

Matrix functions and unitarily invariant norms (行列関数とユニタリ不変ノルム)

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1 Introduction

The eigenvalues of an $n \times n$ Hermitian matrix H are denoted by $\lambda_i(H)$ ($i = 1, 2, \dots, n$) and arranged in increasing order, that is,
 $\lambda_1(H) \leq \lambda_2(H) \leq \dots \leq \lambda_n(H)$.

$$\sigma_{(k)}(H) := \sum_{i=1}^k \lambda_i(H),$$

$$\sigma^{(k)}(H) := \sum_{i=n-k+1}^n \lambda_i(H).$$

A norm $\|\cdot\|$ on the $n \times n$ matrices is called a *unitarily invariant norm* if

$$\|UXV\| = \|X\|$$

for all X and for all unitary matrices U and V .

Example 1.1 The following are typical unitarily invariant norm:

The operator norm $\|X\|$,

Schatten p -norms $\|X\|_p := \{\sum_{i=1}^n \lambda_i(|X|)^p\}^{1/p}$, ($p \geq 1$),

Ky Fan k -norms $\|X\|_{(k)} := \sigma^{(k)}(|X|)$, ($k = 1, 2, \dots, n$).

The following useful result is due to **Ky Fan**:

$\|X\|_{(k)} \leq \|Y\|_{(k)} (\forall k)$ implies $\|X\| \leq \|Y\|$ for every unitarily invariant norm $\|\cdot\|$.

We begin with a simple fact:

Proposition 1.1 Let $f(t)$ be a concave function on an interval I , and let A, B be $n \times n$ Hermitian matrices with the spectra in I . Then for R, S such that $R^*R + S^*S = 1$ and for $k = 1, 2, \dots, n$

$$\sigma_{(k)}(f(R^*AR + S^*BS)) \geq \sigma_{(k)}(R^*f(A)R + S^*f(B)S).$$

Moreover, if $f(t)$ is monotone, then

$$\lambda_k(f(R^*AR + S^*BS)) \geq \lambda_k(R^*f(A)R + S^*f(B)S).$$

Proof. Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of $X^*AX + Y^*BY$ so that $f(\lambda_1) \leq f(\lambda_2) \leq \dots \leq f(\lambda_n)$, and let $\{e_i\}$ be the corresponding eigenvectors. Then the left side of (??) equals $f(\lambda_1) + \dots + f(\lambda_k)$. By the concavity of f , we have

$$\begin{aligned} & \sum_{i=1}^k \langle (X^*f(A)X + Y^*f(B)Y)e_i, e_i \rangle \\ &= \sum_{i=1}^k \left\{ \|Xe_i\|^2 \langle f(A) \frac{Xe_i}{\|Xe_i\|}, \frac{Xe_i}{\|Xe_i\|} \rangle + \|Ye_i\|^2 \langle f(B) \frac{Ye_i}{\|Ye_i\|}, \frac{Ye_i}{\|Ye_i\|} \rangle \right\} \\ &\leq \sum_{i=1}^k \left\{ \|Xe_i\|^2 f\left(\langle A \frac{Xe_i}{\|Xe_i\|}, \frac{Xe_i}{\|Xe_i\|} \rangle\right) + \|Ye_i\|^2 f\left(\langle B \frac{Ye_i}{\|Ye_i\|}, \frac{Ye_i}{\|Ye_i\|} \rangle\right) \right\} \\ &\leq \sum_{i=1}^k f(\langle (X^*AX + Y^*BY)e_i, e_i \rangle) = \sum_{i=1}^k f(\lambda_i). \end{aligned}$$

Thus, by the min-max theorem, we get the first inequality.

If $f(t)$ is increasing, we can arrange eigenvalues $\{\lambda_i\}_{i=1}^n$ as $\lambda_i \leq \lambda_{i+1}$ and $f(\lambda_i) \leq f(\lambda_{i+1})$. For any unit vector x that is a linear combination of $e_1, \dots, e_k$

$$\begin{aligned} \langle (X^*f(A)X + Y^*f(B)Y)x, x \rangle &\leq f(\langle (X^*AX + Y^*BY)x, x \rangle) \\ &\leq f(\lambda_k), \end{aligned}$$

for $\langle (X^*AX + Y^*BY)x, x \rangle \leq \lambda_k$. From this, the second inequality follows. It can be similarly shown even if $f(t)$ is decreasing. $\square$

Corollary 1.2 Let $g(t)$ be a convex function on I . Then for $1 \leq k \leq n$ and for all R, S such that $R^*R + S^*S = 1$

$$\sigma^{(k)}(g(R^*AR + S^*BS)) \leq \sigma^{(k)}(R^*g(A)R + S^*g(B)S).$$

Moreover, if $g(t)$ is monotone,

$$\lambda_k(g(R^*AR + S^*BS)) \leq \lambda_k(R^*g(A)R + S^*g(B)S).$$

Remark: The case $k = n$ of the first inequality in the corollary had been shown by Brown-Kosaki, Hansen-Pedersen.

The second inequality was shown by Bourin.

The case where R, S are scalars are due to Aujla- Silva.

2 Essential results

It is well known that $\text{tr}BA^2B = \text{tr}AB^2A$. But it is difficult to estimate

$$\text{tr}CBA^2BC - \text{tr}CAB^2AC.$$

J. C. Bourin [5] got a nice result to do it. The next special case follows from it, however we can give a simple and direct proof.

Lemma 2.1 Let $A \geq 0$ and $B \geq 0$, and let Q be an orthogonal projection such that $QB = BQ$. If

$$\begin{aligned} & \inf\{\|Bx\| : Qx = x, \|x\| = 1\} \\ & \geq \sup\{\|Bx\| : (1 - Q)x = x, \|x\| = 1\}, \end{aligned}$$

then

$$\begin{aligned} & \text{tr}QBA^2BQ \geq \text{tr}QAB^2AQ, \\ & \text{tr}(1 - Q)AB^2A(1 - Q) \\ & \geq \text{tr}(1 - Q)BA^2B(1 - Q). \end{aligned}$$

Corollary 2.2 Let $A \geq 0$ and $B \geq 0$, and let Q be an orthogonal projection such that $QB = BQ$.

Suppose the strict inequality:

$$\inf\{\|Bx\| : Qx = x, \|x\| = 1\} > \sup\{\|Bx\| : (1 - Q)x = x, \|x\| = 1\}.$$

Then

$$\text{tr } QBA^2BQ = \text{tr } QAB^2AQ \Leftrightarrow QA = AQ.$$

Proposition 2.3 Let $h(t)$ be a continuous function on $[0, \infty)$.

If $h(t)$ is decreasing and $th(t)$ is increasing,

or

if $h(t)$ is increasing and $th(t)$ is decreasing,

then for $A, B \geq 0$ and for every unitarily invariant norm $\|\cdot\|$

$$\|A^{1/2}h(A+B)A^{1/2} + B^{1/2}h(A+B)B^{1/2}\| \geq \|(A+B)h(A+B)\|.$$

Corollary 2.4 Let A and B be non-negative Hermitian matrices such that $A+B$ is invertible. Then the following are equivalent:

- (i) $H := A^{1/2}(A+B)^{-1}A^{1/2} + B^{1/2}(A+B)^{-1}B^{1/2} \leq 1$,
- (ii) $H = 1$,
- (iii) $AB = BA$.

Remark: We give one fact relevant to the above (i).

$$(A+B)^{-1/2}A^{1/2}A^{1/2}(A+B)^{-1/2} + (A+B)^{-1/2}B^{1/2}B^{1/2}(A+B)^{-1/2} = 1.$$

3 Applications

We can easily give another proof of

Theorem A. (Ando and Zhan)

Let $f(t) \geq 0$ be an operator monotone function on $[0, \infty)$ such that $f(t)$ is continuous at $t = 0$. Then

$$\|f(A+B)\| \leq \|f(A) + f(B)\| \quad (1)$$

for every unitarily invariant norm $\|\cdot\|$ and for all $A, B \geq 0$.

Proof We may assume that $A+B$ is invertible. Then, since $(A+B)^{-1/2}A^{1/2}$ is contractive, by Hansen-Pedersen's inequality [7] we have

$$\begin{aligned} \varphi(A) &= \varphi(A^{1/2}(A+B)^{-1/2}(A+B)(A+B)^{-1/2}A^{1/2}) \\ &\geq A^{1/2}(A+B)^{-1/2}\varphi(A+B)(A+B)^{-1/2}A^{1/2}, \\ \varphi(B) &\geq B^{1/2}(A+B)^{-1/2}\varphi(A+B)(A+B)^{-1/2}B^{1/2}. \end{aligned}$$

Since $\varphi(t)$ is increasing and $\varphi(t)/t$ is decreasing, by Proposition 2.3 we get

$$\begin{aligned} \sigma^{(k)}(\varphi(A) + \varphi(B)) &\geq \sigma^{(k)}(A^{1/2}(\varphi/t)(A+B)A^{1/2} + B^{1/2}(\varphi/t)(A+B)B^{1/2}) \\ &\geq \sigma^{(k)}(\varphi(A+B)) \quad (1 \leq k \leq n). \end{aligned}$$

□

Also we can get the following generalization of (1),

For $A_i \geq 0 \quad (1 \leq i \leq k)$

$$\|f(\sum_{i=1}^k A_i)\| \leq \|\sum_{i=1}^k f(A_i)\|$$

Let $f(t)$ be a non-negative concave function on $0 \leq t < \infty$. The following is known:

Theorem B (Rotfel'd [9]) (see also [10, 6, 4, 5])

For $A, B \geq 0$

$$\|f(A + B)\|_1 \leq \|f(A)\|_1 + \|f(B)\|_1.$$

We will extend this to every unitarily invariant norm.

Theorem 3.1 ([12])

$$\|f(|X + Y|)\| \leq \|f(|X|)\| + \|f(|Y|)\| \quad (\forall X, Y).$$

for every unitarily invariant norm $\|\cdot\|$.

Now we can slightly improve this as follows:

Theorem 3.2 Let f be a non-negative (not necessarily continuous) concave function defined on $[0, \infty)$, and let $\{X_i\} (i = 0, 1, \dots, k)$ be a finite set of matrices. Then there are unitary matrices $U_i (i = 1, \dots, k)$ such that the inequality

$$\begin{aligned} & \|f(|X_0 + X_1 + \dots + X_k|)\| \leq \\ & \|f(|X_0|) + U_1^* f(|X_1|) U_1 + \dots + U_k^* f(|X_k|) U_k\| \end{aligned}$$

holds for every unitarily invariant norm.

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