hence Theorem 0 further reduces to

Corollary 1.2. *The following formulae hold:*

$$\log(q;q)_\infty = -\frac{\pi^2}{6t} - \frac{1}{2}\log\frac{t}{2\pi} + \frac{t}{24} - \sum_{l=1}^{\infty} l^{-1}\frac{\widehat{q}^l}{1-\widehat{q}^l}, \tag{1.3.8}$$

or in exponential form

$$(q;q)_\infty = \sqrt{\frac{2\pi}{t}}\exp\left(-\frac{\pi^2}{6t} + \frac{t}{24}\right)(\widehat{q};\widehat{q})_\infty;$$

$$\log(-q;q)_\infty = \frac{\pi^2}{12t} - \frac{1}{2}\log 2 + \frac{t}{24} - \sum_{l=1}^{\infty} l^{-1}\frac{\widehat{q}^{l/2}}{1-\widehat{q}^l}, \tag{1.3.9}$$

or in exponential form

$$(-q;q)_\infty = \frac{1}{\sqrt{2}}\exp\left(\frac{\pi^2}{12t} + \frac{t}{24}\right)(\widehat{q}^{1/2};\widehat{q})_\infty.$$

Remark Formulae (1.3.8) and (1.3.9) are classic; these can be found for e.g., in [4, Chap. 3].

Remark. Formulae (1.3.8) and (1.3.9) both give complete (convergent) asymptotic expansions, since for instance the l-th term of the last infinite series in (1.3.8) is of order $\widehat{q}^l/l + O(\widehat{q}^{2l})$ as $l \to \infty$.

It can be observed that the explicit expression (1.2.5) for the remainder term, in certain specific cases (as in the preceding corollary), further reduces to complete (convergent) asymptotic expansions as $t \to 0$ in $|\arg t| < \pi/2$ (see Corollaries 1.3–1.5 below). If one considers, for instance, the logarithm of the pairing $(q^\alpha; q)_\infty (q^{1-\alpha}; q)_\infty$ with $0 < \alpha < 1$, each term (with $k \geq 2$) in its asymptotic series vanishes again by (1.3.6) and (1.3.7). From (1.2.5) and Theorem 1 we can in fact prove:

Corollary 1.3. *The following formula hold for any real* α *and* μ *with* $0 < \alpha < 1$ *and* $0 < \mu < 1$:

$$\log\{(q^\alpha; q)_\infty (q^{1-\alpha}; q)_\infty\} = -\frac{\pi^2}{3t} + \log(2\sin\pi\alpha) + \frac{1}{2}B_2(\alpha)t - \sum_{l=1}^{\infty} l^{-1}\frac{e((1-\alpha)l)\widehat{q}^l}{1-\widehat{q}^l} - \sum_{l=1}^{\infty} l^{-1}\frac{e(\alpha l)\widehat{q}^l}{1-\widehat{q}^l}, \tag{1.3.10}$$

or in exponential form

$$(q^\alpha; q)_\infty (q^{1-\alpha}; q)_\infty = 2(\sin\pi\alpha)\exp\Big\{-\frac{\pi^2}{3t} + \frac{1}{2}B_2(\alpha)t\Big\} \times (e(1-\alpha)\widehat{q}; \widehat{q})_\infty (e(\alpha)\widehat{q}; \widehat{q})_\infty;$$

$$\begin{aligned}\log\{(e(\mu)q^\alpha; q)_\infty (e(1-\mu)q^{1-\alpha}; q)_\infty\} &= -\{\zeta_\mu(2) + \zeta_{1-\mu}(2)\}t^{-1} \\ &\quad - 2\pi i B_1(\alpha)B_1(\mu) + \frac{1}{2}B_2(\alpha)t \\ &\quad - \sum_{l=1}^{\infty} l^{-1}\frac{e((1-\alpha)l)\widehat{q}^{\mu l}}{1-\widehat{q}^l} - \sum_{l=1}^{\infty} l^{-1}\frac{e(\alpha l)\widehat{q}^{(1-\mu)l}}{1-\widehat{q}^l},\end{aligned} \tag{1.3.11}$$

or in exponential form

$$\begin{aligned}(e(\mu)q^\alpha; q)_\infty (e(1-\mu)q^{1-\alpha}; q)_\infty &= \exp\Big[\{\zeta_\mu(2) + \zeta_{1-\mu}(2)\}t^{-1} - 2\pi i B_1(\alpha)B_1(\mu) \\ &\quad + \frac{1}{2}B_2(\alpha)t\Big](e(1-\alpha)\widehat{q}^\mu; \widehat{q})_\infty (e(\alpha)\widehat{q}^{1-\mu}; \widehat{q})_\infty.\end{aligned}$$

We can now restate Ramanujan's asymptotic formula (1.1.6)–(1.1.8) with explicit error terms. It is known that $F(q) = 1/(q;q)_\infty$ (cf. [38, pp.57–58], and the famous Rogers-Ramanujan identities assert that

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \quad \text{and} \quad H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

(cf. [2, (7.1.6) and (7.1.7)]). Formulae (1.3.8) and (1.3.10) therefore imply

Corollary 1.4. *The following formulae hold for* $F(q)$, $G(q)$ *and* $H(q)$ *defined by* (1.1.4) *and* (1.1.5):

$$F(q) = \Big(\frac{t}{2\pi}\Big)^{1/2} \exp\Big(\frac{\pi^2}{6t} - \frac{t}{24}\Big)\frac{1}{(\widehat{q}; \widehat{q})_\infty}, \tag{1.3.12}$$

or in logarithmic form

$$\log F(q)=\frac{\pi^2}{6t}+\frac{1}{2}\log\frac{t}{2\pi}-\frac{t}{24}+\sum_{l=1}^{\infty}l^{-1}\frac{\widehat{q}^l}{1-\widehat{q}^l};$$

$$G(q)=\left(\frac{2}{5-\sqrt{5}}\right)^{1/2}\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}\right)\frac{1}{(e(1/5)\widehat{q}^{1/5};\widehat{q}^{1/5})_\infty(e(4/5)\widehat{q}^{1/5};\widehat{q}^{1/5})_\infty}, \tag{1.3.13}$$

or in logarithmic form

$$\log G(q)=\frac{\pi^2}{15t}+\frac{1}{2}\log\left(\frac{2}{5-\sqrt{5}}\right)-\frac{t}{60}+\sum_{l=1}^{\infty}l^{-1}\frac{e(l/5)\widehat{q}^{l/5}}{1-\widehat{q}^{l/5}}+\sum_{l=1}^{\infty}l^{-1}\frac{e(4l/5)\widehat{q}^{l/5}}{1-\widehat{q}^{l/5}};$$

$$H(q)=\left(\frac{2}{5+\sqrt{5}}\right)^{1/2}\exp\left(\frac{\pi^2}{15t}+\frac{11t}{60}\right)\frac{1}{(e(2/5)\widehat{q}^{1/5};\widehat{q}^{1/5})_\infty(e(3/5)\widehat{q}^{1/5};\widehat{q}^{1/5})_\infty}, \tag{1.3.14}$$

or in logarithmic form

$$\log H(q)=\frac{\pi^2}{15t}+\frac{1}{2}\log\left(\frac{2}{5+\sqrt{5}}\right)+\frac{11t}{60}+\sum_{l=1}^{\infty}l^{-1}\frac{e(2l/5)\widehat{q}^{l/5}}{1-\widehat{q}^{l/5}}+\sum_{l=1}^{\infty}l^{-1}\frac{e(3l/5)\widehat{q}^{l/5}}{1-\widehat{q}^{l/5}}.$$

We next mention slightly different type of implications from Theorem 1. To this aim several necessary terminologies are prepared. The q-gamma and q-beta functions are defined respectively by

$$\Gamma_q(\alpha)=\frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-\alpha}\qquad\text{and}\qquad B_q(\alpha,\beta)=\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)},$$

whose limits as $q\to 1^-$ are known to be the ordinary gamma function and the beta function $B(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, respectively (cf. [16]). Whilst the basic hypergeometric function ${}_2\phi_1(a,b;c;q,z)$ is defined by

$${}_2\phi_1(a,b;c;q,z)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}z^n,\qquad |z|<1,$$

for any complex a, b and c with $c\neq q^{-n}$ $(n=0,1,2,\ldots)$, whose particular case $a=q^\alpha$, $b=q^\beta$ and $c=q^\gamma$ gives a q-analogue of Gauss' hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;z)$ (cf. [16, 1.2]). It is known that the classical Gauss' and Kummer's summation formulae

$${}_2F_1(\alpha,\beta;\gamma;1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

where $\mathrm{Re}(\gamma-\alpha-\beta)>0$, $\gamma\neq -n$ $(n=0,1,2,\ldots)$, and

$${}_2F_1(\alpha,\beta;1+\alpha-\beta;-1)=\frac{\Gamma(1+\alpha-\beta)\Gamma(1+\alpha/2)}{\Gamma(1+\alpha)\Gamma(1+\alpha/2-\beta)},$$

where $1+\alpha-\beta\neq -n$ $(n=0,1,2,\ldots)$, have q-analogues of the form

$${}_2\phi_1(q^\alpha,q^\beta;q^\gamma;q,q^{\gamma-\alpha-\beta})=\frac{(q^{\gamma-\alpha};q)_\infty(q^{\gamma-\beta};q)_\infty}{(q^\gamma;q)_\infty(q^{\gamma-\alpha-\beta};q)_\infty},$$

$${}_2\phi_1(q^\alpha,q^\beta;q^{1+\alpha-\beta};q,-q^{1-\beta})=\frac{(-q;q)_\infty(q^{1+\alpha};q^2)_\infty(q^{2+\alpha-2\beta};q^2)_\infty}{(q^{1+\alpha-\beta};q)_\infty(-q^{1-\beta};q)_\infty}$$

respectively (cf. [16, 1.5; 1.8]). Combining formulae (1.3.2) and (1.3.4) with appropriate exponents (in place of α) we can prove

Corollary 1.5. *Let α, β, γ be positive real numbers. Then the following formulae hold for any integer $K \geq 1$ when $t \to 0$ through $|\arg t| \leq \pi/2 - \delta$ with any small $\delta > 0$:*

$$\log \Gamma_q(\alpha) = \log \Gamma(\alpha) - \frac{1}{4}(\alpha-1)(\alpha-2)t + \sum_{k=2}^{K-1} \frac{B_k}{kk!}\left\{\frac{(-1)^k B_{k+1}(\alpha)}{k+1} + 1 - \alpha\right\}t^k + O(|t|^K)$$

for $\alpha > 0$;

$$\log B_q(\alpha,\beta) = \log B(\alpha,\beta) + \frac{1}{2}(\alpha\beta - 1)t + \sum_{k=2}^{K-1} \frac{B_k}{kk!}\left\{\frac{(-1)^k C_{k+1}(\alpha,\beta)}{k+1} + 1\right\}t^k + O(|t|^K)$$

for $\alpha > 0$ and $\beta > 0$, where

$$C_k(\alpha,\beta) = B_k(\alpha) + B_k(\beta) - B_k(\alpha+\beta);$$

$$\log {}_2\phi_1(q^\alpha, q^\beta; q^\gamma; q, q^{\gamma-\alpha-\beta}) = \log {}_2F_1(\alpha,\beta;\gamma;1) - \frac{1}{2}\alpha\beta t - \sum_{k=2}^{K-1} \frac{(-1)^k B_k D_{k+1}(\alpha,\beta,\gamma)}{k(k+1)!}t^k + O(|t|^K)$$

for $\gamma - \alpha > 0$, $\gamma - \beta > 0$, $\gamma > 0$ and $\gamma - \alpha - \beta > 0$, where

$$D_k(\alpha,\beta,\gamma) = B_k(\gamma-\alpha) + B_k(\gamma-\beta) - B_k(\gamma) - B_k(\gamma-\alpha-\beta);$$

$$\log {}_2\phi_1(q^\alpha, q^\beta; q^{1+\alpha-\beta}; q, -q^{1-\beta}) = \log {}_2F_1(\alpha,\beta; 1+\alpha-\beta; -1) - \sum_{k=2}^{K-1} \frac{(-1)^k B_k E_{k+1}(\alpha,\beta)}{k(k+1)!}t^k + O(|t|^K)$$

for $1+\alpha > 0$, $2+\alpha-2\beta > 0$, $1+\alpha-\beta > 0$ and $1-\beta > 0$, where

$$E_k(\alpha,\beta) = 2^{k-1}B_k(\alpha/2+1/2) + 2^{k-1}B_k(1+\alpha/2-\beta) - B_k(1+\alpha-\beta) - (2^{k-1}-1)B_k(1-\beta).$$

Here the implied O-constants depend at most on K, α, β, γ and δ.

1.4. **Connections with Ramanujan's formula for $\zeta(2n+1)$.** We next describe that our main theorem implies Ramanujan's formula for $\zeta(2n+1)$ and its several variants. In order to clarify symmetricity of the following results we introduce the new parameter $\tau = t/2\pi$. Then the case $\alpha = \beta = 1$, $\lambda = \mu = 0$ and $s = 2n+1$ ($n = \pm1, \pm2, \ldots$) of Theorem 0 reduces to the following equivalent form of (1.1.9).

Theorem 2 (Ramanujan). *Let $q = e^{-2\pi\tau}$ and $\widehat{q} = e^{-2\pi/\tau}$ with $\operatorname{Re}\tau > 0$. Then for any integer $n \neq 0$ the formula*

$$\begin{aligned}(1.4.1)\qquad & S_{2n+1}\left(\begin{smallmatrix}1,1\\0,0\end{smallmatrix};q\right) + \frac{1}{2}\zeta(2n+1) + \frac{1}{2}(2\pi)^{2n+1}\sum_{k=0}^{n+1}\frac{(-1)^k B_{2n+2-2k}B_{2k}}{(2n+2-2k)!(2k)!}\tau^{2n+1-2k}\\ &= (-1)^n\tau^{2n}\left\{S_{2n+1}\left(\begin{smallmatrix}1,1\\0,0\end{smallmatrix};\widehat{q}\right) + \frac{1}{2}\zeta(2n+1)\right\}\end{aligned}$$

holds.

Theorem 0 further yields the following several variants of (1.1.9).

Theorem 3. *Let q and $\widehat{q}$ be as in Theorem 2. Then the following formulae hold for any integer n and any real α and μ with $0 < \alpha < 1$ and $0 < \mu < 1$:*

$$\begin{aligned}(1.4.2)\qquad & S_{2n+1}\left(\begin{smallmatrix}\alpha,1\\0,\mu\end{smallmatrix};q\right) + S_{2n+1}\left(\begin{smallmatrix}1-\alpha,1\\0,1-\mu\end{smallmatrix};q\right) + (2\pi)^{2n+1}\sum_{k=0}^{2n+2}\frac{(-i)^k B_{2n+2-k}(\alpha)B_k(\mu)}{(2n+2-k)!k!}\tau^{2n+1-k}\\ &= (-1)^n\tau^{2n}\left\{S_{2n+1}\left(\begin{smallmatrix}\mu,1\\0,1-\alpha\end{smallmatrix};\widehat{q}\right) + S_{2n+1}\left(\begin{smallmatrix}1-\mu,1\\0,\alpha\end{smallmatrix};\widehat{q}\right)\right\};\end{aligned}$$

$$\begin{aligned}(1.4.3)\qquad & S_{2n}\left(\begin{smallmatrix}\alpha,1\\0,\mu\end{smallmatrix};q\right) - S_{2n}\left(\begin{smallmatrix}1-\alpha,1\\0,1-\mu\end{smallmatrix};q\right) - (2\pi)^{2n}\sum_{k=0}^{2n+1}\frac{(-i)^k B_{2n+1-k}(\alpha)B_k(\mu)}{(2n+1-k)!k!}\tau^{2n-k}\\ &= i(-1)^n\tau^{2n-1}\left\{S_{2n}\left(\begin{smallmatrix}\mu,1\\0,1-\alpha\end{smallmatrix};\widehat{q}\right) - S_{2n}\left(\begin{smallmatrix}1-\mu,1\\0,\alpha\end{smallmatrix};\widehat{q}\right)\right\},\end{aligned}$$

where $B_k(x)$ denotes the k-th Bernoulli polynomial.

Remark. Eie and Chen [12] recently obtained the same formula as (1.4.2) in a quite different manner, basing on their theorems for multiple zeta functions associated with polynomials.

Theorem 4. *Let q and $\widehat{q}$ be as in Theorem 2. Then the following formulae hold for any integer n and any real β and λ with $0 < \beta < 1$ and $0 < \lambda < 1$:*

$$\begin{aligned}(1.4.4)\qquad & S_{2n+1}\left(\begin{smallmatrix}1,\beta\\\lambda,0\end{smallmatrix};q\right) + S_{2n+1}\left(\begin{smallmatrix}1,1-\beta\\1-\lambda,0\end{smallmatrix};q\right) + \zeta(2n+1,\beta)\\ &\quad + (2\pi)^{2n+1}\sum_{k=0}^{2n+2}\frac{i^k\mathcal{B}_{2n+2-k}(0,e(\lambda))\mathcal{B}_k(0,e(\beta))}{(2n+2-k)!k!}\tau^{2n+1-k}\\ &= (-1)^n\tau^{2n}\left\{S_{2n+1}\left(\begin{smallmatrix}1,\lambda\\1-\beta,0\end{smallmatrix};\widehat{q}\right) + S_{2n+1}\left(\begin{smallmatrix}1,1-\lambda\\\beta,0\end{smallmatrix};\widehat{q}\right) + \zeta(2n+1,1-\lambda)\right\}\end{aligned}$$

except when $n = 0$;

$$\begin{aligned}(1.4.5)\qquad & S_{2n}\left(\begin{smallmatrix}1,\beta\\\lambda,0\end{smallmatrix};q\right) - S_{2n}\left(\begin{smallmatrix}1,1-\beta\\1-\lambda,0\end{smallmatrix};q\right) + \zeta(2n,\beta)\\ &\quad - (2\pi)^{2n}\sum_{k=0}^{2n+1}\frac{i^k\mathcal{B}_{2n+1-k}(0,e(\lambda))\mathcal{B}_k(0,e(\beta))}{(2n+1-k)!k!}\tau^{2n-k}\\ &= i(-1)^n\tau^{2n-1}\left\{S_{2n}\left(\begin{smallmatrix}1,\lambda\\1-\beta,0\end{smallmatrix};\widehat{q}\right) - S_{2n}\left(\begin{smallmatrix}1,1-\lambda\\\beta,0\end{smallmatrix};\widehat{q}\right) - \zeta(2n,1-\lambda)\right\},\end{aligned}$$

where $\mathcal{B}_k(x,y)$ is defined by (1.2.1).

Part II: Asymptotics for multiple q-integrals and q-differentials

2.1. **Introduction (II).** Suppose temporarily that q is a real parameter with $0 < q < 1$. Let $\varphi(u)$ be a function integrable on the interval $[0, x]$. A q-analogue of the ordinary integral $\int_0^x \varphi(u)du$, in the form

$$\int_0^x \varphi(u)d_q u = (1-q)x \sum_{n=0}^{\infty} \varphi(q^n x)q^n, \tag{2.1.1}$$

was introduced by Thomae [34] in 1869 and studied by Jackson [19] during 1910–1951 (see also [16, p.23, Chap.1, 1.11]). The formulation in (2.1.1) is motivated from the fact that

$$\lim_{q\to 1^-} \int_0^x \varphi(u)d_q u = \int_0^x \varphi(u)du \tag{2.1.2}$$

holds for all $\varphi(u)$ continuous on $[0, x]$. On the other hand, a q-analogue of the ordinary differentiation is formulated as

$$\partial_{q,z}\psi(z) = \frac{\psi(z) - \psi(qz)}{(1-q)z} \tag{2.1.3}$$

(cf. [16, p.27, 1.12]), which asserts that

$$\lim_{q\to 1^-} \partial_{q,z}\psi(z) = \psi'(z) = \partial_z \psi(z), \tag{2.1.4}$$

say, for all $\psi(z)$ complex differentiable at z.

Throughout the following, q is a complex parameter with $0 < |q| < 1$, and the substitution $q = e^{-t}$ will be made if necessary, upon transforming the half-plane $\mathrm{Re}\, t > 0$ to the unit disk $|q| < 1$. A complex domain $D \subset \mathbb{C}$ is called *star-shaped* if $0 \in D$ and for any $z \in D$ the line segment $\overline{0, z}$ is included in D. We suppose throughout that $f(z)$ is a function holomorphic in a star-shaped domain D, and ρ_f denotes the distance between 0 and the singularity of $f(z)$ being closest to 0.

We introduce the q-integral and q-differential operators $\mathcal{I}_{q,z}^x$ and $\mathcal{D}_{q,z}^y$ defined for any real $x > 0$ and $y \geq 0$ by

$$\mathcal{I}_{q,z}^x f(z) = \int_0^1 u^{x-1} f(uz)d_q u = z^{-x} \int_0^z w^{x-1} f(w)d_q w, \tag{2.1.5}$$

$$\mathcal{D}_{q,z}^y f(z) = \frac{f(z) - q^y f(qz)}{1-q} = z^{-y}(z\partial_{q,z})\{z^y f(z)\} \tag{2.1.6}$$

for any z in $|z| < \rho_f$, where the latter equalities follow from (2.1.1) and (2.1.3) respectively.

Remark. If the base q is restricted to the range $0 < q < 1$, then the domain of z in which the definitions in (2.1.5) and (2.1.6) are valid is extended to the whole D by its star-shapedness.

Proposition 1. *The operator relations*

$$\mathcal{I}_{q,z}^x \mathcal{D}_{q,z}^x = 1 \qquad \text{and} \qquad \mathcal{D}_{q,z}^x \mathcal{I}_{q,z}^x = 1$$

hold for any $x > 0$, where 1 denotes the identity operation.

It is the main aim of Part II to pursue the directions in (2.1.2) and (2.1.4) further; this leads us to show that complete asymptotic expansions as $t \to 0$ through the sector

$|\arg t| < \pi/2$ exist for the multiple q-integrals $(\mathcal{I}_{q,z}^x)^r f(q^y z)$ (Theorem 5) and the multiple q-differentials $(\mathcal{D}_{q,z}^x)^r f(q^y z)$ (Theorem 6) with any integer $r \geq 1$, under fairly generic situations. A full extension of the domain of z in which Theorems 5 and 6 are valid is possible if $0 < q < 1$ (Theorem 7). Several applications of our main formulae (2.2.4) and (2.2.9) will further be given for the Hurzitz-Lerch zeta-function (Theorems 8 and 9), q-factorials (Corollary 8.1), and q-analogues of the exponential functions (Corollary 8.2), of the binomial functions (Corollary 8.3), and of the poly-logarithmic functions (Corollaries 8.4 and 9.1). As for methodology, it is fundamental to apply a Mellin transform technique in the proofs of Theorems 5 and 6.

2.2. **The main theorems (II).** Let r be any integer, and w a complex variable. To describe our results we introduce the functions $A_{f,k}(x,z)$ and Nörlund's generalized Bernoulli polynomials $B_k^{(r)}(y)$ of rank r (cf. [32]) defined respectively for $k = 0, 1, \ldots$ by the Taylor series expansions

$$e^{xw} f(e^w z) = \sum_{k=0}^{\infty} \frac{A_{f,k}(x,z)}{k!} w^k, \tag{2.2.1}$$

$$e^{yw}\left(\frac{w}{e^w - 1}\right)^r = \sum_{k=0}^{\infty} \frac{B_k^{(r)}(y)}{k!} w^k \tag{2.2.2}$$

near $w = 0$. Note that $B_k^{(1)}(y) = B_k(y)$ is the usual Bernoulli polynomial, and so $B_k(0) = B_k$ is the usual Bernoulli number. We write $B_k^{(r)}(0) = B_k^{(r)}$, and use Euler's differential operator $\vartheta_z = z\partial_z$.

We state our first main result in Part II.

Theorem 5. *Let x and y be real parameters with $x > 0$ and $y \geq 0$, $q = e^{-t}$, and $r \geq 1$ an arbitrary fixed integer. Further let $(\mathcal{I}_{q,z}^x)^r f(z)$ denote the r-times iterated operation of* (2.1.5) *to any function $f(z)$ holomorphic in a star-shaped domain D, and define the coefficients $A_{f,-j}(x,z)$ $(j = 1, 2, \ldots)$ by*

$$A_{f,-j}(x,z) = \int_0^1 u_j^{x-1} \int_0^1 u_{j-1}^{x-1} \cdots \int_0^1 u_1^{x-1} f(u_1 \cdots u_j z) du_1 \cdots du_j. \tag{2.2.3}$$

Then for any integer $K \geq 0$ the formula

$$\begin{aligned} \frac{q^{xy}}{(1-q)^r} (\mathcal{I}_{q,z}^x)^r f(q^y z) = \sum_{j=1}^{r} & \frac{(-1)^{r-j} A_{f,-j}(x,z) B_{r-j}^{(r)}(y)}{(r-j)!} t^{-j} \\ & + \sum_{k=0}^{K-1} \frac{(-1)^{r+k} A_{f,k}(x,z) B_{r+k}^{(r)}(y)}{(r+k)!} t^k + R_{f,K}^{(r)}(x,y;q,z) \end{aligned} \tag{2.2.4}$$

holds in the sector $|\arg t| < \pi/2$ and on the disk $|z| < \rho_f$. Here $R_{f,K}^{(r)}$ is the remainder term expressed by a certain inverse Mellin transform, and satisfies the estimate

$$R_{f,K}^{(r)}(x,y;q,z) = O(|t|^K) \tag{2.2.5}$$

as $t \to 0$ through $|\arg t| \leq \pi/2 - \delta$ with any small $\delta > 0$, where the implied O-constant depends at most on r, x, y, z, K and δ. In particular if $0 \leq y \leq r$ and $K \geq 1$ the *representation*

$$(2.2.6)\qquad R_{f,K}^{(r)}(x,y;q,z)=(-t)^K\sum_{l=0}^{r-1}\frac{(-1)^{r-1-l}B_{r-1-l}^{(r)}(y)}{l!(r-1-l)!}\sum_{n=-\infty}^{\infty}{}'\frac{e(ny)}{(2\pi in)^{K+l}}\times\Big(\frac{\partial}{\partial u}\Big)^l u^{K+l}\int_0^1\xi^{xtu+2\pi in-1}(x+\vartheta_z)^K f(\xi^{tu}z)d\xi\Big|_{u=1}$$

follows, where the primed summation symbol indicates that the term with $n=0$ is to be omitted. with $n=0$.

Remark 3. The explicit expression (2.2.6) will be used to extend the domain of z where (2.2.4) with (2.2.5) is valid (see Theorem 7).

From a point of view of applications it is necessary to establish the asymptotic expansions for $(\mathcal{I}_{q,z}^x)^r f(z)$ both with and without the associated q-multiples (see (2.3.5), (2.3.11) and (2.3.12) below). The case $y=0$ of Theorem 5 in fact yields, in view of the latter equality in (2.1.5), the following corollary.

Corollary 5.1. *Let r and x be as in Theorem 5. Then for any integer $K\geq 0$ the asymptotic formula*

$$(2.2.7)\qquad \int_0^z w_r^{-1}\int_0^{w_r}w_{r-1}^{-1}\cdots w_2^{-1}\int_0^{w_2}w_1^{x-1}f(w_1)d_qw_1\cdots d_qw_r =\sum_{k=0}^{K-1}\frac{(-1)^k C_{f,k}(x,z)}{k!}t^k+O(|t|^K)$$

holds as $t\to 0$ through $|\arg t|\leq\pi/2-\delta$ for any small $\delta>0$, on the disk $|z|<\rho_f$ with $|\arg z|<\pi$, where the implied O-constant depends at most on x, z, K and δ. Here the coefficients $C_{f,k}^{(r)}$ $(k=0,1,\dots)$ are given by

$$(2.2.8)\qquad C_{f,k}^{(r)}(x,z)=\sum_{j=\max(1,r-k)}^{r}\binom{k}{r-j}B_{k-r+j}^{(-r)}B_{r-j}^{(r)}\times\int_0^z w_j^{-1}\int_0^{w_j}w_{j-1}^{-1}\cdots w_2^{-1}\int_0^{w_2}w_1^{x-1}f(w_1)dw_1\cdots dw_j +\sum_{j=0}^{k-r}\binom{k}{r+j}B_{k-r-j}^{(-r)}B_{r+j}^{(r)}\vartheta_z^j\{z^x f(z)\},$$

which is reduced if $r=1$ to

$$C_{f,k}^{(1)}(x,z)=\frac{1}{k+1}\Big[\int_0^z w^{x-1}f(w)dw+\sum_{j=0}^{k-1}\binom{k+1}{j+1}B_{j+1}\vartheta_z^j\{z^x f(z)\}\Big],$$

where the empty sums are to be regarded as null.

The case $K=1$ of Corollary 5.1 implies the following.

Corollary 5.2. *Under the same assumptions as in Corollary 5.1 we have the limiting relation*

$$\lim_{\substack{q\to 1\\ |q|<1}} \int_0^z w_r^{-1}\int_0^{w_r} w_{r-1}^{-1}\cdots w_2^{-1}\int_0^{w_2} w_1^{x-1}f(w_1)d_qw_1\cdots d_qw_r$$
$$= C_{f,0}^{(r)}(x,z) = \int_0^z w_r^{-1}\int_0^{w_r} w_{r-1}^{-1}\cdots w_2^{-1}\int_0^{w_2} w_1^{x-1}f(w_1)dw_1\cdots dw_r.$$

We proceed to state our second main result in Part II. For this, let $\Gamma(s)$ denote the gamma function, and $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n the rising factorial.

Theorem 6. *Let $x \geq 0$ and $y \geq 0$ be real parameters, $q = e^{-t}$, and $r \geq 1$ an arbitrarily fixed integer. Further let $(\mathcal{D}_{q,z}^x)^r f(z)$ denote the r-times iterated operation of (2.1.6) to any function $f(z)$ holomorphic in a star-shaped domain D. Then for any integer $K \geq 0$ the formula*

$$(2.2.9)\quad q^{xy}\Big(\frac{1-q}{t}\Big)^r(\mathcal{D}_{q,z}^x)^r f(q^y z) = \sum_{k=0}^{K-1}\frac{(-1)^k A_{f,r+k}(x,z)B_k^{(-r)}(y)}{k!}t^k + R_{f,K}^{(-r)}(x,y;q,z)$$

holds in the sector $|\arg t| < \pi/2$ and on the disk $|z| < \rho_f$. Here $R_{f,K}^{(-r)}$ is the remainder term expressed by a certain inverse Mellin transform, and satisfies the estimate

$$(2.2.10)\qquad R_{f,K}^{(-r)}(x,y;q,z) = O(|t|^K)$$

as $t \to 0$ through $|\arg t| \leq \pi/2 - \delta$ with any small $\delta > 0$, where the implied O-constant depends at most on r, x, y, z, K and δ. Furthermore, for any real $x \geq 0$ and $y \geq 0$, and any integer $K \geq 0$,

$$(2.2.11)\quad R_{f,K}^{(-r)}(x,y;q,z) = \frac{(-1)^{r+K}t^K}{\Gamma(r+K)}\sum_{n=0}^{r}\frac{(-r)_n}{n!}(y+n)^{r+K}\int_0^1(1-\xi)^{r+K-1}q^{x(y+n)\xi}$$
$$\times (x+\vartheta_z)^{r+K}f(q^{(y+n)\xi}z)d\xi.$$

In view of the latter equality in (2.1.6), the case $y = 0$ of Theorem 6 in fact yields the following corollary.

Corollary 6.1. *Let r and x be as in Theorem 6. Then for any integer $K \geq 0$ the asymptotic formula*

$$(2.2.12)\qquad (z\partial_{q,z})^r\{z^x f(z)\} = \sum_{k=0}^{K-1}\frac{(-1)^k C_{f,k}^{(-r)}(x,z)}{k!}t^k + O(|t|^K)$$

holds as $t \to 0$ through $|\arg t| \leq \pi/2 - \delta$ for any small $\delta > 0$, on the disk $|z| < \rho_f$ with $|\arg z| < \pi$, where the implied O-constant depends at most on r, x, z, K and δ. Here the coefficients $C_{f,k}^{(-r)}$ $(k = 0, 1, \ldots)$ are given by

$$(2.2.13)\qquad C_{f,k}^{(-r)}(x,z) = \sum_{j=0}^{k}\binom{k}{j}B_{k-j}^{(r)}B_j^{(-r)}\vartheta_z^{r+j}\{z^x f(z)\},$$

which reduces if $r = 1$ to

$$C_{f,k}^{(-1)}(x,z) = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j+1}B_{k-j}\vartheta_z^{1+j}\{z^x f(z)\}.$$

The case $K=1$ of Corollary 6.1 implies the following corollary.

Corollary 6.2. *Under the same assumptions as in Corollary 6.1 we have the limiting relation*

$$\lim_{\substack{q\to 1\\ |q|<1}}(z\partial_{q,z})^r f(z)=C_{f,0}^{(-r)}(x,z)=(z\partial_z)^r\{z^x f(z)\}.$$

We lastly proceed to state the full extension of the domain of z in Theorems 5 and 6 under the restriction that $0<q<1$ (see Remark just below of (2.1.6)).

Theorem 7. *Set $q=e^{-t}$ with any real $t>0$, and let $f(z)$ be any function holomorphic in a star-shaped domain D.*

i) *Let x and y be real with $x>0$ and $0\leq y\leq r$. Then the asymptotic expansion* (2.2.4) *with the estimate* (2.2.5) *when $t\to 0^+$, as well as the explicit expression* (2.2.6), *remain valid throughout the domain D*;

ii) *Let $x\geq 0$ and $y\geq 0$ be real. Then the asymptotic expansion* (2.2.9) *with the estimate* (2.2.10) *when $t\to 0^+$, as well as the explicit expression* (2.11), *remain valid throughout the domain D*;

iii) *The asymptotic expansion* (2.2.7) *with* (2.2.8) *when $t\to 0^+$ for $x>0$, and also* (2.2.12) *with* (2.2.13) *when $t\to 0^+$ for $x\geq 0$, remain valid both throughout the domain D.*

2.3. **Applications of Theorems 5 and 6.** We suppose throughout this section that $0<q<1$. Let $[s]_q=(1-q^s)/(1-q)$ be a q-analogue of s, and $[s]_{q;n}=\prod_{m=0}^{n-1}[s+m]_q$ and $[1]_{q;n}=[n]_q!$ for $n=0,1,\ldots$ denote q-analogues of the rising factorial and the factorial of n respectively (cf. [16, p.7, Chap.1]), where the empty products are regarded to be 1. Note that the limiting relation $\lim_{q\to 1^-}[s]_q=s$ implies that

$$\lim_{q\to 1^-}[s]_{q;n}=(s)_n \qquad \text{and} \qquad \lim_{q\to 1^-}[n]_q!=n!. \tag{2.3.1}$$

Recall that the generalized Lerch zeta-function $\Phi(s,x,z)$ is defined by

$$\Phi(s,x,z)=\sum_{m=0}^{\infty}(x+m)^{-s}z^m \tag{2.3.2}$$

for any complex s if $|z|<1$, and for $\operatorname{Re} s>1$ if $|z|=1$ (cf. [13]); this is continued to a holomorphic function of $(s,z)\in\mathbb{C}\times D$, where

$$D=\{z\in\mathbb{C}\mid |\arg(1-z)|<\pi\}=\mathbb{C}\setminus[1,+\infty) \tag{2.3.3}$$

is a complex cut-plane; note here that D is a star-shaped domain. We can therefore apply the part i) of Theorem 7 (upon (2.2.4) with (2.2.5)) to $f(z)=\Phi(s,x,z)$, and obtain the following theorem.

Theorem 8. *Let x and y be real with $x>0$ and $0\leq y\leq r$, and s any complex. Then for any integer $K\geq 0$ the asymptotic expansion*

$$\begin{aligned}\frac{q^{xy}}{(1-q)^r}(\mathcal{I}_{q,z}^x)^r\Phi(s,x,q^y z)&=\sum_{j=1}^{r}\frac{(-1)^{r-j}\Phi(s+j,x,z)B_{r-j}^{(r)}(y)}{(r-j)!}t^{-j}\\&\quad+\sum_{k=0}^{K-1}\frac{(-1)^{r+k}\Phi(s-k,x,z)B_{r+k}^{(r)}(y)}{(r+k)!}t^k+O(t^K)\end{aligned} \tag{2.3.4}$$

holds as $t\to 0^+$, in $|\arg(1-z)|<\pi$, where the implied O-constant depends at most on r, s, x, y, z, and K.

Let $\mathrm{Li}_l(z)$ for any $l \in \mathbb{Z}$ be the poly-logarithmic function defined by $\mathrm{Li}_l(z) = z\Phi(l,1,z)$ for any $z \in D$. It is seen from (2.1.1), (2.1.5), (2.1.8) and the relation $\log(1-z) = -z\Phi(1,1,z)$, by (2.3.2), that

$$\log(q^y z; q)_\infty = -\frac{q^y z}{1-q}\mathcal{I}^1_{q,z}\Phi(1,1,q^y z) \tag{2.3.5}$$

for any real $y \geq 0$ and in $|\arg(1-z)| < \pi$. Then the case $(r,s,x) = (1,1,1)$ of Theorem 7 yields the following corollary.

Corollary 8.1. *Let y be real with $0 \leq y \leq 1$. Then for any integer $K \geq 0$ the asymptotic expansion*

$$\log(q^y z; q)_\infty = -\mathrm{Li}_2(z)t^{-1} - \sum_{k=0}^{K-1} \frac{(-1)^{k+1}\mathrm{Li}_{1-k}(z)B_{k+1}(y)}{(k+1)!}t^k + O(t^K) \tag{2.3.6}$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, where the implied O-constant depends at most on y, z and K.

Remark The assertion (2.3.6) was first established by McIntosh [25][27] in a more general setting.

We next present the applications to q-analogues of the exponential and binomial functions defined respectively by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \qquad \left(|z| < \frac{1}{1-q}\right),$$

$$f_q(y;z) = \sum_{n=0}^{\infty} \frac{[y]_{q;n}}{[n]_q!}z^n \qquad (|z| < 1),$$

from which with (3.1) the limiting relations $\lim_{q\to 1^-} e_q(z) = e^z$ and $\lim_{q\to 1^-} f_q(y;z) = (1-z)^{-y}$ follow. It is known that the q-binomial theorem (cf. [16, p.8, Chap.1, 1.3]) asserts that

$$e_q(z) = \frac{1}{((1-q)z;q)_\infty} \qquad \text{and} \qquad f_q(y;z) = \frac{(q^y z;q)_\infty}{(z;q)_\infty} \tag{2.3.7}$$

for any $y \geq 0$; these further provide the meromorphic continuations of $e_q(z)$ and $f_q(y;z)$ respectively over the whole z-plane.

Corollary 8.1 can therefore be applied to the right sides above on yielding the following corollaries.

Corollary 8.2. *For any integer $K \geq 0$ the asymptotic expansion*

$$\log e_q(z) = z + \sum_{k=1}^{K-1} \alpha_k(z)t^k + O(t^K) \tag{2.3.8}$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, and this further implies that

$$e_q(z) = e^z\left\{1 + \sum_{k=1}^{K-1} \beta_k(z)t^k + O(t^K)\right\}$$

as $t \to 0^+$, where the coefficients $\alpha_k(z)$ and $\beta_k(z)$ are given by

$$\alpha_k(z) = \sum_{j=0}^{k} \frac{(-1)^{k-j} B_{k-j}}{(k-j)!} \sum_{h=0}^{j} (1+h)^{k-j-2} \frac{B_{j-h}^{(-h-1)} z^{1+h}}{(j-h)!}, \tag{2.3.9}$$

$$\beta_k(z) = \sum_{\substack{\sum_{j=1}^{k} j l_j = k \\ l_j \geq 0\ (j=1,\ldots,k)}} \prod_{j=1}^{k} \frac{\alpha_j(z)^{l_j}}{l_j!}$$

for $k = 0, 1, \ldots$, and the implied O-constants depend on z and K.

Corollary 8.3. *Let y be real with $0 \leq y \leq 1$. Then for any integer $K \geq 0$ the asymptotic expansion*

$$\log f_q(y; z) = \sum_{k=0}^{K-1} \frac{(-1)^{k+1} \operatorname{Li}_{1-k}(z)}{(k+1)!} \{B_{k+1} - B_{k+1}(y)\} t^k + O(t^K)$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, and this further implies that

$$f_q(y; z) = (1-z)^{-y} \left\{ 1 + \sum_{k=1}^{K-1} \gamma_k(y, z) t^k + O(t^K) \right\}$$

as $t \to 0^+$, where the coefficients $\gamma_k(y, z)$ are given by

$$\gamma_k(y, z) = (-1)^k \sum_{\substack{\sum_{j=1}^{k} j l_j = k \\ l_j \geq 0\ (j=1,\ldots,k)}} \prod_{j=1}^{k} \frac{1}{l_j!} \left[\frac{\operatorname{Li}_{1-j}(z)}{(j+1)!} \{B_{j+1}(y) - B_{j+1}\} \right]^{l_j}$$

for $k = 0, 1, \ldots$. Here the implied O-constants depend at most on y, z and K.

We thirdly present applications to a q-analogue of the poly-logarithmic function $\operatorname{Li}_{q,l}(z)$ for any $l \in \mathbb{Z}$ defined by

$$\operatorname{Li}_{q,l}(z) = \sum_{m=0}^{\infty} \frac{z^{1+m}}{[1+m]_q^l} \qquad (|z| < 1), \tag{2.3.10}$$

which with (2.3.1) asserts that $\lim_{q \to 1^-} \operatorname{Li}_{q,l}(z) = \operatorname{Li}_l(z)$. We can in fact show

$$\operatorname{Li}_{q,r}(z) = z(\mathcal{I}_{q,z}^1)^r \Phi(0, 1, z) \tag{2.3.11}$$

for any integer $r \geq 0$; this further provides the meromorphic continuation of $\operatorname{Li}_{q,r}(z)$ for all $z \in D$. Corollary 5.1 can therefore be applied upon taking $f(z) = \Phi(0, 1, z)$ to yield the following corollary.

Corollary 8.4. *Let $r \in \mathbb{Z}$ be arbitrarily fixed with $r \geq 1$. Then for any integer $K \geq 0$ the asymptotic expansion*

$$\operatorname{Li}_{q,r}(z) = \sum_{k=0}^{K-1} \frac{(-1)^k C_{f,k}^{(r)}(1, z)}{k!} t^k + O(t^K)$$

holds as $t \to 0^+$, *in* $|\arg(1-z)| < \pi$, *where the coefficients* $C_{f,k}^{(r)}$ *are given by*

$$C_{f,k}^{(r)}(1,z) = \sum_{j=\max(1,r-k)}^{r} \binom{k}{r-j} B_{k-r+j}^{(-r)} B_{r-j}^{(r)} \operatorname{Li}_j(z) + \sum_{j=0}^{k-r} \binom{k}{r+j} B_{k-r-j}^{(-r)} B_{r+j}^{(r)} \operatorname{Li}_{-j}(z)$$

for $k = 0, 1, \ldots$. *Here the implied* O*-constant depends at most on* r, z *and* K.

We fourthly discuss the applications of Theorem 6; this at first yields on taking $f(z) = \Phi(s,x,z)$ the following theorem.

Theorem 9. *Let* $x \geq 0$ *and* $y \geq 0$ *be real, and* s *any complex. Then for any integer* $K \geq 0$ *the asymptotic expansion*

$$q^{xy}\left(\frac{1-q}{t}\right)^r (\mathcal{D}_{q,z}^x)^r \Phi(s,x,q^y z) = \sum_{k=0}^{K-1} \frac{(-1)^k \Phi(s-r-k,x,z) B_k^{(-r)}(y)}{k!} t^k + O(t^K)$$

holds as $t \to 0^+$, *in* $|\arg(1-z)| < \pi$, *where the implied* O*-constant depends at most on* r, s, x, y, z *and* K.

We can in fact show

$$\operatorname{Li}_{q,-r}(z) = z(\mathcal{D}_{q,z}^1)^r \Phi(0,1,z) \tag{2.3.12}$$

for any integer $r \geq 0$. Corollary 6.1 can therefore be applied by taking $f(z) = \Phi(0,1,z)$ to yield the following corollary.

Corollary 9.1. *Let* $r \in \mathbb{Z}$ *be arbitrarily fixed with* $r \geq 1$. *Then for any integer* $K \geq 0$ *the asymptotic expansion*

$$\operatorname{Li}_{q,-r}(z) = \sum_{k=0}^{K-1} \frac{(-1)^k C_{f,k}^{(-r)}(1,z)}{k!} t^k + O(t^K)$$

holds as $t \to 0^+$, *in* $|\arg(1-z)| < \pi$, *where the coefficients* $C_{f,k}^{(-r)}$ *are given by*

$$C_{f,k}^{(-r)}(1,z) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j}^{(r)} B_j^{(-r)} \operatorname{Li}_{-r-j}(z)$$

for $k = 0, 1, \ldots$. *Here the implied* O*-constant depends at most on* r, z *and* K.

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