

KERNEL FUNCTION AND QUANTUM ALGEBRAS

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ABSTRACT. We introduce an analogue $K_n(x, z; q, t)$ of the Cauchy-type kernel function for the Macdonald polynomials, being constructed in the tensor product of the ring $\Lambda_{\mathbb{F}}$ of symmetric functions and the commutative algebra $\mathcal{A}$ over the degenerate $\mathbb{CP}^1$. We show that a certain restriction of $K_n(x, z; q, t)$ with respect to the variable z is neatly described by the tableau sum formula of Macdonald polynomials. Next, we demonstrate that the level m representation of the Ding-Iohara quantum algebra $\mathcal{U}(q, t)$ naturally produces the currents of the deformed $\mathcal{W}_{q,p}(\mathfrak{sl}_n)$. Then we remark that the $K_n(x, z; q, t)$ emerges in the highest-to-highest correlation function of the deformed $\mathcal{W}_{q,p}(\mathfrak{sl}_n)$ algebra.

1. KERNEL FUNCTION

1.1. The algebra $\mathcal{A}$. We briefly recall the definition and the basic facts about the commutative algebra $\mathcal{A}$ introduced in [FHHSY]. Let q_1, q_2 be two independent indeterminates and set $q_3 := 1/q_1 q_2$. We also use the symbols $\mathbb{F} := \mathbb{Q}(q_1, q_2)$, $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ := \{1, 2, \dots\}$.

For $n, k \in \mathbb{N}_+$, we define two operators $\partial^{(0,k)}, \partial^{(\infty,k)}$ acting on the space of symmetric rational functions in n variables $x_1, \dots, x_n$ by

$$\begin{aligned} \partial^{(0,k)} &: f \mapsto \frac{n!}{(n-k)!} \lim_{\xi \rightarrow 0} f(x_1, \dots, x_{n-k}, \xi x_{n-k+1}, \xi x_{n-k+2}, \dots, \xi x_n) \\ \partial^{(\infty,k)} &: f \mapsto \frac{n!}{(n-k)!} \lim_{\xi \rightarrow \infty} f(x_1, \dots, x_{n-k}, \xi x_{n-k+1}, \xi x_{n-k+2}, \dots, \xi x_n) \end{aligned}$$

whenever the limit exists. We also set $\partial^{(0,k)} c = 0, \partial^{(\infty,k)} c = 0$ for $c \in \mathbb{F}$. Finally we define $\partial^{(0,0)}$ and $\partial^{(\infty,0)}$ to be the identity operator.

Definition 1.1. For $n \in \mathbb{N}$, the vector space $\mathcal{A}_n = \mathcal{A}_n(q_1, q_2, q_3)$ is defined by the following conditions (i), (ii), (iii) and (iv).

- (i) $\mathcal{A}_0 := \mathbb{F}$. For $n \in \mathbb{N}_+$, $f(x_1, \dots, x_n) \in \mathcal{A}_n$ is a rational function with coefficients in $\mathbb{F}$, and symmetric with respect to the x_i 's.
- (ii) For $n \in \mathbb{N}$, $0 \leq k \leq n$ and $f \in \mathcal{A}_n$, the limits $\partial^{(\infty,k)} f$ and $\partial^{(0,k)} f$ both exist and coincide: $\partial^{(\infty,k)} f = \partial^{(0,k)} f$ (degenerate $\mathbb{CP}^1$ condition).
- (iii) The poles of $f \in \mathcal{A}_n$ are located only on the diagonal $\{(x_1, \dots, x_n) \mid \exists (i, j), i \neq j, x_i = x_j\}$, and the orders of the poles are at most two.
- (iv) For $n \geq 3$, $f \in \mathcal{A}_n$ satisfies the wheel conditions

$$f(x_1, q_1 x_1, q_1 q_2 x_1, x_4, \dots) = 0, \quad f(x_1, q_2 x_1, q_1 q_2 x_1, x_4, \dots) = 0.$$

Then we set the graded vector space $\mathcal{A} = \mathcal{A}(q_1, q_2, q_3) := \bigoplus_{n \geq 0} \mathcal{A}_n$.

Definition 1.2. For an m -variable symmetric rational function f and an n -variable symmetric rational function g , we define an $(m+n)$ -variable symmetric rational function $f * g$ by

$$(f * g)(x_1, \dots, x_{m+n}) := \text{Sym} \left[f(x_1, \dots, x_m) g(x_{m+1}, \dots, x_{m+n}) \prod_{\substack{1 \leq \alpha \leq m \\ m+1 \leq \beta \leq m+n}} \omega(x_\alpha, x_\beta) \right]. \quad (1.1)$$

Here $\omega(x, y)$ is the rational function

$$\omega(x, y) = \omega(x, y; q_1, q_2, q_3) := \frac{(x - q_1 y)(x - q_2 y)(x - q_3 y)}{(x - y)^3}, \quad (1.2)$$

and the symbol Sym means $\text{Sym}(f(x_1, \dots, x_n)) := (1/n!) \sum_{\sigma \in \mathfrak{S}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

Fact 1.3 ([FHHSY, Theorem 1.5]). $\mathcal{A}$ is closed with respect to $*$, and the pair $(\mathcal{A}, *)$ is a unital associative commutative algebra. The Poincaré series is $\sum_{n \geq 0} (\dim_{\mathbb{F}} \mathcal{A}_n) z^n = \prod_{m \geq 1} (1 - z^m)^{-1}$.

1.2. The ring $\Lambda_{\mathbb{F}}$ of symmetric functions. As for the notations and definitions concerning the partitions, we basically follow the notation in [M]. A partition $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers satisfying $\lambda_1 \geq \lambda_2 \geq \dots$. We define $|\lambda| := \lambda_1 + \lambda_2 + \dots$, $\ell(\lambda) := \#\{i \mid \lambda_i \neq 0\}$, and write $\lambda \vdash n$ if $|\lambda| = n$. We denote the conjugate (transpose) of a partition λ by λ' . We work with the dominance partial ordering defined as : $\lambda \geq \mu \stackrel{\text{def}}{\iff} |\lambda| = |\mu|, \lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for all $i \geq 1$.

We recall some basic facts about the ring of symmetric functions. As was in [FHHSY], we set $q_1 = q^{-1}, q_2 = t$ (hence $q_3 = qt^{-1}$) and $\mathbb{F} = \mathbb{Q}(q_1, q_2) = \mathbb{Q}(q, t)$. Let $\Lambda_{\mathbb{F}}$ be the ring of symmetric functions over the base field $\mathbb{F}$, constructed in the category of graded ring with the projection operators $\rho_{m,n} : f(x_1, \dots, x_m) \mapsto f(x_1, \dots, x_n, 0, \dots, 0)$.

Let $p_n(x) := \sum_i x_i^n$ be the power sum function. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, the monomial symmetric function is defined by $m_\lambda(x) := \sum_{\alpha} x^\alpha$, where α runs over all the distinct permutations of λ . The elementary symmetric function $e_n(x)$ is defined by the generating function $E(y) := \prod_i (1 + x_i y) = \sum_{n \geq 0} e_n(x) y^n$. Set $G(y) := \prod_i \{(tx_i y; q)_\infty / (x_i y; q)_\infty\} = \sum_{n \geq 0} g_n(x; q, t) y^n$, where $(x; q)_\infty := \prod_{i \geq 0} (1 - q^i x)$. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ set $p_\lambda := p_{\lambda_1} p_{\lambda_2} \dots$. Similarly we write $e_\lambda := e_{\lambda_1} e_{\lambda_2} \dots$ and $g_\lambda := g_{\lambda_1} g_{\lambda_2} \dots$. It is known that $\{p_\lambda\}$, $\{m_\lambda\}$, $\{e_\lambda\}$ and $\{g_\lambda\}$ form bases of $\Lambda_{\mathbb{F}}$.

Recall Macdonald's scalar product $\langle p_\lambda, p_\mu \rangle_{q,t} := \delta_{\lambda,\mu} \prod_{i \geq 1} i^{m_i} m_i! \prod_{j \geq 1} (1 - q^{\lambda_j}) / (1 - t^{\lambda_j})$, where m_i denotes the number of parts i in the partition λ . For any dual bases $\{u_\lambda\}$ and $\{v_\lambda\}$, we have

$$\Pi(x, y; q, t) := \prod_{i,j} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \sum_{\lambda} u_\lambda(x) v_\lambda(y). \quad (1.3)$$

It is known that $\{m_\lambda\}$ and $\{g_\lambda\}$ form dual bases, namely we have $\langle m_\lambda, g_\mu \rangle_{q,t} = \delta_{\lambda,\mu}$.

Macdonald polynomials $P_\lambda(x; q, t)$ are uniquely characterized by (a) the triangular expansion $P_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu$ ($a_{\lambda\mu} \in \mathbb{F}$), and (b) the orthogonality $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ if $\lambda \neq \mu$.

Se set

$$b_\lambda(q, t) := \langle P_\lambda(z; q, t), P_\lambda(z; q, t) \rangle_{q,t}^{-1}, \quad Q_\lambda(z; q, t) := b_\lambda(q, t) P_\lambda(z; q, t). \quad (1.4)$$

Then $\{Q_\lambda\}$ forms a dual basis to $\{P_\lambda\}$.

1.3. The isomorphism $\iota : \Lambda_{\mathbb{F}} \rightarrow \mathcal{A}$. Both $\Lambda_{\mathbb{F}}$ and $\mathcal{A}$ are commutative rings having the same Poincaré series $\sum_{n \geq 0} (\dim_{\mathbb{F}} \Lambda_{\mathbb{F}}^n) z^n = \sum_{n \geq 0} (\dim_{\mathbb{F}} \mathcal{A}_n) z^n = \prod_{m \geq 1} (1 - z^m)^{-1}$, where $\Lambda_{\mathbb{F}}^n$ denotes the ring of symmetric functions of degree n . Moreover it was shown in [FHHSY] that there is a natural way to identify $\Lambda_{\mathbb{F}}$ and $\mathcal{A}$ from the point of view of the free field construction of the Macdonald operators. Based on the finding in [FHHSY] we give an isomorphism $\iota : \Lambda_{\mathbb{F}} \rightarrow \mathcal{A}$ as follows.

For $p \in \mathbb{F}$, let

$$\epsilon_n(z_1, z_2, \dots, z_n; p) := \prod_{1 \leq i < j \leq n} \frac{(z_i - pz_j)(z_i - p^{-1}z_j)}{(z_i - z_j)^2}, \quad (1.5)$$

and set $\epsilon_\lambda(z; p) := (\epsilon_{\lambda_1} * \epsilon_{\lambda_2} * \dots * \epsilon_{\lambda_l})(z; p)$ for a multi-index $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$.

Fact 1.4 ([FHHSY, Propositions 2.20 & 2.23]). For $i = 1, 2, 3$, $\{\epsilon_\lambda(z; q_i)\}_{\lambda \vdash n}$ forms a basis of $\mathcal{A}_n$.

Let us write the expansions of P_λ in the bases $\{e_\mu\}$ and $\{g_\mu\}$ by

$$P_\lambda(z; q, t) = \sum_{\mu \geq \lambda'} c_{\lambda\mu}^{e \rightarrow P}(q, t) e_\mu(z; q, t), \quad P_\lambda(x; q, t) = \sum_{\mu \geq \lambda} c_{\lambda\mu}^{g \rightarrow P}(q, t) g_\mu(x; q, t). \quad (1.6)$$

A detailed study of the algebra $\mathcal{A}$ with the help of the free field representation allowed us to establish the following equality.

Fact 1.5 ([FHHSY, §3 E]). Set the next two elements in $\mathcal{A}$.

$$f_\lambda^{(q^{-1})}(z; q, t) := \frac{t^{-|\lambda|}}{(1-t^{-1})^{|\lambda|} |\lambda|!} \sum_{\mu \geq \lambda'} c_{\lambda\mu}^{e \rightarrow P}(q, t) \epsilon_\mu(z; q) \frac{|\mu|!}{\prod_{i=1}^{\ell(\mu)} \mu_i!}, \quad (1.7)$$

$$f_\lambda^{(t)}(z; q, t) := \frac{(-1)^{|\lambda|}}{(1-q)^{|\lambda|} |\lambda|!} \sum_{\mu \geq \lambda} c_{\lambda\mu}^{g \rightarrow P}(q, t) \epsilon_\mu(z; t) \frac{|\mu|!}{\prod_{i=1}^{\ell(\mu)} \mu_i!}. \quad (1.8)$$

Then we have $f_\lambda^{(q^{-1})}(z; q, t) = f_\lambda^{(t)}(z; q, t)^1$.

Definition 1.6. Let $F_\lambda(z; q, t) := f_\lambda^{(q^{-1})}(z; q, t) = f_\lambda^{(t)}(z; q, t)$.

Definition 1.7. Define the isomorphism $\iota : \Lambda_{\mathbb{F}} \rightarrow \mathcal{A}$ by

$$\iota(e_\lambda) = \frac{t^{-|\lambda|}}{(1-t^{-1})^{|\lambda|}} \frac{1}{\prod_{i=1}^{\ell(\mu)} \lambda_i!} \epsilon_\lambda(z; q).$$

Proposition 1.8. (1) We have

$$\iota(g_\lambda) = \frac{(-1)^{-|\lambda|}}{(1-q)^{|\lambda|}} \frac{1}{\prod_{i=1}^{\ell(\mu)} \lambda_i!} \epsilon_\lambda(z; t).$$

(2) We have $\iota(P_\lambda) = F_\lambda(z; q, t)$.

Proof. (1) By the Wronski relation given in [FHHSY, Proposition 3.11].

(2) By (1.6) and the definitions of ι and F_λ . □

Remark 1.9. To explain the importance of the element $F_\lambda(z; q, t)$, we recall the Gordon filtration on $\mathcal{A}$. For $p \in \mathbb{F}$ and $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$, we defined a linear map

$$\begin{aligned} \varphi_\lambda^{(p)} : \quad \mathcal{A}_n &\longrightarrow \mathbb{F}(y_1, \dots, y_l) \\ f(z_1, \dots, z_n) &\mapsto f(y_1, py_1, \dots, p^{\lambda_1-1} y_1, \\ &\quad y_2, py_2, \dots, p^{\lambda_2-1} y_2, \\ &\quad \dots \\ &\quad y_l, py_l, \dots, p^{\lambda_l-1} y_l), \end{aligned} \quad (1.9)$$

called the *specialization map*. The Gordon filtration is given by $\mathcal{A}_{n,\lambda}^{(q_i)} := \bigcap_{\mu \leq \lambda} \ker \varphi_\mu^{(q_i)}$ for $i = 1, 2, 3$.

Then by [FHHSY, Theorem 1.19], $\mathcal{A}_{n,\mu}^{(q^{-1})} \cap \mathcal{A}_{n,\mu'}^{(t)}$ is one dimensional and is spanned by $F_\lambda(z; q, t)$.

1.4. The kernel function. Now we are ready to study the kernel function from the point of view of the algebra $\mathcal{A}$.

Definition 1.10. Introduce $K_n(x, z; q, t) \in \Lambda_{\mathbb{F}}^n \otimes \mathcal{A}_n$ as

$$K_n(x, z; q, t) := \sum_{\lambda \vdash n} Q_\lambda(x) F_\lambda(z; q, t).$$

¹Note that the first and second lines of Page 25 in [FHHSY] contains typos and should be read as (1.7) and (1.8).

B. FEIGIN, A. HOSHINO, J. SHIBAHARA, J. SHIRAISHI AND S. YANAGIDA

Remark 1.11. The name “kernel function” comes from $\Pi(x, y)$ in (1.3). By Proposition 1.8 (2), we have, in a suitable completion of $\Lambda_{\mathbb{F}} \otimes \mathcal{A}$,

$$\sum_{n \geq 0} K_n(x, z; q, t) = \sum_{\lambda} Q_{\lambda}(x) \iota(P_{\lambda}(y)),$$

where λ runs over all the partitions of every non-negative integer. Thus K_n is a homogeneous component of the analogue of $\Pi(x, y)$.

Proposition 1.12. In $\Lambda_{\mathbb{F}} \otimes \mathcal{A}$ we have

$$K_n(x, z; q, t) = \frac{(-1)^n}{(1-q)^n n!} \sum_{\lambda \vdash n} m_{\lambda}(x) \epsilon_{\lambda}(z; t) \frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} \quad (1.10)$$

Proof. First we show

$$m_{\lambda}(x) = \sum_{\mu \leq \lambda} c_{\mu\lambda}^{g \rightarrow P}(q, t) Q_{\mu}(x; q, t). \quad (1.11)$$

Since $\{Q_{\mu}(x; q, t)\}$ is a basis of $\Lambda_{\mathbb{F}}$, we can expand $m_{\lambda}(x) = \sum_{\nu} c_{\nu\lambda} Q_{\nu}(x; q, t)$ with $c_{\nu\lambda} \in \mathbb{F}$. Then the pairing $\langle m_{\lambda}, P_{\mu} \rangle_{q,t}$ is calculated as $\langle m_{\lambda}, P_{\mu} \rangle_{q,t} = \langle \sum_{\nu} c_{\nu\lambda} Q_{\nu}(z; q, t), P_{\mu} \rangle_{q,t} = c_{\mu\lambda}$, where we used the fact that $\{P_{\lambda}\}$ and $\{Q_{\lambda}\}$ are dual. On the other hand, by (1.6), we have $\langle m_{\lambda}, P_{\mu} \rangle_{q,t} = \langle m_{\lambda}, \sum_{\nu \geq \mu} c_{\mu\nu}^{g \rightarrow P}(q, t) g_{\nu} \rangle_{q,t} = c_{\mu\lambda}^{g \rightarrow P}(q, t)$. Comparing both expressions, we obtain (1.11).

Then we have

$$\begin{aligned} \text{RHS of (1.10)} &= \frac{(-1)^n}{(1-q)^n n!} \sum_{\lambda \vdash n} \sum_{\mu \leq \lambda} c_{\mu\lambda}^{g \rightarrow P}(q, t) Q_{\mu}(x; q, t) \epsilon_{\lambda}(z; t) \frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} \\ &= \frac{(-1)^n}{(1-q)^n n!} \sum_{\mu \vdash n} Q_{\mu}(x; q, t) \sum_{\lambda \geq \mu} c_{\mu\lambda}^{g \rightarrow P}(q, t) \epsilon_{\lambda}(z; t) \frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} = \sum_{\mu \vdash n} Q_{\mu}(x; q, t) F_{\mu}(z; q, t). \end{aligned}$$

□

Consider the case of finitely many variables and set $x = (x_1, x_2, \dots, x_m)$. Also let $z = (z_1, z_2, \dots, z_n)$ be the set of variables for the elements in $\mathcal{A}_n$.

Proposition 1.13. We have

$$K_n(x, z; q, t) = \frac{(-1)^n}{(1-q)^n n!} \sum_{i_1=1}^m \sum_{i_2=1}^m \cdots \sum_{i_n=1}^m x_{i_1} x_{i_2} \cdots x_{i_n} \prod_{1 \leq \alpha < \beta \leq n} \gamma_{i_{\alpha}, i_{\beta}}(z_{\alpha}, z_{\beta}; q, t), \quad (1.12)$$

where the function $\gamma_{i,j}(z, w; q, t)$ is given by

$$\gamma_{i,j}(z, w; q, t) := \begin{cases} \frac{(z - tw)(z - t^{-1}w)}{(z - w)^2} & i = j, \\ \frac{(z - q^{-1}w)(z - tw)(z - qt^{-1}w)}{(z - w)^3} & i < j, \\ \frac{(z - qw)(z - t^{-1}w)(z - q^{-1}tw)}{(z - w)^3} & i > j. \end{cases} \quad (1.13)$$

Proof. Note that we have

$$\gamma_{i,j}(z, w; q, t) = \begin{cases} \epsilon_2(z, w; t) & i = j, \\ \omega(z, w; q^{-1}, t, qt^{-1}) & i < j, \\ \omega(z, w; q, t^{-1}, q^{-1}t) = \omega(w, z; q^{-1}, t, qt^{-1}) & i > j, \end{cases} \quad (1.14)$$

which is obtained from (1.2), (1.5) and (1.13). Thus we have

$$\begin{aligned} & \sum_{i_1=1}^m \sum_{i_2=1}^m \cdots \sum_{i_n=1}^m x_{i_1} x_{i_2} \cdots x_{i_n} \prod_{1 \leq \alpha < \beta \leq n} \gamma_{i_\alpha, i_\beta}(z_\alpha, z_\beta; q, t) \\ &= \sum_{I_1, \dots, I_m} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} \prod_{k=1}^m \epsilon_{a_k}(z_{I_k}; t) \prod_{1 \leq i < j \leq m} \prod_{\alpha \in I_i, \beta \in I_j} \omega(z_\alpha, z_\beta; q^{-1}, t, qt^{-1}), \end{aligned}$$

where I_k ($k = 1, 2, \dots, m$) is a subset of $\{1, 2, \dots, n\}$ such that $|I_k| = a_k$, $I_1 \cup I_2 \cup \cdots \cup I_m = \{1, \dots, n\}$. Using the multi-index notation $a = (a_1, \dots, a_m) \in \mathbb{N}^m$, we have

$$= \sum_{a \in \mathbb{N}^m, |a|=n} x^a \frac{n!}{\prod_{k=1}^m a_k!} \epsilon_a(z; t)$$

with $|a| := a_1 + \cdots + a_m$. Applying $\mathfrak{S}_m$ on the running index a and averaging them, we have

$$= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_m} \sum_{a \in \mathbb{N}^m, |a|=n} x^{\sigma(a)} \frac{n!}{\prod_{k=1}^m a_k!} \epsilon_{\sigma(a)}(z; t).$$

Dividing $\mathfrak{S}_m$ by the stabilizer $\text{Stab}(a)$ of $a \in \mathbb{N}^m$ and using the commutativity of $\mathcal{A}$, we have

$$= \frac{1}{n!} \sum_{a \in \mathbb{N}^m, |a|=n} \# \text{Stab}(a) \frac{n!}{\prod_{k=1}^m a_k!} \left(\sum_{\bar{\sigma} \in \mathfrak{S}_m / \text{Stab}(a)} x^{\bar{\sigma}(a)} \right) \epsilon_a(z; t).$$

Then we obtain the result by taking a partition λ as the running index. □

1.5. Macdonald's tableau sum formula. We recall the tableau sum formula for the Macdonald polynomials.

Let $\text{Tb}(\lambda; m)$ denotes the set of all the ways of drawing numbers $1, 2, \dots, m$ into the Young diagram of shape λ *without any conditions*. Reading the numbers from left to right then top to bottom, namely in the English reading manner, we get a bijection between $\text{Tb}(\lambda; m)$ and the set $\{1, 2, \dots, m\}^n$.

Let $\text{RTb}(\lambda; m)$ denotes the subset of $\text{Tb}(\lambda; m)$ in which the numbers in each row are arranged in non-decreasing manner. The element of $\text{RTb}(\lambda; m)$ is uniquely specified by the set of numbers $\theta_{i,j}$ which denote the number of j in the i -th row. We have $\lambda_i = \sum_{k=1}^n \theta_{i,k}$ for $1 \leq i \leq n$. Next we introduce a sequence $\lambda^{(j)} = (\lambda_1^{(j)}, \lambda_2^{(j)}, \dots)$ by setting $\lambda_i^{(j)} := \sum_{k=1}^j \theta_{i,k}$. It is clear that we have $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(m)} = \lambda$. Note that $\lambda^{(j)}$ may not be a partition.

Let $\text{SSTb}(\lambda; m)$ be the set of semi-standard tableaux. A semi-standard tableau T is expressed as a sequence of partitions $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(m)} = \lambda$, where the skew diagrams $\lambda^{(k)} / \lambda^{(k-1)}$ ($k = 1, 2, \dots, m$) are horizontal strips. We have $\theta_{i,j} = 0$ for $i > j$, $\lambda_i = \sum_{k=i}^n \theta_{i,k}$ for $1 \leq i \leq n$, and

$$0 \leq \theta_{i,j} \leq \lambda_i - \lambda_{i+1} - \sum_{k=j+1}^{\ell(\lambda)} (\theta_{i,k} - \theta_{i+1,k})$$

for $1 \leq i < j \leq \ell(\lambda)$.

It is known that the $b_\lambda(q, t)$ in (1.4) has the factorized form.

$$Q_\lambda(x; q, t) = b_\lambda(q, t) P_\lambda(x; q, t), \quad b_\lambda(q, t) = \prod_{s \in \lambda} \frac{1 - q^{a(s)} t^{\ell(s)+1}}{1 - q^{a(s)+1} t^{\ell(s)}}, \quad (1.15)$$

where for a box $s = (i, j)$ of λ , $a(s) := \lambda_i - j$ is the arm-length and $\ell(s) := \lambda'_j - i$ is the leg-length.

The $P_\lambda(x; q, t)$ has the tableau sum formula:

$$P_\lambda(x; q, t) = \sum_{T \in \text{SSTb}(\lambda; m)} x^T \psi_T(q, t).$$

B. FEIGIN, A. HOSHINO, J. SHIBAHARA, J. SHIRAISHI AND S. YANAGIDA

Here the coefficient $\psi(q, t) \in \mathbb{F}$ is determined by

$$\begin{aligned} \psi_T(q, t) &:= \prod_{k=1}^m \psi_{\lambda^{(k)}/\lambda^{(k-1)}}(q, t), \\ \psi_{\lambda/\mu}(q, t) &:= \prod_{1 \leq i \leq j \leq \ell(\mu)}^n \frac{f(q^{\mu_i - \mu_j} t^{j-i}) f(q^{\lambda_i - \lambda_{j+1}} t^{j-i})}{f(q^{\lambda_i - \mu_j} t^{j-i}) f(q^{\mu_i - \lambda_{j+1}} t^{j-i})}, \quad f(u) := \frac{(tu; q)_\infty}{(qu; q)_\infty}. \end{aligned} \quad (1.16)$$

The next proposition is obtained by simple combinatorics, and we omit the proof for lack of space.

Proposition 1.14. Let $T \in \text{RTb}(\lambda; m) \setminus \text{SSTb}(\lambda; m)$ and regard T as a sequence $\lambda^{(j)}$ explained as above. Then $\psi_T(q, t)$ calculated from (1.16) vanishes.

1.6. Tableau sum formula and $K_n(x, z; q, t)$. Now we investigate the relationship between the function $K_n(x, z; q, t)$ and the tableau formula of Macdonald polynomial. We fix a natural number m and consider the case $x = (x_1, \dots, x_m)$.

In order to state the main result, we need to consider the composition of the specialization maps $\varphi_\lambda^{(p)}$ of (1.9). For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of n and $\zeta \in \mathbb{F}$, we define the map $\tilde{\varphi}_\lambda^{(\zeta)}$ by

$$\begin{aligned} \tilde{\varphi}_\lambda^{(\zeta)} &:= \varphi_{(l)}^{(\zeta)} \circ \varphi_\lambda^{(q^{-1})} : \mathbb{F}(z_1, \dots, z_n) \longrightarrow \mathbb{F}(y) \\ f(z_1, \dots, z_n) &\mapsto f(y, q^{-1}y, \dots, q^{-(\lambda_1-1)}y, \\ &\quad \zeta y, q^{-1}\zeta y, \dots, q^{-(\lambda_2-1)}\zeta y, \\ &\quad \vdots, \\ &\quad \zeta^{l-1}y, q^{-1}\zeta^{l-1}y, \dots, q^{-(\lambda_l-1)}\zeta^{l-1}y). \end{aligned} \quad (1.17)$$

Here the map $\varphi_{(l)}^{(\zeta)}$ denotes the substitution $\varphi_{(l)}^{(\zeta)} g(y_1, \dots, y_l) = g(y, \zeta y, \dots, \zeta^{l-1}y)$.

Theorem 1.15. For partitions μ, λ of n , $\tilde{\varphi}_\lambda^{(\zeta)}(F_\mu/F_\lambda)$ is regular at $\zeta = t$ and its value is $\delta_{\lambda, \mu}$.

Our proof uses the tableau sum formula of $P_\lambda(x; q, t)$. Let us express the statement as

$$\lim_{\zeta \rightarrow t} \tilde{\varphi}_\lambda^{(\zeta)} \frac{F_\mu(z; q, t)}{F_\lambda(z; q, t)} = \delta_{\lambda, \mu}.$$

Then by using Proposition 1.12, it can be rewritten into the next equivalent form.

$$\lim_{\zeta \rightarrow t} \tilde{\varphi}_\lambda^{(\zeta)} \frac{K_n(x, z; q, t)}{F_\lambda(z; q, t)} = Q_\lambda(x; q, t). \quad (1.18)$$

Regard $T = (i_1, i_2, \dots, i_n) \in \{1, 2, \dots, m\}^n$ as an element of $\text{Tb}(\lambda; m)$. For simplicity we set

$$\gamma_T(z) := \prod_{1 \leq \alpha < \beta \leq n} \gamma_{i_\alpha, i_\beta}(z_\alpha, z_\beta; q, t).$$

We also use the same symbol for the cases $T \in \text{RTb}(\lambda; m)$ and $T \in \text{SSTb}(\lambda; m)$. By Proposition 1.12, (1.18) is equivalent to

$$\frac{(-1)^n}{(1-q)^n n!} \sum_{T \in \text{Tb}(\lambda; m)} x^T \lim_{\zeta \rightarrow t} \tilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_T(z)}{F_\lambda(z; q, t)} = Q_\lambda(x; q, t).$$

It is easy to see from the definition of $\gamma_{i,j}$ that all the terms with $T \in \text{Tb}(\lambda; m) \setminus \text{RTb}(\lambda; m)$ vanish after the first specialization $\varphi_\lambda^{(q^{-1})}$. Thus we may replace $\sum_{T \in \text{Tb}(\lambda; m)}$ by $\sum_{T \in \text{RTb}(\lambda; m)}$.

Hence it is enough to show that for $T \in \text{RTb}(\lambda; m)$ we have

$$\frac{(-1)^n}{(1-q)^n n!} \lim_{\zeta \rightarrow t} \tilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_T(z)}{F_\lambda(z; q, t)} = b_\lambda(q, t) \psi_T(q, t).$$

We prove this in two steps.

Proposition 1.16. Let $D \in \text{SSTb}(\lambda; m)$ given by $\theta_{i,i} = \lambda_i$ and $\theta_{i,j} = 0$ for $i \neq j$. Then we have

$$\frac{(-1)^n}{(1-q)^n n!} \lim_{\zeta \rightarrow t} \tilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_D(z)}{F_\lambda(z; q, t)} = b_\lambda(q, t), \quad (1.19)$$

$$\lim_{\zeta \rightarrow t} \tilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_T(z)}{\gamma_D(z)} = \psi_T(q, t). \quad (1.20)$$

Proof. The proof is postponed until §3.1. □

2. DING-IOHARA ALGEBRA AND KERNEL FUNCTION

In this section all objects are defined on $\tilde{\mathbb{F}} := \mathbb{Q}(q^{1/2}, t^{1/2})$. We will also use $p := q/t$.

2.1. Review of the Ding-Iohara algebra $\mathcal{U}(q, t)$. Recall that the Ding-Iohara algebra [DI] was introduced as a generalization of the quantum affine algebra, which respects the structure of the Drinfeld coproduct. In [FHHSY, Appendix A], the authors introduced a version $\mathcal{U}(q, t)$ of the Ding-Iohara algebra having two parameters q and t .

Definition 2.1. Set

$$g(z) := \frac{G^+(z)}{G^-(z)}, \quad G^\pm(z) := (1 - q^{\pm 1}z)(1 - t^{\mp 1}z)(1 - q^{\mp 1}t^{\pm 1}z).$$

Then we define $\mathcal{U}(q, t)$ to be a unital associative algebra generated by the Drinfeld currents

$$x^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n}, \quad \psi^\pm(z) = \sum_{\pm n \in \mathbb{N}} \psi_n^\pm z^{-n},$$

and the central element $\gamma^{\pm 1/2}$, satisfying the defining relations

$$\begin{aligned} \psi^\pm(z)\psi^\pm(w) &= \psi^\pm(w)\psi^\pm(z), & \psi^+(z)\psi^-(w) &= \frac{g(\gamma^{+1}w/z)}{g(\gamma^{-1}w/z)}\psi^-(w)\psi^+(z), \\ \psi^+(z)x^\pm(w) &= g(\gamma^{\mp 1/2}w/z)^{\mp 1}x^\pm(w)\psi^+(z), & \psi^-(z)x^\pm(w) &= g(\gamma^{\mp 1/2}z/w)^{\pm 1}x^\pm(w)\psi^-(z), \\ [x^+(z), x^-(w)] &= \frac{(1-q)(1-1/t)}{1-q/t}(\delta(\gamma^{-1}z/w)\psi^+(\gamma^{1/2}w) - \delta(\gamma z/w)\psi^-(\gamma^{-1/2}w)), \\ G^\mp(z/w)x^\pm(z)x^\pm(w) &= G^\pm(z/w)x^\pm(w)x^\pm(z). \end{aligned}$$

Fact 2.2 ([FHHSY, Proposition A.2]). The algebra $\mathcal{U}(q, t)$ has a Hopf algebra structure with Coproduct Δ :

$$\begin{aligned} \Delta(\gamma^{\pm 1/2}) &= \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2}, & \Delta(x^+(z)) &= x^+(z) \otimes 1 + \psi^-(\gamma_{(1)}^{1/2}z) \otimes x^+(\gamma_{(1)}z), \\ \Delta(\psi^\pm(z)) &= \psi^\pm(\gamma_{(2)}^{\pm 1/2}z) \otimes \psi^\pm(\gamma_{(1)}^{\mp 1/2}z), & \Delta(x^-(z)) &= x^-(\gamma_{(2)}z) \otimes \psi^+(\gamma_{(2)}^{1/2}z) + 1 \otimes x^-(z), \end{aligned}$$

where $\gamma_{(1)}^{\pm 1/2} = \gamma^{\pm 1/2} \otimes 1$ and $\gamma_{(2)}^{\pm 1/2} = 1 \otimes \gamma^{\pm 1/2}$.

Counit ε :

$$\varepsilon(\gamma^{\pm 1/2}) = 1, \quad \varepsilon(\psi^\pm(z)) = 1, \quad \varepsilon(x^\pm(z)) = 0.$$

Antipode a :

$$\begin{aligned} a(\gamma^{\pm 1/2}) &= \gamma^{\mp 1/2}, & a(x^+(z)) &= -\psi^-(\gamma^{-1/2}z)^{-1}x^+(\gamma^{-1}z), \\ a(\psi^\pm(z)) &= \psi^\pm(z)^{-1}, & a(x^-(z)) &= -x^-(\gamma^{-1}z)\psi^+(\gamma^{-1/2}z)^{-1}. \end{aligned}$$

2.2. Level one representation of $\mathcal{U}(q, t)$. We say a representation of $\mathcal{U}(q, t)$ is of level k , if the central element γ is realized by the constant $(t/q)^{k/2} = p^{-k/2}$.

Fact 2.3 ([FHHSY, Proposition A.6]). Consider the Heisenberg Lie algebra $\mathfrak{h}$ over $\mathbb{F}$ with the generators a_n ($n \in \mathbb{Z}$) and the relations

$$[a_m, a_n] = m \frac{1 - q^{|m|}}{1 - t^{|m|}} \delta_{m+n,0} a_0. \quad (2.1)$$

Let $\mathfrak{h}^{\geq 0}$ (resp. $\mathfrak{h}^{< 0}$) be the subalgebra generated by a_n for $n \geq 0$ (resp. $n < 0$). Consider the one dimensional representation $\tilde{\mathcal{F}}$ of $\mathfrak{h}^{\geq 0}$, where a_n ($n > 0$) acts trivially and a_0 acts by some fixed element of $\tilde{\mathcal{F}}$. Then one has the induced Fock representation $\mathcal{F} := \text{Ind}_{\mathfrak{h}^{\geq 0}}^{\mathfrak{h}} \tilde{\mathcal{F}}$ of $\mathfrak{h}$. Let us also introduce the following four vertex operators [FHHSY, (1.7), (3.23), (3.27), (3.28)].

$$\begin{aligned} \eta(z) &:= \exp \left(\sum_{n>0} \frac{1 - t^{-n}}{n} a_{-n} z^n \right) \exp \left(- \sum_{n>0} \frac{1 - t^n}{n} a_n z^{-n} \right), \\ \xi(z) &:= \exp \left(- \sum_{n>0} \frac{1 - t^{-n}}{n} p^{-n/2} a_{-n} z^n \right) \exp \left(\sum_{n>0} \frac{1 - t^n}{n} p^{-n/2} a_n z^{-n} \right), \\ \varphi^+(z) &:= \exp \left(- \sum_{n>0} \frac{1 - t^n}{n} (1 - p^{-n}) p^{n/4} a_n z^{-n} \right), \\ \varphi^-(z) &:= \exp \left(\sum_{n>0} \frac{1 - t^{-n}}{n} (1 - p^{-n}) p^{n/4} a_{-n} z^n \right). \end{aligned}$$

Then for a fixed $c \in \tilde{\mathbb{F}}^\times$, we have a level one representation $\rho_c(\cdot)$ of $\mathcal{U}(q, t)$ on $\mathcal{F}$ by setting

$$\rho_c(\gamma^{\pm 1/2}) = p^{\mp 1/4}, \quad \rho_c(\psi^\pm(z)) = \varphi^\pm(z), \quad \rho_c(x^+(z)) = c \eta(z), \quad \rho_c(x^-(z)) = c^{-1} \xi(z).$$

Remark 2.4. We can rephrase this fact as follows. Let us define b_n 's by the expansion of $\psi^\pm$:

$$\psi^+(z) = \psi_0^+ \exp \left(+ \sum_{n>0} b_n \gamma^{n/2} z^{-n} \right), \quad \psi^-(z) = \psi_0^- \exp \left(- \sum_{n>0} b_{-n} \gamma^{n/2} z^n \right). \quad (2.2)$$

Then we have

$$[b_m, b_n] = \frac{1}{m} (1 - q^{-m})(1 - t^m)(1 - p^m)(\gamma^m - \gamma^{-m}) \gamma^{-|m|} \delta_{m+n,0}, \quad (2.3)$$

and the coproduct for b_n reads

$$\Delta(b_n) = b_n \otimes \gamma^{-|n|} + 1 \otimes b_n. \quad (2.4)$$

Then the representation ρ_c is given by $\gamma^{\pm 1/2} \mapsto p^{\mp 1/4}$ and

$$b_n \mapsto -\frac{1 - t^n}{|n|} (p^{|n|/2} - p^{-|n|/2}) a_n, \quad \psi_0^\pm \mapsto 1, \quad x^+(z) \mapsto c \eta(z), \quad x^-(z) \mapsto c^{-1} \xi(z).$$

Definition 2.5. Consider the m -fold tensor representation $\rho_{y_1} \otimes \cdots \otimes \rho_{y_m}$ on $\mathcal{F}^{\otimes m}$ for $m \in \mathbb{Z}_{\geq 2}$. Define $\Delta^{(m)}$ inductively by

$$\Delta^{(2)} := \Delta, \quad \Delta^{(m)} := (\text{id} \otimes \cdots \otimes \text{id} \otimes \Delta) \circ \Delta^{(m-1)}.$$

Since we have $\rho_{y_1} \otimes \cdots \otimes \rho_{y_m} \Delta^{(m)}(\gamma) = \gamma_{(1)} \cdots \gamma_{(m)} = p^{-m/2}$, the level is m . We also define

$$\rho_y^{(m)} := \rho_{y_1} \otimes \cdots \otimes \rho_{y_m} \circ \Delta^{(m)} : \mathcal{U}(q, t) \rightarrow \mathcal{F}^{\otimes m}. \quad (2.5)$$

Lemma 2.6. We have

$$\rho_y^{(m)}(x^+(z)) = \sum_{i=1}^m y_i \tilde{\Lambda}_i(z), \quad \rho_y^{(m)}(x^-(z)) = \sum_{i=1}^m y_i^{-1} \tilde{\Lambda}_i^*(z),$$

where the $\tilde{\Lambda}_i(z)$, $\tilde{\Lambda}_i^*(z)$ are defined to be

$$\tilde{\Lambda}_i(z) := \varphi^-(p^{-1/4}z) \otimes \varphi^-(p^{-3/4}z) \otimes \cdots \otimes \varphi^-(p^{-(2i-3)/4}z) \otimes \eta(p^{-(i-1)/2}z) \otimes 1 \otimes \cdots \otimes 1, \quad (2.6)$$

$$\tilde{\Lambda}_i^*(z) := 1 \otimes \cdots \otimes 1 \otimes \xi(p^{-(m-i)/2}z) \otimes \varphi^+(p^{-(2m-2i-1)/4}z) \otimes \cdots \otimes \varphi^+(p^{-1/4}z), \quad (2.7)$$

where $\eta(p^{-(i-1)/2}z)$ and $\xi(p^{-(m-i)/2}z)$ sit in the i -th tensor component.

Proof. By the definition (2.5), Fact 2.3 and Remark 2.4. □

2.3. New currents $t(z)$ and $t^*(z)$.

Definition 2.7. We define

$$t(z) := \alpha(z)x^+(z)\beta(z), \quad t^*(z) := \alpha(p^{-1}z)^{-1}x^-(p^{-1}\gamma^{-1}z)\beta(\gamma^{-2}p^{-1}z)^{-1}. \quad (2.8)$$

Here we used auxiliary vertex operators

$$\alpha(z) := \exp\left(-\sum_{n=1}^{\infty} \frac{1}{\gamma^n - \gamma^{-n}} b_{-n} z^n\right), \quad \beta(z) := \exp\left(\sum_{n=1}^{\infty} \frac{1}{\gamma^n - \gamma^{-n}} b_n z^{-n}\right). \quad (2.9)$$

Here the part $1/(\gamma^n - \gamma^{-n})$ is considered to be the formal power sum $\sum_{i=0}^{\infty} \gamma^{-(2i+1)n}$.

Remark 2.8. The definition of $t^*(z)$ can be read as

$$t^*(\gamma pz) = \alpha(\gamma z)^{-1}x^-(z)\beta(\gamma^{-1}z)^{-1}.$$

This form is convenient in the actual calculations.

Proposition 2.9. (1) The elements $t(z)$ and $t^*(z)$ commutes with $\alpha(w)$, $\beta(w)$ and $\psi^\pm(w)$:

$$\begin{aligned} [t(z), \alpha(w)] &= [t(z), \beta(w)] = [t^*(z), \alpha(w)] = [t^*(z), \beta(w)] = 0, \\ [t(z), \psi^\pm(w)] &= [t^*(z), \psi^\pm(w)] = 0. \end{aligned}$$

(2) Set

$$A(z) := \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})(1-p^{-n}\gamma^{-2n})}{1-\gamma^{-2n}} z^n\right), \quad (2.10)$$

where the part $1/(1-\gamma^{-2n})$ is considered to be the formal power sum $\sum_{i=0}^{\infty} \gamma^{-2in}$. Then we have

$$A\left(\frac{w}{z}\right)t(z)t(w) - A\left(\frac{z}{w}\right)t(w)t(z) = \frac{(1-q)(1-t^{-1})}{1-p} [\delta(p^{-1}\frac{w}{z})t^{(2)}(z) - \delta(p\frac{w}{z})t^{(2)}(w)], \quad (2.11)$$

$$A\left(\frac{w}{z}\right)t^*(z)t^*(w) - A\left(\frac{z}{w}\right)t^*(w)t^*(z) = \frac{(1-q^{-1})(1-t)}{1-p^{-1}} [\delta(p\frac{w}{z})t^{*(2)}(z) - \delta(p^{-1}\frac{w}{z})t^{*(2)}(w)], \quad (2.12)$$

where $\delta(z) := \sum_{n \in \mathbb{N}} z^n + z^{-1} \sum_{n \in \mathbb{N}} z^{-n}$ is the formal delta function, and

$$\begin{aligned} t^{(2)}(z) &:= \alpha(pz)\alpha(z)x^+(pz)x^+(z)\beta(pz)\beta(z), \\ t^{*(2)}(z) &:= \alpha(\gamma pz)^{-1}\alpha(\gamma z)^{-1}x^-(pz)x^-(z)\beta(\gamma^{-1}pz)^{-1}\beta(\gamma^{-1}z)^{-1}. \end{aligned}$$

(3) As in (2), set

$$B(z) := \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})(p^{-2n}\gamma^{-2n} - p^{-n}\gamma^{-2n})}{1-\gamma^{-2n}} z^n\right). \quad (2.13)$$

Then

$$B(\frac{w}{z})t(z)t^*(w) - B(\gamma^2 p^2 \frac{z}{w})t^*(w)t(z) = \frac{(1-q)(1-t^{-1})}{1-p} \left(\delta(p^{-1} \frac{w}{z})\psi_0^+ - \delta(\gamma^{-2} p^{-1} \frac{w}{z})\psi_0^- \right). \quad (2.14)$$

Proof. See §3.2. $\square$

In the next subsection we show that the currents $t(z)$, $t^*(z)$ are connected to the realization of deformed $\mathcal{W}$ algebra in the Fock representation of $\mathcal{U}(q, t)$.

2.4. Deformed algebra $\mathcal{W}_{q,p}(\mathfrak{sl}_m)$. We basically follow the description of $\mathcal{W}_{q,p}(\mathfrak{sl}_m)$ in [FF, §4]. As for the connection between the singular vectors of the $\mathcal{W}_{q,p}(\mathfrak{sl}_m)$ and the Macdonald polynomials, see [SKAO, AKOS].

Definition 2.10. Set

$$f_{k,l}(z) := \exp \left(\sum_{n=1}^{\infty} \frac{(1-q^n)(1-t^{-n})(p^{(k-1)n} - p^{(l-1)n})}{1-p^{ln}} z^n \right).$$

Remark 2.11. Our functions $A(z)$ and $B(z)$ give special cases of this function under $\rho_y^{(m)}$, that is,

$$\rho_y^{(m)}(A(z)) = f_{1,m}(z), \quad \rho_y^{(m)}(B(z)) = f_{m-1,m}(z).$$

Definition 2.12. Set

$$T(z) = T_1(z) := \rho_y^{(m)}(t(z)), \quad T^*(z) = T_1^*(z) := \rho_y^{(m)}(t^*(z)). \quad (2.15)$$

Let us also define

$$\Lambda_i(z) := \rho_y^{(m)}(\alpha(z)) \tilde{\Lambda}_i(z) \rho_y^{(m)}(\beta(z)), \quad \Lambda_i^*(z) := \rho_y^{(m)}(\alpha(p^{-1}z)^{-1}) \tilde{\Lambda}_i^*(p^{(m-2)/2}z) \rho_y^{(m)}(\beta(\gamma^{-2}p^{-1}z)^{-1}). \quad (2.16)$$

Then by Definition 2.7 and Lemma 2.6 we have

$$T_1(z) = \sum_{i=1}^m y_i \Lambda_i(z), \quad T_1^*(z) = \sum_{i=1}^m y_i^{-1} \Lambda_i^*(z). \quad (2.17)$$

For $i = 2, \dots, m$, we further define

$$T_i(z) := \sum_{1 \leq j_1 < \dots < j_i \leq m} y_{j_1} y_{j_2} \cdots y_{j_i} : \Lambda_{j_1}(z) \Lambda_{j_2}(zp) \cdots \Lambda_{j_i}(zp^{i-1}) :, \quad (2.18)$$

$$T_i^*(z) := \sum_{1 \leq j_1 < \dots < j_i \leq m} y_{j_1}^{-1} y_{j_2}^{-1} \cdots y_{j_i}^{-1} : \Lambda_{j_1}^*(z) \Lambda_{j_2}^*(zp^{-1}) \cdots \Lambda_{j_i}^*(zp^{-i+1}) :. \quad (2.19)$$

Proposition 2.13. (1) The operator product of $\Lambda_i(z)$ and $\Lambda_j(w)$ is given by

$$f_{1,m}(\frac{w}{z}) \Lambda_i(z) \Lambda_j(w) = : \Lambda_i(z) \Lambda_j(w) : \times \begin{cases} 1 & i = j, \\ \gamma_+(z, w; q, p) & i < j, \\ \gamma_-(z, w; q, p) & i > j. \end{cases} \quad (2.20)$$

Here we used the symbol

$$\gamma_+(z, w; q, t) := \frac{(z - q^{-1}w)(z - qt^{-1}w)}{(z - w)(z - t^{-1}w)}, \quad \gamma_-(z, w; q, t) := \frac{(z - qw)(z - q^{-1}tw)}{(z - w)(z - tw)}. \quad (2.21)$$

(2) We have

$$: \Lambda_1(z) \Lambda_2(pz) \cdots \Lambda_m(p^{m-1}z) : = 1.$$

Therefore $T_m(z) = y_1 y_2 \cdots y_m$.

(3) The $\Lambda_i(z)$ and $\Lambda_j^*(z)$ are connected by the following equation.

$$\Lambda_k^*(z) =: \prod_{i=1}^{k-1} \Lambda_i(p^{k-1}z) \prod_{l=k+1}^m \Lambda_l(p^{l-2}z) :. \quad (2.22)$$

Thus we also have

$$T_1^*(z) = y_1^{-1} y_2^{-1} \cdots y_m^{-1} T_{m-1}(z). \quad (2.23)$$

(4) The operator product of $\Lambda_i^*(z)$ and $\Lambda_j^*(w)$ is given by

$$f_{1,m}(\frac{w}{z}) \Lambda_i^*(z) \Lambda_j^*(w) =: \Lambda_i^*(z) \Lambda_j^*(w) : \times \begin{cases} 1 & i = j, \\ \gamma_-(z, w; q, p) & i < j, \\ \gamma_+(z, w; q, p) & i > j. \end{cases} \quad (2.24)$$

(5) We have

$$: \Lambda_1^*(z) \Lambda_2^*(p^{-1}z) \cdots \Lambda_m^*(p^{-m+1}z) := 1.$$

Therefore $T_m^*(z) = y_1^{-1} y_2^{-1} \cdots y_m^{-1}$.

Proof. See §3.3. □

Proposition 2.14. We have

$$f_{1,m}(\frac{w}{z}) T_1(z) T_i(w) - f_{1,m}(p^{1-i} \frac{z}{w}) T_i(w) T_1(z) = \frac{(1-q)(1-t^{-1})}{1-p} [\delta(p^{-1} \frac{w}{z}) T_{i+1}(z) - \delta(p^i \frac{w}{z}) T_{i+1}(w)], \quad (2.25)$$

$$\begin{aligned} f_{1,m}(\frac{w}{z}) T_{m-1}(z) T_{m-1}(w) - f_{1,m}(\frac{z}{w}) T_{m-1}(w) T_{m-1}(z) \\ = \frac{(1-q^{-1})(1-t)}{1-p^{-1}} [\delta(p \frac{w}{z}) T_2^*(z) - \delta(p^{-1} \frac{w}{z}) T_2^*(w)]. \end{aligned} \quad (2.26)$$

Proof. (2.25) follows from (2.17), (2.18) and (2.20). See [FF, Theorem 2] for detail².

(2.26) is also shown by the same method using (2.23), (2.19) and (2.24). □

2.5. Deformed $\mathcal{W}$ algebra and kernel function. Our final consequence of this paper relates the vacuum expectation values of the deformed algebra $\mathcal{W}_{q,p}$ with the finite kernel function.

Theorem 2.15. Let $|0\rangle$ be the vacuum of $\mathcal{F}$, that is, $a_0|0\rangle = |0\rangle$ and $a_n|0\rangle = 0$ for $n > 0$. Let $\langle 0|$ to be the dual vacuum. We denote the tensor $|0\rangle^{\otimes m} \in \mathcal{F}^{\otimes m}$ by the same symbol $|0\rangle$. We use the similar abbreviation for the tensored dual vacuum. Then, denoting $y = (y_1, \dots, y_m)$, we have

$$\frac{(-1)^n}{(1-q)^n n!} \prod_{i < j} f_{1,m}(z_i/z_j) \langle 0| T_1(z_1) T_1(z_2) \cdots T_1(z_n) |0\rangle = K_n(y, z; q, p).$$

Proof. This follows from (2.17), the operator product (2.20) and the definition (1.12). □

²It seems that [FF] contains some typo. In (6.2) of that paper, the term $f_{m,N}(\frac{z}{w})$ should be $f_{m,N}(p^{1-m} \frac{z}{w})$.

3. PROOFS OF THE PROPOSITIONS

3.1. **Proof of Proposition 1.16.** Using the $\gamma_{\pm}$ defined in (2.21), we have

$$\frac{\omega(z, w)}{\epsilon_2^{(t)}(z, w)} = \gamma_+(z, w; q, t), \quad \frac{\omega(w, z)}{\epsilon_2^{(t)}(w, z)} = \gamma_-(z, w; q, t), \quad \frac{\omega(w, z)}{\omega(z, w)} = \frac{\gamma_-(z, w; q, t)}{\gamma_+(z, w; q, t)}. \quad (3.1)$$

For later purpose, we prepare the following formulae. Let θ and ρ be natural numbers. Then

$$\prod_{1 \leq i < j \leq \theta} \gamma_+(q^{-i}z, q^{-j}w; q, t) = \left(\frac{1 - tz/w}{1 - qz/w} \right)^{\theta} \frac{(qz/w)_{\theta}}{(tz/w)_{\theta}} \quad (3.2)$$

$$\prod_{1 \leq i < j \leq \theta} \gamma_-(q^{-i}z, q^{-j}w; q, t) = \left(\frac{1 - z/w}{1 - qt^{-1}z/w} \right)^{\theta} \frac{(qt^{-1}z/w)_{\theta}}{(z/w)_{\theta}} \quad (3.3)$$

$$\prod_{l=1}^{\theta} \prod_{k=1}^{\rho} \gamma_+(q^{-l}z, q^{-k}w; q, t) = \frac{(q^{-\rho}w/z)_{\theta}}{(w/z)_{\theta}} \frac{(qt^{-1}w/z)_{\theta}}{(q^{-\rho+1}t^{-1}w/z)_{\theta}} \quad (3.4)$$

$$= \frac{(q^{\rho-\theta+1}z/w)_{\theta}}{(q^{-\theta+1}z/w)_{\theta}} \frac{(q^{-\theta}tz/w)_{\theta}}{(q^{\rho-\theta}tz/w)_{\theta}}, \quad (3.5)$$

$$\prod_{l=1}^{\theta} \prod_{k=1}^{\rho} \gamma_-(q^{-l}z, q^{-k}w; q, t) = \frac{(qw/z)_{\theta}}{(q^{-\rho+1}w/z)_{\theta}} \frac{(q^{-\rho}tw/z)_{\theta}}{(tw/z)_{\theta}} \quad (3.6)$$

$$= \frac{(q^{-\theta}z/w)_{\theta}}{(q^{\rho-\theta}z/w)_{\theta}} \frac{(q^{\rho-\theta+1}t^{-1}z/w)_{\theta}}{(q^{-\theta+1}t^{-1}z/w)_{\theta}}. \quad (3.7)$$

Here we used $(u)_n := (u; q)_n = \prod_{i=1}^n (1 - uq^{i-1})$. These equations are checked by simple calculations.

3.1.1. *Proof of (1.19).* By (1.7) we have

$$\frac{(1-q)^{n!}}{(-1)^n} F_{\lambda}(z; q, t) = \left(\frac{1-q}{1-t} \right)^{|\lambda|} \sum_{\mu \geq \lambda'} c_{\lambda\mu}^{e \rightarrow P}(q, t) \epsilon_{\mu}(z; q) \frac{|\mu|!}{\prod_{i=1}^{\ell(\mu)} \mu_i!}.$$

Recalling the argument of [FHHSY, Proposition 2.19], we find that under the specialization $\varphi_{\lambda}^{(q^{-1})}$ only the term $\epsilon_{\lambda'}$ in $F_{\lambda}(z; q, t)$ survives and the other terms ϵ_{μ} vanish. The specialization result is ³

$$\begin{aligned} \varphi_{\lambda}^{(q^{-1})} \epsilon_{\lambda'}(y) &= \frac{\prod_{h=1}^{\ell(\lambda)} \lambda'_h!}{n!} \prod_{i=1}^{\ell(\lambda)} \epsilon_{\lambda'_i}(y_1, \dots, y_{\lambda'_i}; q) \prod_{1 \leq j < k \leq \ell(\lambda')} \prod_{\alpha=1}^{\lambda'_j} \prod_{\beta=1}^{\lambda'_k} \omega(q^{-j+1}y_{\alpha}, q^{-k+1}y_{\beta}) \\ &= \frac{\prod_{h=1}^{\ell(\lambda)} \lambda'_h!}{n!} \prod_{i=1}^{\ell(\lambda)} \epsilon_{\lambda'_i}(y_1, \dots, y_{\lambda'_i}; q) \times \prod_{\alpha=1}^{\ell(\lambda)} \prod_{1 \leq i < j \leq \lambda_{\alpha}} \omega(q^{-i+1}y_{\alpha}, q^{-j+1}y_{\alpha}) \\ &\quad \times \prod_{1 \leq \alpha < \beta \leq \ell(\lambda)} \left[\prod_{1 \leq i < j \leq \lambda_{\beta}} \omega(q^{-i+1}y_{\alpha}, q^{-j+1}y_{\beta}) \omega(q^{-i+1}y_{\beta}, q^{-j+1}y_{\alpha}) \right. \\ &\quad \left. \times \prod_{i=1}^{\lambda_{\beta}} \prod_{j=\lambda_{\beta}+1}^{\lambda_{\alpha}} \omega(q^{-i+1}y_{\beta}, q^{-j+1}y_{\alpha}) \right]. \end{aligned}$$

We also note that $c_{\lambda\lambda'}^{e \rightarrow P}(q, t) = 1$.

³This expression is given at the last equation in the proof of [FHHSY, Proposition 2.19], although it contains a typo. The range “ $1 \leq j < k \leq l$ ” of the third product should be “ $1 \leq j < k \leq \ell(\lambda')$ ”

Recalling (1.14), we can also calculate the first specialization $\varphi_\lambda^{(q^{-1})}$ of the numerator in (1.19) as

$$\begin{aligned} \varphi_\lambda^{(q^{-1})} \gamma_D(z) &= \left[\prod_{k=1}^{\ell(\lambda)} \prod_{1 \leq i < j \leq \lambda_k} \epsilon_2(q^{-i}, q^{-j}; t) \right] \left[\prod_{\alpha=1}^{\ell(\lambda)} \prod_{\beta=\alpha}^{\ell(\lambda)} \prod_{i=1}^{\lambda_\alpha} \prod_{j=1}^{\lambda_\beta} \omega(q^{-i} y_\alpha, q^{-j} y_\beta) \right] \\ &= \left[\prod_{k=1}^{\ell(\lambda)} \prod_{1 \leq i < j \leq \lambda_k} \epsilon_2(q^{-i}, q^{-j}; t) \right] \prod_{1 \leq \alpha < \beta \leq \ell(\lambda)} \left[\left(\prod_{1 \leq i < j \leq \lambda_\beta} \omega(q^{-i} y_\alpha, q^{-j} y_\beta) \right) \left(\prod_{1 \leq j < i \leq \lambda_\beta} \omega(q^{-i} y_\alpha, q^{-j} y_\beta) \right) \right. \\ &\quad \left. \left(\prod_{j=1}^{\lambda_\beta} \prod_{i=\lambda_\beta+1}^{\lambda_\alpha} \omega(q^{-i} y_\alpha, q^{-j} y_\beta) \right) \left(\prod_{i=1}^{\lambda_\alpha} \omega(q^{-i} y_\alpha, q^{-i} y_\beta) \right) \right]. \end{aligned}$$

Thus we have

$$\begin{aligned} \frac{(-1)^n}{(1-q)^n n!} \tilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_D(z)}{F_\lambda(z; q, t)} &= \left(\frac{1-t}{1-q} \right)^{|\lambda|} \prod_{\alpha=1}^{\ell(\lambda)} \prod_{1 \leq i < j \leq \lambda_\alpha} \frac{\epsilon_2^{(t)}(q^{-i+1} y_\alpha, q^{-j+1} y_\beta)}{\omega(q^{-i+1} y_\alpha, q^{-j+1} y_\beta)} \times \\ &\quad \prod_{1 \leq \alpha < \beta \leq \ell(\lambda)} \left[\left(\prod_{1 \leq i < j \leq \lambda_\beta} \frac{\omega(q^{-i+1} y_\alpha, q^{-j+1} y_\beta)}{\omega(q^{-i+1} y_\beta, q^{-j+1} y_\alpha)} \right) \left(\prod_{i=1}^{\lambda_\beta} \prod_{j=\lambda_\beta+1}^{\lambda_\alpha} \frac{\omega(q^{-j+1} y_\alpha, q^{-i+1} y_\beta)}{\omega(q^{-i+1} y_\beta, q^{-j+1} y_\alpha)} \right) \left(\frac{\omega(y_\alpha, y_\beta)}{\epsilon_2^{(q)}(y_\beta, y_\alpha)} \right)^{\lambda_\beta} \right]. \end{aligned}$$

Then recalling (3.1) and using (3.2) and (3.3), one has

$$\begin{aligned} \prod_{1 \leq i < j \leq \lambda_\alpha} \frac{\epsilon_2^{(t)}(q^{-i+1} y_\alpha, q^{-j+1} y_\beta)}{\omega(q^{-i+1} y_\alpha, q^{-j+1} y_\beta)} &= \left(\frac{1-q}{1-t} \right)^{\lambda_\alpha} \frac{(t)_{\lambda_\alpha}}{(q)_{\lambda_\alpha}}, \\ \prod_{1 \leq i < j \leq \lambda_\beta} \frac{\omega(q^{-i+1} y_\alpha, q^{-j+1} y_\beta)}{\omega(q^{-i+1} y_\beta, q^{-j+1} y_\alpha)} \left[\frac{\omega(y_\alpha, y_\beta)}{\epsilon_2^{(q)}(y_\beta, y_\alpha)} \right]^{\lambda_\beta} &= \frac{(ty_\beta/y_\alpha)_{\lambda_\alpha}}{(qy_\beta/y_\alpha)_{\lambda_\alpha}} \frac{(qt^{-1}y_\beta/y_\alpha)_{\lambda_\alpha}}{(y_\beta/y_\alpha)_{\lambda_\alpha}}, \\ \prod_{i=1}^{\lambda_\beta} \prod_{j=\lambda_\beta+1}^{\lambda_\alpha} \frac{\omega(q^{-j+1} y_\alpha, q^{-i+1} y_\beta)}{\omega(q^{-i+1} y_\beta, q^{-j+1} y_\alpha)} &= \frac{(qy_\beta/y_\alpha)_{\lambda_\beta}}{(ty_\beta/y_\alpha)_{\lambda_\beta}} \frac{(y_\beta/y_\alpha)_{\lambda_\beta}}{(qt^{-1}y_\beta/y_\alpha)_{\lambda_\beta}} \frac{(q^{\lambda_\alpha-\lambda_\beta} ty_\beta/y_\alpha)_{\lambda_\beta}}{(q^{\lambda_\alpha-\lambda_\beta} y_\beta/y_\alpha)_{\lambda_\beta}} \frac{(q^{\lambda_\alpha-\lambda_\beta+1} t^{-1} y_\beta/y_\alpha)_{\lambda_\beta}}{(q^{\lambda_\alpha-\lambda_\beta+1} y_\beta/y_\alpha)_{\lambda_\beta}}. \end{aligned}$$

Combining these factors, we obtain

$$\begin{aligned} \frac{(-1)^n}{(1-q)^n n!} \lim_{\zeta \rightarrow t} \tilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_D(z; t)}{F_\lambda(z; q, t)} &= \prod_{\alpha=1}^{\ell(\lambda)} \frac{(t)_{\lambda_\alpha}}{(q)_{\lambda_\alpha}} \prod_{1 \leq \alpha < \beta \leq \ell(\lambda)} \frac{(q^{\lambda_\alpha-\lambda_\beta} ty_\beta/y_\alpha)_{\lambda_\beta}}{(q^{\lambda_\alpha-\lambda_\beta} y_\beta/y_\alpha)_{\lambda_\beta}} \frac{(q^{\lambda_\alpha-\lambda_\beta+1} t^{-1} y_\beta/y_\alpha)_{\lambda_\beta}}{(q^{\lambda_\alpha-\lambda_\beta+1} y_\beta/y_\alpha)_{\lambda_\beta}} \\ &= \prod_{1 \leq \alpha < \beta \leq \ell(\lambda)} \frac{(q^{\lambda_\alpha-\lambda_\beta} t^{\beta-\alpha+1})_{\lambda_\beta-\lambda_{\beta+1}}}{(q^{\lambda_\alpha-\lambda_\beta+1} t^{\beta-\alpha})_{\lambda_\beta-\lambda_{\beta+1}}}. \end{aligned}$$

But one can easily find that the last expression equals to $b_\lambda(q, t)$ using the form (1.15).

3.1.2. Proof of (1.20). For a tableau $T \in \text{RTb}(\lambda; m)$, define $\theta_{\alpha, k}$ and $\lambda_\alpha^{(k)}$ as explained in §1.5. Then by the direct calculation we have

$$\tilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_T(z)}{\gamma_D(z)} = \prod_{1 \leq \alpha < \beta \leq \ell(\lambda)} \prod_{i=1}^{\lambda_\alpha} \prod_{j=1}^{\lambda_\beta} \gamma_+(q^{-i} \zeta^\alpha, q^{-j} \zeta^\beta)^{-1} \quad (3.8)$$

B. FEIGIN, A. HOSHINO, J. SHIBAHARA, J. SHIRAISHI AND S. YANAGIDA

$$\times \prod_{k=1}^m \prod_{\alpha=1}^{\ell(\lambda)} \prod_{\beta=\alpha+1}^{\ell(\lambda)} \prod_{i=1}^{\theta_{\alpha,k}} \prod_{j=1}^{\lambda_{\beta}^{(k-1)}} \gamma_{-}(q^{-i-\lambda_{\alpha}^{(k-1)}} \zeta^{\alpha}, q^{-j} \zeta^{\beta}) \quad (3.9)$$

$$\times \prod_{k=1}^m \prod_{\alpha=1}^{\ell(\lambda)} \prod_{\beta=\alpha}^{\ell(\lambda)} \prod_{i=1}^{\theta_{\alpha,k}} \prod_{j=1+\lambda_{\beta}^{(k)}}^{\lambda_{\beta}} \gamma_{+}(q^{-i-\lambda_{\alpha}^{(k-1)}} \zeta^{\alpha}, q^{-j} \zeta^{\beta}) \quad (3.10)$$

By the formula (3.4) we find that

$$\lim_{\zeta \rightarrow t} (3.8) = \prod_{1 \leq \alpha < \beta \leq \ell(\lambda)} \frac{(t^{\beta-\alpha})_{\lambda_{\alpha}}}{(q^{-\lambda_{\beta}} t^{\beta-\alpha})_{\lambda_{\alpha}}} \frac{(q^{-\lambda_{\beta}+1} t^{\beta-\alpha-1})_{\lambda_{\alpha}}}{(qt^{\beta-\alpha-1})_{\lambda_{\alpha}}}. \quad (3.11)$$

Note that the regularity of (3.8) at $\zeta = t$ is included in this equation. Similarly by the formula (3.6), (3.9) is regular at $\zeta = t$ and its value is

$$\lim_{\zeta \rightarrow t} (3.9) = \prod_{k=1}^m \prod_{\alpha=1}^{\ell(\lambda)} \prod_{\beta=\alpha+1}^{\ell(\lambda)} \frac{(q^{\lambda_{\alpha}^{(k-1)}+1} t^{\beta-\alpha})_{\theta_{\alpha,k}}}{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}^{(k-1)}} t^{\beta-\alpha+1})_{\theta_{\alpha,k}}} \frac{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}^{(k-1)}} t^{\beta-\alpha+1})_{\theta_{\alpha,k}}}{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}^{(k-1)}+1} t^{\beta-\alpha})_{\theta_{\alpha,k}}}. \quad (3.12)$$

The rest term (3.10) is calculated by the formula (3.4) and (3.5):

$$\begin{aligned} \lim_{\zeta \rightarrow t} (3.10) = & \prod_{k=1}^m \prod_{\alpha=1}^{\ell(\lambda)} \left[\frac{(t)_{\theta_{\alpha,k}}}{(q)_{\theta_{\alpha,k}}} \frac{(q^{\lambda_{\alpha}-\lambda_{\alpha}^{(k)}}+1)_{\theta_{\alpha,k}}}{(q^{\lambda_{\alpha}-\lambda_{\alpha}^{(k)}})_{\theta_{\alpha,k}}} \right. \\ & \left. \prod_{\beta=\alpha+1}^{\ell(\lambda)} \frac{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}^{(k-1)}+1} t^{\beta-\alpha-1})_{\theta_{\alpha,k}}}{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}^{(k)}} t^{\beta-\alpha})_{\theta_{\alpha,k}}} \frac{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta} t^{\beta-\alpha}})_{\theta_{\alpha,k}}}{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}+1} t^{\beta-\alpha-1})_{\theta_{\alpha,k}}} \right]. \end{aligned} \quad (3.13)$$

Note that some parts of (3.12) and (3.13) are combined into the next form.

$$\begin{aligned} & \left[\prod_{k=1}^m \prod_{\alpha=1}^{\ell(\lambda)} \prod_{\beta=\alpha+1}^{\ell(\lambda)} \frac{(q^{\lambda_{\alpha}^{(k-1)}+1} t^{\beta-\alpha})_{\theta_{\alpha,k}}}{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}^{(k-1)}} t^{\beta-\alpha+1})_{\theta_{\alpha,k}}} \right] \\ & \times \left[\prod_{k=1}^m \prod_{\alpha=1}^{\ell(\lambda)} \frac{(q^{\lambda_{\alpha}-\lambda_{\alpha}^{(k)}}+1)_{\theta_{\alpha,k}}}{(q^{\lambda_{\alpha}-\lambda_{\alpha}^{(k)}})_{\theta_{\alpha,k}}} \prod_{\beta=\alpha+1}^{\ell(\lambda)} \frac{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta} t^{\beta-\alpha}})_{\theta_{\alpha,k}}}{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}+1} t^{\beta-\alpha-1})_{\theta_{\alpha,k}}} \right] \\ & = \left[\prod_{\alpha=1}^{\ell(\lambda)} \frac{(q)_{\lambda_{\alpha}}}{(t)_{\lambda_{\alpha}}} \right] \left[\prod_{1 \leq \alpha < \beta \leq \ell(\lambda)} \frac{(q^{-\lambda_{\beta}} t^{\beta-\alpha})_{\lambda_{\alpha}}}{(q^{-\lambda_{\beta}+1} t^{\beta-\alpha-1})_{\lambda_{\alpha}}} \frac{(qt^{\beta-\alpha})_{\lambda_{\alpha}}}{(t^{\beta-\alpha+1})_{\lambda_{\alpha}}} \right] \\ & = \left[\prod_{1 \leq \alpha < \beta \leq \ell(\lambda)} \frac{(q^{-\lambda_{\beta}} t^{\beta-\alpha})_{\lambda_{\alpha}}}{(q^{-\lambda_{\beta}+1} t^{\beta-\alpha-1})_{\lambda_{\alpha}}} \right] \left[\prod_{1 \leq \alpha < \beta \leq \ell(\lambda)+1} \frac{(qt^{\beta-\alpha+1})_{\lambda_{\alpha}}}{(t^{\beta-\alpha})_{\lambda_{\alpha}}} \right] = (3.13)^{-1} \times \prod_{\alpha=1}^{\ell(\lambda)} \frac{(qt^{\ell(\lambda)-\alpha})_{\lambda_{\alpha}}}{(t^{\ell(\lambda)-\alpha+1})_{\lambda_{\alpha}}}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \lim_{\zeta \rightarrow t} \tilde{\varphi}_{\lambda}^{(\zeta)} \frac{\gamma_T(z)}{\gamma_D(z)} = & \left[\prod_{\alpha=1}^{\ell(\lambda)} \frac{(qt^{\ell(\lambda)-\alpha})_{\lambda_{\alpha}}}{(t^{\ell(\lambda)-\alpha+1})_{\lambda_{\alpha}}} \prod_{k=1}^m \frac{(t)_{\theta_{\alpha,k}}}{(q)_{\theta_{\alpha,k}}} \right] \times \\ & \prod_{k=1}^m \prod_{1 \leq \alpha < \beta \leq \ell(\lambda)} \frac{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}^{(k-1)}} t^{\beta-\alpha+1})_{\theta_{\alpha,k}}}{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}^{(k-1)}+1} t^{\beta-\alpha})_{\theta_{\alpha,k}}} \frac{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}^{(k-1)}+1} t^{\beta-\alpha+1})_{\theta_{\alpha,k}}}{(q^{\lambda_{\alpha}^{(k-1)}-\lambda_{\beta}^{(k)}} t^{\beta-\alpha})_{\theta_{\alpha,k}}} \end{aligned}$$

$$= \prod_{k=1}^m \prod_{1 \leq \alpha \leq \beta \leq \ell(\lambda)} \frac{(q^{\lambda_\alpha^{(k-1)} - \lambda_\beta^{(k-1)}} t^{\beta - \alpha + 1})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)} - \lambda_\beta^{(k-1)} + 1} t^{\beta - \alpha})_{\theta_{\alpha,k}}} \times \prod_{k=1}^m \prod_{1 \leq \alpha \leq \beta \leq \ell(\lambda)} \frac{(q^{\lambda_\alpha^{(k-1)} - \lambda_{\beta+1}^{(k)} + 1} t^{\beta - \alpha})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)} - \lambda_{\beta+1}^{(k)} + 1} t^{\beta - \alpha + 1})_{\theta_{\alpha,k}}}. \quad (3.14)$$

Note that the function $f(u) := (tu)_\infty / (qu)_\infty$ satisfies $f(u)/f(q^{-\theta}u) = (q^{-\theta+1}u)_\infty / (q^{-\theta}tu)_\infty$. Then (3.14) can be rewritten into

$$(3.14) = \prod_{k=1}^m \prod_{1 \leq \alpha \leq \beta \leq \ell(\lambda)} \frac{f(q^{\lambda_\alpha^{(k-1)} - \lambda_\beta^{(k-1)}} t^{\beta - \alpha})}{f(q^{\lambda_\alpha^{(k)} - \lambda_\beta^{(k-1)}} t^{\beta - \alpha})} \frac{f(q^{\lambda_\alpha^{(k)} - \lambda_{\beta+1}^{(k)}} t^{\beta - \alpha})}{f(q^{\lambda_\alpha^{(k-1)} - \lambda_{\beta+1}^{(k)}} t^{\beta - \alpha})}. \quad (3.15)$$

Finally, if $T \in \text{SSTb}(\lambda; m)$, we have $k \geq \ell(\lambda^{(k)})$, Therefore if $\beta \geq k$ then $\lambda_{\beta+1}^{(k)} = \lambda_\beta^{(k-1)} = 0$. Thus one can see that

$$(3.15) = \prod_{k=1}^m \prod_{1 \leq \alpha \leq \beta \leq \ell(\lambda^{(k-1)})} \frac{f(q^{\lambda_\alpha^{(k-1)} - \lambda_\beta^{(k-1)}} t^{\beta - \alpha})}{f(q^{\lambda_\alpha^{(k)} - \lambda_\beta^{(k-1)}} t^{\beta - \alpha})} \frac{f(q^{\lambda_\alpha^{(k)} - \lambda_{\beta+1}^{(k)}} t^{\beta - \alpha})}{f(q^{\lambda_\alpha^{(k-1)} - \lambda_{\beta+1}^{(k)}} t^{\beta - \alpha})} = \prod_{k=1}^m \psi_{\lambda^{(k)}/\lambda^{(k-1)}}(q, t),$$

which is $\psi_T(q, t)$. On the other hand if $T \in \text{RTb}(\lambda; m) \setminus \text{SSTb}(\lambda; m)$, one can see that (3.15) = 0. Using Proposition 1.14, we have the desired equality.

3.2. Proof of Proposition 2.9. First we rewrite the relation of $\psi^\pm(z)$ and $x^\pm(w)$ given in Definition 2.1 into the next adjoint form.

$$\begin{aligned} \exp\left(\sum_{n>0} \text{ad}_{b_n} \gamma^{n/2} z^{-n}\right) x^\pm(w) &= \exp\left(\mp \sum_{n>0} \frac{1}{n} (1 - q^n)(1 - t^{-n})(1 - p^{-n}) \gamma^{\mp n/2} \left(\frac{w}{z}\right)^n\right) x^\pm(w), \\ \exp\left(-\sum_{n>0} \text{ad}_{b_{-n}} \gamma^{n/2} z^n\right) x^\pm(w) &= \exp\left(\pm \sum_{n>0} \frac{1}{n} (1 - q^n)(1 - t^{-n})(1 - p^{-n}) \gamma^{\mp n/2} \left(\frac{w}{z}\right)^n\right) x^\pm(w). \end{aligned}$$

Here we used the exponential form (2.2) of $\psi^\pm$. Then we see that

$$\begin{aligned} \alpha(z) x^\pm(w) \alpha(z)^{-1} &= \exp\left(-\sum_{n>0} \text{ad}_{b_{-n}} \frac{z^n}{\gamma^n - \gamma^{-n}}\right) x^\pm(w) \\ &= \exp\left(\pm \sum_{n>0} \frac{1}{n} \frac{(1 - q^n)(1 - t^{-n})(1 - p^{-n})}{\gamma^n - \gamma^{-n}} \gamma^{-n/2 \mp n/2} \left(\frac{z}{w}\right)^n\right) x^\pm(w), \end{aligned} \quad (3.16)$$

$$\begin{aligned} \beta(z) x^\pm(w) \beta(z)^{-1} &= \exp\left(\sum_{n>0} \text{ad}_{b_n} \frac{z^{-n}}{\gamma^n - \gamma^{-n}}\right) x^\pm(w) \\ &= \exp\left(\mp \sum_{n>0} \frac{1}{n} \frac{(1 - q^n)(1 - t^{-n})(1 - p^{-n})}{\gamma^n - \gamma^{-n}} \gamma^{-n/2 \mp n/2} \left(\frac{w}{z}\right)^n\right) x^\pm(w). \end{aligned} \quad (3.17)$$

We also prepare the operator product of $\alpha(w)$ and $\beta(z)$, which is easily obtained from the definitions (2.9) and the commutation relations (2.3) of b_n 's:

$$\beta(z) \alpha(w) = \alpha(w) \beta(z) \exp\left(\sum_{n>0} \frac{1}{n} \frac{(1 - q^n)(1 - t^{-n})(1 - p^{-n})}{\gamma^n - \gamma^{-n}} \gamma^{-n} \left(\frac{w}{z}\right)^n\right). \quad (3.18)$$

3.2.1. Proof of (1). Using (3.18) and (3.16), we see that

$$\begin{aligned} \alpha(z) t(w) \alpha(z)^{-1} &= \alpha(z) \alpha(w) x^+(w) \beta(w) \alpha(z)^{-1} \\ &= \alpha(w) \alpha(z) x^+(w) \alpha(z)^{-1} \beta(w) \times \exp\left(-\sum_{n>0} \frac{1}{n} \frac{(1 - q^n)(1 - t^{-n})(1 - p^{-n})}{\gamma^n - \gamma^{-n}} \gamma^{-n} \left(\frac{z}{w}\right)^n\right) \\ &= \alpha(w) x^+(w) \beta(w) \times \exp\left(-\sum_{n>0} \frac{1}{n} \frac{(1 - q^n)(1 - t^{-n})(1 - p^{-n})}{\gamma^n - \gamma^{-n}} \gamma^{-n} \left(\frac{z}{w}\right)^n\right) \end{aligned}$$

B. FEIGIN, A. HOSHINO, J. SHIBAHARA, J. SHIRAISHI AND S. YANAGIDA

$$+ \sum_{n>0} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})(1-p^{-n})}{\gamma^n - \gamma^{-n}} \gamma^{-n} \left(\frac{z}{w}\right)^n = t(w).$$

Thus we have $[t(z), \alpha(w)] = 0$. The other relations $[t(z), \beta(w)] = 0$, $[t^*(z), \alpha(w)] = [t^*(z), \beta(w)] = 0$, $[t(z), \psi^\pm(w)] = [t^*(z), \psi^\pm(w)] = 0$ also follow from equations (3.16)-(3.18) and we omit the detail.

3.2.2. *Proof of (2)*. Using the commutativity $[t(z), \alpha(w)] = 0$ given in (1), we have

$$\begin{aligned} A(w/z)t(z)t(w) &= A(w/z)\alpha(z)x^+(z)\beta(z)\alpha(w)x^+(w)\beta(w) = \alpha(z)\alpha(w)x^+(z)x^+(w)\beta(z)\beta(w) \\ &\times \exp\left(\sum_{n>0} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})(1-p^{-n}\gamma^{-2n})}{1-\gamma^{-2n}} \left(\frac{w}{z}\right)^n - \sum_{n>0} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})(1-p^{-n})}{\gamma^n - \gamma^{-n}} \gamma^{-n} \left(\frac{w}{z}\right)^n\right). \end{aligned}$$

Here the first summation in the exponential comes from the $A(w/z)$, and the second from transposition of $\beta(w)x^+(z)$ using (3.17). Thus we have

$$\begin{aligned} A(w/z)t(z)t(w) &= \alpha(z)\alpha(w)x^+(z)x^+(w)\beta(z)\beta(w) \times \exp\left(\sum_{n>0} \frac{1}{n} (1-q^n)(1-t^{-n}) \left(\frac{w}{z}\right)^n\right) \\ &= \frac{(1-q\frac{w}{z})(1-t^{-1}\frac{w}{z})}{(1-\frac{w}{z})(1-p\frac{w}{z})} \alpha(z)\alpha(w)x^+(z)x^+(w)\beta(z)\beta(w). \end{aligned}$$

Then

$$\begin{aligned} A(w/z)t(z)t(w) - A(z/w)t(w)t(z) &= \alpha(z)\alpha(w) \\ &\times \left[\frac{(1-q\frac{w}{z})(1-t^{-1}\frac{w}{z})}{(1-\frac{w}{z})(1-p\frac{w}{z})} x^+(z)x^+(w) - \frac{(1-q\frac{z}{w})(1-t^{-1}\frac{z}{w})}{(1-\frac{z}{w})(1-p\frac{z}{w})} x^+(w)x^+(z) \right] \\ &\times \beta(z)\beta(w). \end{aligned} \quad (3.19)$$

Now recall the relation of $x^+(z)$ and $x^+(w)$ given in Definition 2.1:

$$-(\frac{z}{w})^3 G^+(\frac{z}{w})x^+(z)x^+(w) = G^+(\frac{z}{w})x^+(w)x^+(z). \quad (3.20)$$

Using this equation, the line (3.19) is rewritten into

$$\begin{aligned} (3.19) &= \left[\frac{1}{(1-\frac{w}{z})(1-p\frac{w}{z})(1-p^{-1}\frac{w}{z})} + \frac{(\frac{z}{w})^3}{(1-\frac{z}{w})(1-p\frac{z}{w})(1-p^{-1}\frac{z}{w})} \right] G^+(\frac{w}{z})x^+(w)x^+(z) \\ &= \left[\frac{\delta(\frac{w}{z})}{(1-p^{-1})(1-p)} + \frac{\delta(p\frac{w}{z})}{(1-p^{-1})(1-p^{-2})} + \frac{\delta(p^{-1}\frac{w}{z})}{(1-p)(1-p^2)} \right] G^+(\frac{w}{z})x^+(w)x^+(z). \end{aligned}$$

Now from (3.20) and $G^+(1) \neq 0$, we see that $\delta(w/z)G^+(w/z)x^+(w)x^+(z) = 0$. We also find from (3.20) and $G^+(p^{-1}) = 0$ that $\delta(p\frac{w}{z})G^+(\frac{w}{z})x^+(w)x^+(z) = \delta(p\frac{w}{z})G^+(p^{-1})x^+(pw)x^+(w)$. Similarly from (3.20) and $G^+(p) \neq 0$ we have $\delta(p^{-1}\frac{w}{z})G^+(\frac{w}{z})x^+(w)x^+(z) = \delta(p^{-1}\frac{w}{z})G^+(p)x^+(pz)x^+(z)$. Thus after a short calculation we have

$$(3.19) = \frac{(1-t^{-1})(1-q)}{1-p} \left[\delta(p^{-1}\frac{w}{z})x^+(pz)x^+(z) - \delta(p\frac{w}{z})x^+(pw)x^+(w) \right].$$

Then we have the desired consequence (2.11).

The equation (2.12) can be similarly shown, so that we omit the detail.

3.2.3. *Proof of (3)*. We apply the same method as in (2). Recalling Remark (2.8), we calculate $B(\gamma pw/z)t(z)t^*(\gamma pw) - B(\gamma^{-1}p^{-1}z/w)t^*(\gamma pw)t(z)$. From the definition (2.13) of $B(z)$, the commutativity $[t(z), \alpha(w)] = 0$ given in (1) and the formula (3.18), we have

$$\begin{aligned} B(\gamma pw/z)t(z)t^*(\gamma pw) &= B(\gamma p\frac{w}{z})\alpha(z)x^+(z)\beta(z)\alpha(\gamma w)^{-1}x^-(w)\beta(\gamma^{-1}w)^{-1} \\ &= \alpha(z)\alpha(\gamma w)^{-1}x^+(z)x^-(w)\beta(z)\beta(\gamma^{-1}w)^{-1} \end{aligned}$$

$$\begin{aligned}
& \times \exp \left(\sum_{n>0} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})(1-p^{-n})}{\gamma^n - \gamma^{-n}} \left(\frac{w}{z}\right)^n \right. \\
& \quad \left. + \sum_{n>0} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})(\gamma^{-2n}p^{-2n} - \gamma^{-2n}p^{-n})}{1 - \gamma^{-2n}} \gamma^n p^{-n} \left(\frac{w}{z}\right)^n \right) \\
& = \alpha(z)\alpha(\gamma w)^{-1}x^+(z)x^-(w)\beta(z)\beta(\gamma^{-1}w)^{-1}.
\end{aligned}$$

A similar calculation shows that $B(\gamma p \frac{z}{w})t^*(\gamma pw)t(z) = \alpha(z)\alpha(\gamma w)^{-1}x^-(w)x^+(z)\beta(z)\beta(\gamma^{-1}w)^{-1}$. Thus we have

$$\begin{aligned}
& B(\gamma p \frac{w}{z})t(z)t^*(\gamma pw) - B(\gamma p \frac{z}{w})t^*(\gamma pw)t(z) \\
& = \alpha(z)\alpha(\gamma w)^{-1}[x^+(z)x^-(w) - x^-(w)x^+(z)]\beta(z)\beta(\gamma^{-1}w)^{-1}
\end{aligned}$$

Using the expression of $[x^+(z), x^-(w)]$ given in Definition 2.1, the expansion (2.2) and the definition (2.9), one may immediately find that

$$B(\gamma p \frac{w}{z})t(z)t^*(\gamma pw) - B(\gamma p \frac{z}{w})t^*(\gamma pw)t(z) = \frac{(1-q)(1-t^{-1})}{1-p} \left(\delta(\gamma^{-1} \frac{z}{w})\psi_0^+ - \delta(\gamma \frac{z}{w})\psi_0^- \right).$$

Replacing w in the above equation with $\gamma^{-1}p^{-1}w$, we have the desired equation (2.14).

3.3. Proof of Proposition 2.13. Let us define $a_{n,(i)} := 1 \otimes \cdots \otimes 1 \otimes a_n \otimes 1 \otimes \cdots \otimes 1$, where a_n sits in the i -th tensor component. Then from (2.4) and (2.5) one finds that $\rho_y^{(m)}(b_n) = -\sum_{i=1}^m a_{n,(i)}(1-t^n)(1-p^{-|n|})p^{(m-i+1)|n|/2}/|n|$. Thus we have

$$\rho_y^{(m)}(\alpha(z)) = \prod_{i=1}^m \alpha_{(i)}^m(z), \quad \alpha_{(i)}^m(z) := \exp \left(\sum_{n>0} \frac{1}{n} \frac{p^{(m-i+1)n/2}(1-t^{-n})(1-p^{-n})}{p^{-mn/2} - p^{mn/2}} a_{-n,(i)} z^n \right), \quad (3.21)$$

$$\rho_y^{(m)}(\beta(z)) = \prod_{i=1}^m \beta_{(i)}^m(z), \quad \beta_{(i)}^m(z) := \exp \left(- \sum_{n>0} \frac{1}{n} \frac{p^{(m-i+1)n/2}(1-t^n)(1-p^{-n})}{p^{-mn/2} - p^{mn/2}} a_{n,(i)} z^{-n} \right). \quad (3.22)$$

3.3.1. Proof of (1). We calculate each tensor component of $\Lambda_i(z)\Lambda_j(w)$. First assume $i = j$.

If $k > i$, then the k -th tensor component comes from $\alpha_{(k)}^m(z)\beta_{(k)}^m(z)\alpha_{(k)}^m(w)\beta_{(k)}^m(w)$. Under the normal ordering, the following coefficient arises.

$$\exp \left(- \sum_{n>0} \frac{1}{n} (1-q^n)(1-t^{-n}) \left(\frac{1-p^{-n}}{1-p^{mn}} \right)^2 p^{(2m-k+1)n} \left(\frac{w}{z} \right)^n \right). \quad (3.23)$$

For $k = i$, the i -th tensor component comes from $\alpha_{(k)}^m(z)\eta(p^{-(i-1)/2}z)\beta_{(k)}^m(z)\alpha_{(k)}^m(w)\eta(p^{-(i-1)/2}w)\beta_{(k)}^m(w)$. Under the normal ordering, the following coefficient arises.

$$\begin{aligned}
& \exp \left(- \sum_{n>0} \frac{1}{n} (1-q^n)(1-t^{-n}) \left(\frac{w}{z} \right)^n \right. \\
& \quad \left. \left\{ \left(\frac{1-p^{-n}}{1-p^{mn}} \right)^2 p^{(2m-i+1)n} + \frac{1-p^{-n}}{1-p^{mn}} p^{mn} + \frac{1-p^{-n}}{1-p^{mn}} p^{(m-i+1)n} + 1 \right\} \right).
\end{aligned} \quad (3.24)$$

If $k < i$, then the k -th tensor component is $\alpha_{(k)}^m(z)\varphi^-(p^{-(2k-1)/4}z)\beta_{(k)}^m(z)\alpha_{(k)}^m(w)\varphi^-(p^{-(2k-1)/4}w)\beta_{(k)}^m(w)$. The normal ordering coefficient is

$$\exp \left(- \sum_{n>0} \frac{1}{n} (1-q^n)(1-t^{-n}) \left(\frac{w}{z} \right)^n \left\{ \left(\frac{1-p^{-n}}{1-p^{mn}} \right)^2 p^{(2m-k+1)n} + \frac{(1-p^{-n})^2}{1-p^{mn}} p^{(m-k+1)n} \right\} \right). \quad (3.25)$$

By simple calculations, the product of (3.23), (3.24) and (3.25) is shown to be $f_{1,m}(w/z)^{-1}$. Thus the statement holds.

B. FEIGIN, A. HOSHINO, J. SHIBAHARA, J. SHIRAISHI AND S. YANAGIDA

Next we consider the case $i < j$. If $k < i$, then the normal order coefficient is the same as (3.25). For $k = i$, the normal order coefficient is

$$\exp \left(- \sum_{n>0} \frac{1}{n} (1 - q^n) (1 - t^{-n}) \left(\frac{w}{z} \right)^n \right. \\ \left. \left\{ \left(\frac{1 - p^{-n}}{1 - p^{mn}} \right)^2 p^{(2m-i+1)n} + \frac{1 - p^{-n}}{1 - p^{mn}} p^{mn} + \frac{(1 - p^{-n})^2}{1 - p^{mn}} p^{(m-i+1)n} + 1 - p^{-n} \right\} \right). \quad (3.26)$$

If $i < k < j$, then the normal order coefficient is

$$\exp \left(- \sum_{n>0} \frac{1}{n} (1 - q^n) (1 - t^{-n}) \left(\frac{w}{z} \right)^n \left\{ \left(\frac{1 - p^{-n}}{1 - p^{mn}} \right)^2 p^{(2m-k+1)n} + \frac{(1 - p^{-n})^2}{1 - p^{mn}} p^{(m-k+1)n} \right\} \right). \quad (3.27)$$

If $k = j$, then the normal order coefficient is

$$\exp \left(- \sum_{n>0} \frac{1}{n} (1 - q^n) (1 - t^{-n}) \left(\frac{w}{z} \right)^n \left\{ \left(\frac{1 - p^{-n}}{1 - p^{mn}} \right)^2 p^{(2m-j+1)n} + \frac{1 - p^{-n}}{1 - p^{mn}} p^{(m-j+1)n} \right\} \right). \quad (3.28)$$

If $k > j$, then the normal order coefficient is (3.23). The product of (3.25), (3.26), (3.27), (3.28), (3.23) is equal to $f_{1,m}(w/z)^{-1} \gamma_+(z, w; q, p)$. Thus we obtain the result.

The case $i > j$ is similar, so we omit the detail.

3.3.2. *Proof of (2)*. The desired equation is equivalent to

$$\rho_y^{(m)}(\alpha(z) \cdots \alpha(p^{m-1}z)) : \prod_{k=1}^m \tilde{\Lambda}_k(p^{k-1}z) : \rho_y^{(m)}(\beta(z) \cdots \beta(p^{m-1}z)) = 1.$$

We will show this equation by comparing each tensor component.

By (3.21), the k -th tensor component of $\rho_y^{(m)}(\alpha(z) \cdots \alpha(p^{m-2}z))$ is equal to

$$\exp \left(- \sum_{n>0} \frac{1}{n} (1 - t^{-n}) p^{(2m-k-1)n/2} a_{-n} z^n \right). \quad (3.29)$$

Similarly, the k -th tensor component of $\rho_y^{(m)}(\beta(z) \cdots \beta(p^{m-1}z))$ is equal to

$$\exp \left(\sum_{n>0} \frac{1}{n} (1 - t^{-n}) p^{(-k+1)n/2} a_n z^{-n} \right). \quad (3.30)$$

The k -th tensor component of $: \prod_{k=1}^m \tilde{\Lambda}_k(p^{k-1}z) :$ is

$$: \eta(p^{-(k-1)/2} p^{k-1}z) \varphi^-(p^{-(2k-1)/4} p^kz) \varphi^-(p^{-(2k-1)/4} p^{k+1}z) \cdots \varphi^-(p^{-(2k-1)/4} p^{m-1}z) : \\ = \exp \left(\sum_{n>0} \frac{1 - t^{-n}}{n} p^{n(2m-k-1)/2} a_{-n} z^n \right) \exp \left(- \sum_{n>0} \frac{1 - t^n}{n} p^{-n(k-1)/2} a_n z^{-n} \right) \quad (3.31)$$

It is easy to see that (3.29), (3.30) and (3.31) cancel. Thus we have the consequence.

3.3.3. *Proof of (3)*. The desired equation is equivalent to

$$\rho_y^{(m)}(\alpha(p^{-1}z) \cdots \alpha(p^{m-2}z)) : \prod_{k=1}^{i-1} \tilde{\Lambda}_k(p^{k-1}z) \prod_{l=i+1}^m \tilde{\Lambda}_l(p^{l-2}z) : \rho_y^{(m)}(\beta(z) \cdots \beta(p^{m-1}z)) \\ = \tilde{\Lambda}_i^*(p^{(m-2)/2}z). \quad (3.32)$$

We will show this equation by comparing each tensor component.

As in (3.29), the k -th tensor component of $\rho_y^{(m)}(\alpha(p^{-1}z) \cdots \alpha(p^{m-2}z))$ is equal to

$$\exp \left(- \sum_{n>0} \frac{1}{n} (1 - t^n) p^{(2m-k-3)n/2} a_{-n} z^n \right). \quad (3.33)$$

The k -th tensor component of $\rho_y^{(m)}(\beta(z) \cdots \beta(p^{m-1}z))$ is given by (3.30).

The k -th tensor component of $:\prod_{k=1}^{i-1} \tilde{\Lambda}_k(p^{k-1}z) \prod_{l=i+1}^m \tilde{\Lambda}_l(p^{l-2}z):$ depends on k . If $k = i$, then by Lemma 2.6 and some simple calculations, the component turns out to be

$$\begin{aligned} & \varphi^-(p^{-(2i-1)/4} p^{i-1} z) \varphi^-(p^{-(2i-1)/4} p^i z) \cdots \varphi^-(p^{-(2i-1)/4} p^{m-2} z) \\ &= \exp \left(- \sum_{n>0} \frac{1-t^n}{n} p^{(i-3)/2} (1 - p^{n(m-i)}) a_{-n} z^n \right). \end{aligned} \quad (3.34)$$

Similarly, if $k < i$, then by Lemma 2.6 the component is

$$\begin{aligned} & : \eta(p^{-(k-1)/2} p^{k-1} z) \varphi^-(p^{-(2k-1)/4} p^k z) \varphi^-(p^{-(2k-1)/4} p^{k+1} z) \cdots \varphi^-(p^{-(2k-1)/4} p^{m-2} z) : \\ &= \exp \left(\sum_{n>0} \frac{1-t^n}{n} p^{n(2m-k-3)/2} a_{-n} z^n \right) \exp \left(- \sum_{n>0} \frac{1-t^n}{n} p^{-n(k-1)/2} a_n z^{-n} \right) : \end{aligned} \quad (3.35)$$

If $k > i$, then by Lemma 2.6 the component is

$$\begin{aligned} & : \eta(p^{-(k-1)/2} p^{k-2} z) \varphi^-(p^{-(2k-1)/4} p^{k-1} z) \varphi^-(p^{-(2k-1)/4} p^k z) \cdots \varphi^-(p^{-(2k-1)/4} p^{m-2} z) : \\ &= \exp \left(\sum_{n>0} \frac{1-t^n}{n} p^{n(2m-k-3)/2} a_{-n} z^n \right) \exp \left(- \sum_{n>0} \frac{1-t^n}{n} p^{-n(k-3)/2} a_n z^{-n} \right) : \end{aligned} \quad (3.36)$$

Then the i -th tensor component of (3.32) is the product of (3.33), (3.30) and (3.34). After a short calculation, one finds that it is $\xi(p^{(i-2)/2} z)$, which is the i -th component of $\tilde{\Lambda}_i^*(p^{(m-2)/2} z)$.

If $k < i$, then the k -th tensor component of (3.32) is the product of (3.33), (3.30) and (3.35). It is 1, that is, the k -th component of $\tilde{\Lambda}_i^*(p^{(m-2)/2} z)$.

Finally, for $k > i$, the k -th tensor component of (3.32) is the product of (3.33), (3.30) and (3.36). It turns out to be $\varphi^-(p^{(2j-5)/4} z)$, which is the k -th component of $\tilde{\Lambda}_i^*(p^{(m-2)/2} z)$.

3.3.4. Proof of (4). From the known identities (2.20) and (2.22), it is not difficult to calculate $\left[\prod_{k,l=1}^{m-1} f_{1,m}(p^{-k+l} w/z) \right] \Lambda_i^*(z) \Lambda_j^*(w)$ in terms of Λ_k 's.

First we consider the case $i = j$. From the operator product (2.20), we have

$$\begin{aligned} & \left[\prod_{k,l=1}^{m-1} f_{1,m}(p^{-k+l} \frac{w}{z}) \right] \Lambda_i^*(z) \Lambda_i^*(w) \\ &= \left[\prod_{k=1}^{m-2} \prod_{l=k+1}^{m-1} \gamma_+(p^{-k+l} \frac{w}{z}) \right] \left[\prod_{k=2}^{m-1} \prod_{l=1}^{k-1} \gamma_-(p^{-k+l} \frac{w}{z}) \right] : \Lambda_i^*(z) \Lambda_i^*(w) : \\ &= \exp \left(\sum_{n>0} \frac{1}{n} (1 - q^n) (1 - t^{-n}) \frac{1 - p^{-n(m-2)}}{1 - p^{-n}} \frac{1 - p^{n(m-1)}}{1 - p^n} \left(\frac{w}{z} \right)^n \right) : \Lambda_i^*(z) \Lambda_i^*(w) : . \end{aligned}$$

Here we used the abbreviation $\gamma_{\pm}(\frac{w}{z}) := \gamma_{\pm}(z, w; q, p)$. Then we have

$$\begin{aligned} & \left[\prod_{k,l=1}^{m-1} f_{1,m}(p^{-k+l} \frac{w}{z}) \right] \times \exp \left(- \sum_{n>0} \frac{1}{n} (1 - q^n) (1 - t^{-n}) \frac{1 - p^{-n(m-2)}}{1 - p^{-n}} \frac{1 - p^{n(m-1)}}{1 - p^n} \left(\frac{w}{z} \right)^n \right) \\ &= \exp \left(- \sum_{n>0} \frac{1}{n} \frac{(1 - q^n) (1 - t^{-n}) (1 - p^{(m-1)n})}{1 - p^{mn}} \left(\frac{w}{z} \right)^n \right) = f_{1,m}(\frac{w}{z}). \end{aligned}$$

B. FEIGIN, A. HOSHINO, J. SHIBAHARA, J. SHIRAISHI AND S. YANAGIDA

Thus the desired equation $f_{1,m}(\frac{w}{z})\Lambda_i^*(z)\Lambda_i^*(w) =: \Lambda_i^*(z)\Lambda_i^*(w) :$ is proved.

Next, note that the calculation of the case $i \neq j$ reduces to that of $k = i$. If $i < j$, then

$$f_{1,m}(\frac{w}{z})\Lambda_i^*(z)\Lambda_j^*(w) =: \Lambda_i^*(z)\Lambda_j^*(w) : \frac{\gamma_-(\frac{w}{z})^{j-i}}{\gamma_+(p\frac{w}{z})^{j-i-1}} =: \Lambda_i^*(z)\Lambda_j^*(w) : \gamma_-(\frac{w}{z}).$$

At the last line we used the formula $\gamma_-(z)/\gamma_+(pz) = 1$. For the final case $i > j$, we have

$$f_{1,m}(\frac{w}{z})\Lambda_i^*(z)\Lambda_j^*(w) =: \Lambda_i^*(z)\Lambda_j^*(w) : \frac{\gamma_+(\frac{w}{z})^{i-j}}{\gamma_-(p^{-1}\frac{w}{z})^{i-j-1}} =: \Lambda_i^*(z)\Lambda_j^*(w) : \gamma_+(\frac{w}{z}).$$

Thus all the cases are proved.

3.3.5. *Proof of (5)*. This is similary shown as (2) and (3), so we omit the detail.

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