

On ANR(stratifiable)-spaces

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In metric spaces, the closed embedding theorem of Kuratowski-Wojdyslawski plays an important role in the development of retract theory. In generalized metric spaces, for a paracomplex (i.e. Hyman's M-space) X , Hyman [11] proved that X can be embedded in an AR(paracomplex)-space as a closed subset. For a stratifiable space X , Cauty [5] constructed a space $Z(X)$, and proved that $Z(X)$ is stratifiable and X is AR(stratifiable) (resp. ANR(stratifiable)) if and only if X is a retract (resp. neighborhood retract) of $Z(X)$. However, it is not known whether $Z(X)$ is ANR(stratifiable). In previous paper [16], we constructed a space $E(X)$ for a space X , and showed that a stratifiable space X can be embedded in the AR(stratifiable)-space $E(X)$ as a closed subset.

In this paper, in section 1 we briefly give some contents in [16], and prove some applications of the closed embedding theorem. In section 2, we consider some problems which arise in and around ANR(stratifiable)-spaces.

Throughout this paper, we assume that all spaces are regular and all maps are continuous. For the definitions of AR, ANR, AE and ANE, see Hu [10].

1. The closed embedding theorem and some applications

For a full simplicial complex, we introduced the locally convex topology in [15]; i.e. the strongest topology, which is locally convex, contained in the Whitehead topology. The introduction of this topology is valid by the fact that a full simplicial complex with the Whitehead topology need not be locally convex (cf. [9; pp. 416, 4.3]). Note that a full simplicial complex (with the Whitehead topology) with countable vertices is locally convex (cf. [6; Lemma 4.4]). By using the locally convex topology, we can construct $E(X)$ for a space X as follows:

CONSTRUCTION 1.1 ([16; 3.1]). Let X be a space. $A(X)$ denotes the full simplicial complex with the locally convex topology which has all points of X as the set of vertices. Let i be a canonical bijection from the 0-skeleton of $A(X)$ onto X . Then $E(X)$ is the set $A(X)$ equipped with the topology generated by sets U such that

(C1) U is open in $A(X)$ and $i(U \cap X)$ is open in X , and

(C2) U is convex in $A(X)$.

By (C1), it is clear that X is closed in $E(X)$. Note the difference of $E(X)$ and Cauty's $Z(X)$ [5]. We obtain the following:

THEOREM 1.2([16; Theorem 3.4]). If X is stratifiable, $E(X)$ is $AR(\text{stratifiable})$.

Next, the notion of L -space was introduced in [14]. This is defined as follows: A Hausdorff space is called an L -space if it can be mapped onto some stratifiable space by a perfect

map. As easily seen by this definition, L -spaces have some analogous properties as paracompact M -spaces. For example:

THEOREM 1.3([14;1.3]). L -spaces are precisely the homeomorphic images of closed subsets of product $S \times I^\tau$, where S is stratifiable, $I = [0,1]$ and τ is an arbitrary cardinal.

Now, we give some applications of the closed embedding theorem. The following theorem is an analogous result of [19; Theorem 4].

THEOREM 1.4. Let $\mathcal{Q}$ be a closed hereditary class consisting of normal spaces. If every $\text{ANR}(\text{stratifiable})$ -space is $\text{ANE}(\mathcal{Q})$, then every $\text{ANR}(L)$ -space is $\text{ANE}(\mathcal{Q})$.

Proof. Let X be any $\text{ANR}(L)$ -space. Then by Theorem 1.3, X is a closed subset of product $S \times I^\tau$ where S is a stratifiable space and $\tau = w(X)$. By Theorem 1.2, S is considered as a closed subset of an $\text{AR}(\text{stratifiable})$ -space $E(S)$. Therefore, we can assume that X is a closed subset of product $E(S) \times I^\tau$. Since $E(S) \times I^\tau$ is an L -space, X is a neighborhood retract of $E(S) \times I^\tau$. Since $E(S)$ is $\text{ANE}(\mathcal{Q})$ by assumption and I^τ is clearly $\text{AE}(\mathcal{Q})$, $E(S) \times I^\tau$ is $\text{ANE}(\mathcal{Q})$. Therefore X is $\text{ANE}(\mathcal{Q})$. This completes the proof.

For $\mathcal{Q}$ satisfying the condition in this theorem, see Theorem 2.3 and Problem 2.5 in later section.

For pairs of ANR 's, see [13], and for closed embeddings of pairs, see [18].

THEOREM 1.5. Every pair (X, X_0) of stratifiable spaces, where X_0 is a closed subset of X , has a closed embedding in a pair of $\text{AR}(\text{stratifiable})$'s.

Proof. As easily seen from Construction 1.1, $(E(X), E(X_0))$ is a required pair of $\text{AR}(\text{stratifiable})$'s.

Note that, in [18;Theorem 1.1], metric case was considered.

2. Problems

In general, the following problem naturally arises.

PROBLEM 2.1. For any two classes $\mathcal{Q}$ and $\mathcal{Q}'$ with $\mathcal{Q} \subset \mathcal{Q}'$, is an $\text{ANR}(\mathcal{Q})$ -space $\text{ANR}(\mathcal{Q}')$?

For example, the case $\mathcal{Q} = \text{metric}$ has been studied as follows:

THEOREM 2.2. (1) (Dowker [8]) Every $\text{ANR}(\text{metric})$ -space is $\text{ANR}(\text{collectionwise normal and perfectly normal})$.

(2) (Lisica [12]) Every $\text{ANR}(\text{metric})$ -space is $\text{ANR}(\text{paracompact } M)$.

(3) (Mardešić and Šostak [14]) Every $\text{ANR}(\text{metric})$ -space is $\text{ANR}(L)$.

For the case $\mathcal{Q} = \text{stratifiable}$, the following result was shown so far as I know.

THEOREM 2.3 (Borges [1],[2]) Every $\text{ANR}(\text{stratifiable})$ -space is $\text{ANR}(\text{linearly stratifiable})$.

This theorem is easily seen by [1;Theorem 4.1 and 4.4], [2;Theorem 2.1] and [5;Theorem 1.8].

In connection with Theorem 2.2(2) and (3), the following problem was posed in [14].

PROBLEM 2.4 (Mardešić and Šostak). Is an $\text{ANR}(\text{stratifiable})$ -space $\text{ANR}(L)$?

For this problem, a partial answer was proved in [14;Theorem 2.4 and 2.5]. In connection with Theorem 2.3 and Problem 2.4, we pose

PROBLEM 2.5. Find a class $\mathcal{Q}$ containing stratifiable spaces such that every $\text{ANR}(\text{stratifiable})$ -space is $\text{ANR}(\mathcal{Q})$.

The class of all paracompact σ -spaces is not such a class (cf. [7;Example 2] and [19;Example 2]). In connection with this, Tsuda [19] posed

PROBLEM 2.6 (Tsuda). If an $\text{ANR}(\text{stratifiable})$ -space X is $\text{ANR}(\text{paracompact } \sigma)$, then is X metrizable?

Finally, let X and Y be two spaces, A a closed subset of X and $f:A \rightarrow Y$ a map. It is well known [10;pp. 178] that if X , A and Y are $\text{ANR}(\text{metric})$'s, then the adjunction space $X \cup_f Y$ is $\text{ANR}(\text{metric})$ provided that it is metrizable. This result was essentially proved in successive stages by Borsuk [3], Whitehead [20] and Hanner [9]. For attempt to generalize this theorem, Hyman [11] proved the case of paracomplex spaces. Cauty [4] announced the case of stratifiable spaces, but his proof was false. (This was pointed out by San-nou [17].) So, the case of stratifiable spaces is still open:

PROBLEM 2.7. Let X and Y be two stratifiable spaces, A a closed subset of X and $f:A \rightarrow Y$ a map. If X , A and Y are $\text{ANR}(\text{stratifiable})$'s, is the adjunction space $X \cup_f Y$ $\text{ANR}(\text{stratifiable})$?

References

- [1] C.R. Borges: A study of absolute extensor spaces, Pacific J. Math., 31(1969), 609-617.
- [2] ———: Absolute extensor spaces: A correction and an answer, Pacific J. Math., 50(1974), 29-30.
- [3] K. Borsuk: Quelques rétracts singuliers, Fund. Math., 24 (1935), 249-258.
- [4] R. Cauty: Une généralisation du théorème de Borsuk-Whitehead-Hanner aux espaces stratifiables, C.R. Acad. Sci. Paris, 275(1972), 271-274.
- [5] ———: Rétractions dans les espaces stratifiables, Bull. Soc. Math. France, 102(1974), 129-149.
- [6] D.W. Curtis: Some theorems and examples on local equiconnect- edness and its specializations, Fund. Math., 72(1971), 101-113.
- [7] E.K. van Douwen and R. Pol: Countable spaces without exten- sion properties, Bull. Acad. Pol. Sci., 25(1977), 987-991.
- [8] C.H. Dowker: On a theorem of Hanner, Arkiv Mat., 2(1952), 307-313.
- [9] J. Dugundji: Topology, Allyn and Bacon, Boston, 1966.
- [10] S.T. Hu: Theory of retracts, Wayne State University Press, Detroit, 1965.
- [11] D.M. Hyman: A category slightly larger than the metric and CW-categories, Michigan Math. J., 15(1968), 193-214.
- [12] J.T. Lisica: Extension of continuous mappings and a factor- ization theorem, Sibirsk. Mat. Ž., 14(1973), 128-139;

- Siberian Math. J., 14(1973), 90-96.
- [13] S. Mardešić and J. Segal: Shape theory, North-Holland Publ. Comp., 1982.
- [14] ————— and A. Šostak: On perfect preimages of stratifiable spaces, Uspehi Mat. Nauk, 35(1980), 84-93; Russian Math. Surveys, 35(1980), 99-108.
- [15] T. Miwa: A locally convex topology of simplicial complexes, Mem. Fac. Sci. Shimane Univ., 20(1986), 25-30.
- [16] —————: Embeddings to AR-spaces, Bull. Pol. Acad. Sci., 35 No. 7-8 (1987).
- [17] S. San-nou: A note on E-product, J. Math. Soc. Japan, 29 (1977), 281-285.
- [18] Yu. M. Smirnov: Shape theory for G-pairs, Uspehi Mat. Nauk, 40(1985), 151-165; Russian Math. Surveys, 40(1985), 185-203.
- [19] K. Tsuda: ANR(paracompact M) versus ANR(stratifiable), Q and A in Gen. Top., 3(1985/86), 87-94.
- [20] J.H.C. Whitehead: Note on a theorem due to Borsuk, Bull. Amer. Math. Soc., 54(1948), 1125-1132.