

Submetrizability and Interpolations

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All topological spaces considered here are Tychonoff. For a topological space X , $C(X)$ is the Banach space of all bounded real-valued continuous functions with the sup norm: $\|f\| = \sup\{|f(x)| : x \in X\}$ for $f \in C(X)$. The space $F(X \times \mathbf{R})$ is the hyperspace consisting of all finite subsets of the product space $X \times \mathbf{R}$. Of course, its topology is the Vietoris topology. Let $S(X)$ be the subspace of $F(X \times \mathbf{R})$ defined by

$$S(X) = \{(x_1, r_1), \dots, (x_n, r_n) : x_i \neq x_j \text{ for } i \neq j\}.$$

For each $n = 1, 2, \dots$, define $F_n(X \times \mathbf{R})$ and $S_n(X)$ by:

$$F_n(X \times \mathbf{R}) = \{D \in F(X \times \mathbf{R}) : D \text{ has at most } n \text{ points}\},$$

$$S_n(X) = S(X) \cap F_n(X \times \mathbf{R}).$$

For a point $D = \{(x_1, r_1), (x_2, r_2), \dots, (x_n, r_n)\} \in S(X)$, a function f_D in $C(X)$ is called an interpolation function for D if

$$f_D(x_1) = r_1, f_D(x_2) = r_2, \dots, f_D(x_n) = r_n$$

are satisfied. A map $\Theta : S(X) \rightarrow C(X)$ is called interpolation (algorithm) if $\Theta(D) = f_D$ is an interpolation function for each $D \in S(X)$. Further, if Θ satisfies the condition that the restriction $\Theta|_{S_n(X) - S_{n-1}(X)}$ is continuous for each $n = 1, 2, \dots$, we call Θ to be a weakly continuous interpolation. Let a topology τ_1 on a set X is stronger than a topology τ_2 on the same set. If Θ is a weakly continuous interpolation on τ_2 , then the same Θ is a weakly continuous interpolation on τ_1 . Every metrizable space has a weakly continuous interpolation, and hence every submetrizable space has a weakly continuous interpolation. In [1], a weakly continuous interpolation on a metric space (X, d) is constructed as follows: For any $D = \{(x_1, r_1), \dots, (x_n, r_n)\} \in S(X)$, let

$$m = \min\{d(x_i, x_j) : i \neq j\}$$

and

$$f_D(x) = \begin{cases} 0 & \text{if } d(x, x_i) \geq m/4 \text{ for each } i = 1, \dots, n \\ r_i - \frac{4r_i}{m}d(x, x_i) & \text{if } d(x, x_i) < m/4 \text{ for some } i = 1, \dots, n. \end{cases}$$

Then the map $\Theta : S(X) \rightarrow C(X)$ defined by $\Theta(D) = f_D$ is weakly continuous. For this interpolation, we can show the following. Here, for any distinct $p, q \in X$ D_{pq} denotes the sample $\{(p, -1), (q, 1)\}$ in $S_2(X) - S_1(X)$.

Proposition 1 *The weakly continuous interpolation $\Theta : S(X) \rightarrow C(X)$ on a metric space (X, d) constructed above satisfies the following condition: There exists a constant M such that*

$$\|f_{D_{wx}} - f_{D_{xy}}\| \leq M \max\{\|f_{D_{wy}} - f_{D_{xy}}\|, \|f_{D_{zx}} - f_{D_{xy}}\|\}$$

for any $x, y, z, w \in X$.

Proof. We can assume that metric function is bounded. Assume that $d(x, y) < 1$ for any $x, y \in X$. It suffices to show that $\|f_{D_{wy}} - f_{D_{xy}}\| < \epsilon, \|f_{D_{zx}} - f_{D_{xy}}\| < \epsilon$ imply $\|f_{D_{wx}} - f_{D_{xy}}\| < 6\epsilon$.

Claim 1. Let $0 < \alpha < \frac{1}{2}$. If $d(z, y) < \alpha d(w, y)$, then $\|f_{D_{wy}} - f_{D_{wx}}\| \leq 9\alpha$.

Since interpolation functions defined above are piecewise linear on the distance from data points, $\|f_{D_{wy}} - f_{D_{wx}}\|$ does not exceed the following 5 values:

- (1) At z , $|f_{D_{wy}}(z) - f_{D_{wx}}(z)| = \frac{4}{d(w, y)}d(y, z) < \frac{4\alpha d(w, y)}{d(w, y)} = 4\alpha$.
- (2) At y , $|f_{D_{wy}}(y) - f_{D_{wx}}(y)| = \frac{4}{d(w, z)}d(y, z) < \frac{4d(y, z)}{d(w, y) - d(y, z)} < \frac{4\alpha d(w, y)}{(1-\alpha)d(w, y)} = \frac{4\alpha}{1-\alpha} < 8\alpha$.
- (3) When the distance from y is $\frac{d(w, y)}{4}$, $\frac{4}{d(w, z)}(d(y, z) + \frac{d(w, z)}{4} - \frac{d(w, y)}{4}) = \frac{4d(y, z)}{d(w, z)} + 1 - \frac{d(w, y)}{d(w, z)} < 8\alpha + 1 - \frac{d(w, y)}{d(w, z)} \leq 8\alpha + 1 - \frac{d(w, y)}{d(w, y) + d(y, z)} < 8\alpha + 1 - \frac{1}{1+\alpha} < 9\alpha$.
- (4) When the distance from z is $\frac{d(w, z)}{4}$, $\frac{4}{d(w, y)}(d(y, z) + \frac{d(w, y)}{4} - \frac{d(w, z)}{4}) = (\frac{4d(y, z)}{d(w, y)} + 1 - \frac{d(w, z)}{d(w, y)}) < \alpha + 1 - \frac{d(w, y) - d(y, z)}{d(w, y)} = \alpha + 1 - 1 + \frac{d(y, z)}{d(w, y)} < 2\alpha$.
- (5) When the distance from w is $\frac{d(w, z)}{4}$ in case $d(w, z) \leq d(w, y)$, $\frac{4}{d(w, y)}(\frac{d(w, y)}{4} - \frac{d(w, z)}{4}) = 1 - \frac{d(w, z)}{d(w, y)} \leq 1 - \frac{d(w, y) - d(y, z)}{d(w, y)} = \frac{d(y, z)}{d(w, y)} < \alpha$. When the distance from w is $\frac{d(w, y)}{4}$ in case $d(w, y) < d(w, z)$, $\frac{4}{d(w, z)}(\frac{d(w, z)}{4} - \frac{d(w, y)}{4}) = 1 - \frac{d(w, y)}{d(w, z)} \leq 1 - \frac{d(w, y)}{d(w, y) + d(y, z)} = 1 - \frac{1}{1+\alpha} = \frac{\alpha}{1+\alpha} < \alpha$.

Claim 2. Let $0 < \epsilon < 1$. If $\|f_{D_{wy}} - f_{D_{xy}}\| < \epsilon, \|f_{D_{zx}} - f_{D_{xy}}\| < \epsilon$, then $d(y, z) < \frac{\epsilon}{2}d(w, y)$.

Since $\|f_{D_{wy}} - f_{D_{xy}}\| < \epsilon$, estimating the value at x , we obtain $d(x, w)\frac{4}{d(w, y)} < \epsilon$. Hence $d(x, w) < \frac{\epsilon}{4}d(w, y)$. Similarly, the value of $|f_{D_{zx}} - f_{D_{xy}}|$ at z is $d(y, z)\frac{4}{d(x, y)}$. Hence it follows that $d(y, z) < \frac{\epsilon}{4}d(x, y)$. Then $d(y, z) < \frac{\epsilon}{4}(d(x, w) + d(w, y)) < \frac{\epsilon}{4}(\frac{\epsilon}{4}d(w, y) + d(w, y)) = ((\frac{\epsilon}{4})^2 + \frac{\epsilon}{4})d(w, y) < \frac{\epsilon}{2}d(w, y)$.

By claim 2, $d(y, z) < \frac{\varepsilon}{2}d(w, y)$. Then $\|f_{D_{wy}} - f_{D_{wz}}\| < \frac{9}{2}\varepsilon < 5\varepsilon$ by claim 1. Hence $\|f_{D_{wz}} - f_{D_{xy}}\| \leq \|f_{D_{wz}} - f_{D_{wy}}\| + \|f_{D_{wy}} - f_{D_{xy}}\| < 5\varepsilon + \varepsilon = 6\varepsilon$.

Let us call a weakly continuous interpolation to be regular when the condition in the proposition is satisfied. A sequence $\{\mathcal{G}_n\}$ of open covers of X is called a G_δ -diagonal if for any $x \neq y \in X$, there exists n such that $y \notin st(x, \mathcal{G}_n)$. Further G_δ -diagonal sequence $\{\mathcal{G}_n\}$ is called regular if for $G, G' \in \mathcal{G}_{n+1}$, $G \cap G' \neq \emptyset$ implies that there exists $G \in \mathcal{G}_n$ such that $G \cup G' \subset G$. It is well known that a space X is submetrizable if and only if X has a regular G_δ -diagonal sequence (see [2]).

Theorem 1 *The following are equivalent.*

- (1) X is submetrizable.
- (2) $X \times (\omega + 1)$ has a weakly continuous regular interpolation.

Proof. If X is submetrizable, then $X \times (\omega + 1)$ is also submetrizable. Hence it follows that (1) implies (2) from the proposition above.

Assume that (2) is satisfied. We will show that X has a regular G_δ -diagonal sequence. Let $\Theta : S(X \times (\omega + 1)) \rightarrow C(X \times (\omega + 1))$ be a weakly continuous interpolation which satisfies the regular condition. Let us recall the notation that $D_{(p,m)(q,n)} = \{((p, m), -1), ((q, n), 1)\} \in S_2(X \times (\omega + 1)) - S_1(X \times (\omega + 1))$ for any $(p, m), (q, n) \in X \times (\omega + 1)$ and $f_{D_{(p,m)(q,n)}} = \Theta(D_{(p,m)(q,n)})$. Then there exists a constant M such that

$$\|f_{D_{(w,l)(z,k)}} - f_{D_{(x,i)(y,j)}}\| \leq M \max\{\|f_{D_{(w,l)(y,j)}} - f_{D_{(x,i)(y,j)}}\|, \|f_{D_{(x,i)(z,k)}} - f_{D_{(x,i)(y,j)}}\|\}$$

for any $D_{(x,i)(y,j)}, D_{(w,l)(y,j)}, D_{(x,i)(z,k)}, D_{(w,l)(z,k)} \in S_2(X \times (\omega + 1)) - S_1(X \times (\omega + 1))$. It can be also assumed that $M > 1$.

Let $\mathcal{G}_n$ be the family of all open subsets U which satisfies

$$\|f_{D_{(x,i)(y,\omega)}} - f_{D_{(x',i)(y',\omega)}}\| < \frac{1}{(2M)^n}$$

for any $x, y, x', y' \in U$ and any $i = 0, 1, \dots, n$.

Claim 1. $\mathcal{G}_n$ is an open cover of X .

For any $x \in X$, consider $D_{(x,i)(x,\omega)}$ for $i = 0, 1, \dots, n$. Since Θ is weakly continuous, for each $i = 0, 1, \dots, n$ there exists an open neighborhood U_i of x such that

$$\|f_{D_{(u,i)(v,\omega)}} - f_{D_{(x,i)(x,\omega)}}\| < \frac{1}{2(2M)^n}$$

for any $u, v \in U_i$. Then $U = \bigcap_{i=0}^n U_i \in \mathcal{G}_n$ is an open neighborhood of x in X .

Claim 2. $\{\mathcal{G}_n\}$ is a G_δ -diagonal sequence of X .

Assume that $\{\mathcal{G}_n\}$ is not a G_δ -diagonal sequence. Then there exist two distinct point x_0, y_0 such that for each n , $x_0, y_0 \in U_n$ for some $U_n \in \mathcal{G}_n$. Let us take $D_{(x_0, \omega)(y_0, \omega)} = \{((x_0, \omega), -1), ((y_0, \omega), 1)\}$. Then $f_{D_{(x_0, \omega)(y_0, \omega)}}((x_0, \omega)) = -1$. Let W be a neighborhood of $D_{(x_0, \omega)(y_0, \omega)}$ in $S_2(X \times (\omega + 1)) - S_1(X \times (\omega + 1))$ such that $\|f_D - f_{D_{(x_0, \omega)(y_0, \omega)}}\| < 1$ for any $D \in W$. Then there exist n such that $D_{(x_0, i)(y_0, \omega)} \in W$ for any $i \geq n$. Especially, for such $D_{(x_0, i)(y_0, \omega)}$, the value $|f_{D_{(x_0, i)(y_0, \omega)}}((x_0, \omega)) - f_{D_{(x_0, \omega)(y_0, \omega)}}((x_0, \omega))| < 1$ at (x_0, ω) , and hence

$$f_{D_{(x_0, i)(y_0, \omega)}}((x_0, \omega)) < 0.$$

On the other hand, since $x_0, y_0 \in U_i$ for $i \geq n$, $\|f_{D_{(x_0, i)(y_0, \omega)}} - f_{D_{(x_0, i)(x_0, \omega)}}\| < \frac{1}{(2M)^i}$ and $f_{D_{(x_0, i)(x_0, \omega)}}((x_0, \omega)) = 1$, it must be

$$f_{D_{(x_0, i)(y_0, \omega)}}((x_0, \omega)) > 0.$$

This is a contradiction.

Claim 3. $\{\mathcal{G}_n\}$ is regular.

Assume that $U_1, U_2 \in \mathcal{G}_{n+1}$ satisfy $U_1 \cap U_2 \neq \emptyset$. Then there exist $p \in U_1 \cap U_2$. For any $x, y \in U_1 \cup U_2$, it is shown that $\|f_{D_{(x, i)(y, \omega)}} - f_{D_{(p, i)(p, \omega)}}\| < \frac{1}{2^{n+1}M^n}$ for any $i = 0, 1, \dots, n$. In fact, in case $x, y \in U_1$ or $x, y \in U_2$, it is obvious. In other case, since $\|f_{D_{(x, i)(p, \omega)}} - f_{D_{(p, i)(p, \omega)}}\| < \frac{1}{(2M)^{n+1}}$, $\|f_{D_{(p, i)(y, \omega)}} - f_{D_{(p, i)(p, \omega)}}\| < \frac{1}{(2M)^{n+1}}$, it follows that $\|f_{D_{(x, i)(y, \omega)}} - f_{D_{(p, i)(p, \omega)}}\| < \frac{1}{2^{n+1}M^n}$ by the regularity condition of Θ . Then for $i = 0, 1, \dots, n$ and for any $x, y, x', y' \in U_1 \cup U_2$,

$$\begin{aligned} \|f_{D_{(x, i)(y, \omega)}} - f_{D_{(x', i)(y', \omega)}}\| &= \|f_{D_{(x, i)(y, \omega)}} - f_{D_{(p, i)(p, \omega)}}\| + \|f_{D_{(p, i)(p, \omega)}} - f_{D_{(x', i)(y', \omega)}}\| \\ &\leq \frac{1}{2^{n+1}M^n} + \frac{1}{2^{n+1}M^n} < \frac{1}{(2M)^n}. \end{aligned}$$

This shows that $U_1 \cup U_2 \in \mathcal{G}_n$.

In the proof of the above theorem, we used the regularity condition on the interpolation only to show the regularity of the G_δ -diagonal sequence. Hence the following theorem is also obtained.

Theorem 2 *If $X \times (\omega + 1)$ has a weakly continuous interpolation, then X has a G_δ -diagonal. In particular, for a paracompact space X , $X \times (\omega + 1)$ has a weakly continuous interpolation if and only if X has a G_δ -diagonal.*

Further, we used interpolation functions essentially for only $D \in S_2(X \times (\omega + 1)) - S_1(X \times (\omega + 1))$ to show the submetrizability of X in Theorem 1. Let us call a map $\Theta_2 : S_2(X) - S_1(X) \rightarrow C(X)$ to be a continuous S_2 -interpolation if it is continuous and $\Theta_2(D)$ is an interpolation function for every $D \in S_2(X) - S_1(X)$. Theorem 1 can be rewritten as the following.

Theorem 3 *The following are equivalent.*

- (1) X is submetrizable.
- (2) $X \times (\omega + 1)$ has a weakly continuous regular interpolation.
- (3) $X \times (\omega + 1)$ has a continuous regular S_2 -interpolation.

Remark. It may be generally shown that a space X has a weakly continuous interpolation if and only if X has a continuous S_2 -interpolation.

参考文献

- [1] G. Gruenhage. *Generalized metric spaces*. Handbook of Set-theoretic Topology, 423-502, North-Holland, 1984.
- [2] T. Terada. *Continuity of interpolations*. Tsukuba J. Math., vol. 30, (2006), 225-236.