

順序を保存する作用素不等式のある拡張などについて
**On further extensions of order preserving
operator inequality**

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1. Introduction

Each capital letter means a bounded linear operator on a Hilbert space. An operator T is said to be positive semidefinite (denoted by $0 \leq T$) if $0 \leq (Tx, x)$ for all vectors x .

The readers should pay attention to that the statements cited here might be neither the precise repetition of nor the full strength as in their original articles.

Theorem (Löwner-Heinz).

Let $0 \leq p \leq 1$. $0 \leq B \leq A$

$\implies$

$$B^p \leq A^p.$$

It is well-known that for $1 < p$, $0 \leq B \leq A$ does not always ensure $B^p \leq A^p$.

Example. Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $0 \leq B \leq A$ and $B^2 \not\leq A^2$.

Therefore, $0 \leq B \leq A$ does not always imply $AB^2A \leq A^4$.
(Consider multiplying A^{-1} from both sides.)

Conjecture (Chan and Kwong '85).

$$0 \leq B \leq A$$

$\stackrel{?}{\implies}$

$$(AB^2A)^{\frac{1}{2}} \leq A^2.$$

The Furuta inequality was epochmaking on this direction.

Theorem (Furuta '87).

Let $0 \leq p$, $1 \leq q$, $0 \leq r$ and $p + r \leq (1 + r)q$.

$$0 \leq B \leq A$$

$\implies$

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1}{q}} \leq A^{\frac{p+r}{q}}.$$

Remark.

- $r = 0$: Löwner-Heinz
- $p = q = r = 2$: Chan-Kwong's conjecture

(essential case)

Let $1 \leq p$, $0 \leq r$.

$$0 \leq B \leq A$$

$\implies$

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1+r}{p+r}} \leq A^{1+r}.$$

Theorem (Ando and Hiai '94).

Let $1 \leq p, 1 \leq r$.

$0 \leq B \leq A$ and $\exists A^{-1}$

$\implies$

$$\left\{ A^{\frac{r}{2}} \left(A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}} \right)^r A^{\frac{r}{2}} \right\}^{\frac{1}{p}} \leq A^r.$$

Theorem (Furuta '95).

Let $1 \leq p, 1 \leq s, 0 \leq t \leq 1, t \leq r$.

$0 \leq B \leq A$ and $\exists A^{-1}$

$\implies$

$$\left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}.$$

Remark.

- $t = 0, s = 1$: Furuta '87
- $t = 1, s = r$: Ando-Hiai

Theorem (Furuta '08).

Let $1 \leq p_1, \dots, p_{2n}, 0 \leq t \leq 1, t \leq r$.

$0 \leq B \leq A$ and $\exists A^{-1}$

$\Rightarrow$

$$\left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} \dots \left(A^{\frac{t}{2}} \left(A^{-\frac{t}{2}} B^{p_1} A^{-\frac{t}{2}} \right)^{p_2} A^{\frac{t}{2}} \right)^{p_3} \dots A^{-\frac{t}{2}} \right)^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{\varphi[2n;r,t]}}$$

$$\leq A^{1-t+r}.$$

Remark. $n = 1$: Furuta '95

Definition.

$$\varphi[2n; r, t] = (\dots (((p_1 - t)p_2 + t)p_3 - t)p_4 + \dots - t)p_{2n} + r.$$

Up to here, we are concerned with 2 operators. The following theorem treats 3 operators.

Theorem (Uchiyama '03).

Let $1 \leq p_1, p_2, 0 \leq t_1 \leq 1, t_1 \leq t_2$.

$0 \leq B \leq A_1 \leq A_2$ and $\exists A_1^{-1}$

$\Rightarrow$

$$\left\{ A_2^{\frac{t_2}{2}} \left(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}} \right)^{p_2} A_2^{\frac{t_2}{2}} \right\}^{\frac{1-t_1+t_2}{(p_1-t_1)p_2+t_2}} \leq A_2^{1-t_1+t_2}.$$

Remark. $A_1 = A_2$: Furuta '95

Theorem (Yang and Wang '10).

$$1 \leq p_1, \dots, p_{2n},$$

$$0 \leq t_1, \dots, t_n \leq 1, \quad t_n \leq r,$$

$$0 \leq B \leq A_1 \leq A_2 \leq \dots \leq A_{2n-1} \leq A_{2n} \quad \text{and} \quad \exists A_1^{-1}$$

$\Rightarrow$

$$\begin{aligned} & \left\{ A_{2n}^{\frac{r}{2}} \left(A_{2n-1}^{-\frac{t_n}{2}} \left(A_{2n-2}^{\frac{t_{n-1}}{2}} \dots A_4^{\frac{t_2}{2}} \right. \right. \right. \\ & \quad \left. \left. \left[A_3^{-\frac{t_2}{2}} \left\{ A_2^{\frac{t_1}{2}} \left(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}} \right)^{p_2} A_2^{\frac{t_1}{2}} \right\}^{p_3} A_3^{-\frac{t_2}{2}} \right]^{p_4} \right. \right. \\ & \quad \left. \left. \left. A_4^{\frac{t_2}{2}} \dots A_{2n-2}^{\frac{t_{n-1}}{2}} \right)^{p_{2n-1}} A_{2n-1}^{-\frac{t_n}{2}} \right)^{p_{2n}} A_{2n}^{\frac{r}{2}} \right\}^{\frac{1-t_n+r}{\mathfrak{O}[2n]-t_n+r}} \\ & \leq A_{2n}^{1-t_n+r}. \end{aligned}$$

Remark.

- $n = 1$: Uchiyama '03
- $A_1 = \dots = A_{2n}, t_1 = \dots = t_n$: Furuta'08

Definition.

$$\begin{aligned} \mathfrak{O}[2n] = \{ & \dots (((((p_1 - t_1)p_2 + t_1)p_3 - t_2)p_4 + t_2)p_5 - \dots \\ & - t_n\} p_{2n} + t_n. \end{aligned}$$

2. Some extensions of operator inequalities

Theorem 1(KW).

$$1 \leq p_1, \dots, p_{2n},$$

$$0 \leq t_{2k-1} \leq 1, \quad t_{2k-1} \leq t_{2k} \quad (k = 1, \dots, n),$$

$$0 \leq B \leq A_1 \leq A_2, \quad \exists A_1^{-1} \quad \text{and}$$

$$A_{2k-2}^{\alpha(2k-2)} \leq A_{2k-1}^{\alpha(2k-2)} \leq A_{2k}^{\alpha(2k-2)} \quad (k = 2, \dots, n)$$

$\Rightarrow$

$$\left\{ A_{2n}^{\frac{t_{2n}}{2}} \left(A_{2n-1}^{-\frac{t_{2n-1}}{2}} \dots \right. \right. \\ \left. \left(A_2^{\frac{t_2}{2}} \left(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}} \right)^{p_2} A_2^{\frac{t_2}{2}} \right)^{p_3} \right. \\ \left. \left. \dots A_{2n-1}^{-\frac{t_{2n-1}}{2}} \right)^{p_{2n}} A_{2n}^{\frac{t_{2n}}{2}} \right\}^{\frac{\alpha(2n)}{\psi(2n)}} \\ \leq A_{2n}^{\alpha(2n)}.$$

Definition.

$$\alpha(2n) = 1 - t_1 + t_2 - \dots - t_{2n-1} + t_{2n}$$

$$\psi(2n) = \{ \dots (((p_1 - t_1)p_2 + t_2)p_3 - t_3)p_4 + \dots - t_{2n-1} \} p_{2n} + t_{2n}.$$

Operator inequalities in the assumption of Theorem 1:

$$0 \leq B \leq A_1 \leq A_2$$

$$A_2^{1-t_1+t_2} \leq A_3^{1-t_1+t_2} \leq A_4^{1-t_1+t_2} \quad (*)$$

$$A_4^{1-t_1+t_2-t_3+t_4} \leq A_5^{1-t_1+t_2-t_3+t_4} \leq A_6^{1-t_1+t_2-t_3+t_4}$$

•

•

•

$$A_{2n-2}^{\alpha(2n-2)} \leq A_{2n-1}^{\alpha(2n-2)} \leq A_{2n}^{\alpha(2n-2)}.$$

For the condition $(*)$, it is sufficient that $A_2 \leq A_3 \leq A_4$ and

(i) $t_1 = t_2$

or

(ii) commute, especially $A_2 = A_3 = A_4$.

The full proof of Theorem 1 is just a mathematical induction. It is so natural and simple that one can understand the whole if he/she once see the proof for the case of $n = 2$.

Proof of Theorem 1 for $n = 2$.

Let $1 \leq p_1, p_2, p_3, p_4$,

$$0 \leq t_1, t_3, \leq 1, \quad t_1 \leq t_2, \quad t_3 \leq t_4,$$

$$0 \leq B \leq A_1 \leq A_2, \quad \exists A_1^{-1} \quad \text{and}$$

$$A_2^{1-t_1+t_2} \leq A_3^{1-t_1+t_2} \leq A_4^{1-t_1+t_2}.$$

By Uchiyama '03,

$$\left\{ A_2^{\frac{t_2}{2}} \left(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}} \right)^{p_2} A_2^{\frac{t_2}{2}} \right\}^{\frac{1-t_1+t_2}{(p_1-t_1)p_2+t_2}} \leq A_2^{1-t_1+t_2}.$$

Denote the left hand side by B_1 , then

$$B_1 \leq A_2^{1-t_1+t_2} \leq A_3^{1-t_1+t_2} \leq A_4^{1-t_1+t_2}.$$

Idea: Apply once again Uchiyama '03.

Put

$$p = \frac{(p_1 - t_1)p_2 + t_2}{1 - t_1 + t_2} p_3, \quad t = \frac{t_3}{1 - t_1 + t_2},$$

$$r = \frac{t_4}{1 - t_1 + t_2}, \quad s = p_4.$$

Then

$$1 \leq p, \quad 1 \leq s, \quad 0 \leq t \leq 1, \quad t \leq r.$$

So we may apply Uchiyama '03 to

$$0 \leq B_1 \leq A_3^{1-t_1+t_2} \leq A_4^{1-t_1+t_2},$$

which yields that

$$\left\{ \left(A_4^{1-t_1+t_2} \right)^{\frac{r}{2}} \left(\left(A_3^{1-t_1+t_2} \right)^{-\frac{t}{2}} B_1^p \left(A_3^{1-t_1+t_2} \right)^{-\frac{t}{2}} \right)^s \left(A_4^{1-t_1+t_2} \right)^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}} \\ \leq \left(A_4^{1-t_1+t_2} \right)^{1-t+r}.$$

At first,

$$\begin{aligned} & (1 - t_1 + t_2)(1 - t + r) \\ &= (1 - t_1 + t_2) \left(1 - \frac{t_3}{1 - t_1 + t_2} + \frac{t_4}{1 - t_1 + t_2} \right) \\ &= 1 - t_1 + t_2 - t_3 + t_4, \end{aligned}$$

so the right hand side is $A_4^{1-t_1+t_2-t_3+t_4}$.

By the definition of B_1 and p ,

$$B_1^p = \left(A_2^{\frac{t_2}{2}} \left(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}} \right)^{p_2} A_2^{\frac{t_2}{2}} \right)^{p_3}.$$

Obviously,

$$\left(A_3^{1-t_1+t_2} \right)^{-\frac{t}{2}} = A_3^{-\frac{t_3}{2}} \quad \text{and} \quad \left(A_4^{1-t_1+t_2} \right)^{\frac{r}{2}} = A_4^{\frac{t_4}{2}}.$$

Moreover,

$$\begin{aligned}
& \frac{1 - t + r}{(p - t)s + r} \\
&= \frac{1 - \frac{t_3}{1-t_1+t_2} + \frac{t_4}{1-t_1+t_2}}{\left(\frac{(p_1-t_1)p_2+t_2}{1-t_1+t_2} - \frac{t_3}{1-t_1+t_2}\right)p_4 + \frac{t_4}{1-t_1+t_2}} \\
&= \frac{1 - t_1 + t_2 - t_3 + t_4}{(((p_1 - t_1)p_2 + t_2)p_3 - t_3)p_4 + t_4} \\
&= \frac{\alpha(4)}{\psi(4)}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
& \left\{ A_4^{\frac{t_4}{2}} \left(A_3^{-\frac{t_3}{2}} \left(A_2^{\frac{t_2}{2}} \left(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}} \right)^{p_2} A_2^{\frac{t_2}{2}} \right)^{p_3} A_3^{-\frac{t_3}{2}} \right)^{p_4} A_4^{\frac{t_4}{2}} \right\}^{\frac{\alpha(4)}{\psi(4)}} \\
& \leq A_4^{\alpha(4)}. \quad \square
\end{aligned}$$

Remark to Theorem 1.

We can't reduce the part of the assumption

$$A_{2k-2}^{\alpha(2k-2)} \leq A_{2k-1}^{\alpha(2k-2)} \leq A_{2k}^{\alpha(2k-2)} \quad (k = 2, \dots, n)$$

to

$$A_2 \leq \dots \leq A_{2n}.$$

Easy counter example even in $n = 2$.

Take $I \leq C_1 \leq C_2$ such that $C_1^2 \not\leq C_2^2$.

Put

$$p_1 = \dots = p_4 = 1,$$

$$t_1 = t_3 = 1, \quad t_2 = t_4 = 2,$$

$$B = A_1 = I, \quad A_2 = C_1, \quad A_3 = A_4 = C_2.$$

In this case, $\alpha(4) = \psi(4) = 3$.

If the inequality of the conclusion of Theorem 1 holds, we would have

$$C_2 C_2^{-\frac{1}{2}} C_1^2 C_2^{-\frac{1}{2}} C_2 \leq C_2^3,$$

which leads to $C_1^2 \leq C_2^2$, a contradiction.

Theorem 2(KW).

$$1 \leq p_1, \dots, p_{2n+1},$$

$$0 \leq t_1, \quad 0 \leq t_{2k} \leq 1, \quad t_{2k} \leq t_{2k+1} \quad (k = 1, \dots, n),$$

$$0 \leq B \leq A_1, \quad \exists A_1^{-1} \quad \text{and}$$

$$A_{2k-1}^{\beta(2k-1)} \leq A_{2k}^{\beta(2k-1)} \leq A_{2k+1}^{\beta(2k-1)} \quad (k = 1, \dots, n)$$

$$\implies$$

$$\begin{aligned} & \left\{ A_{2n+1}^{\frac{t_{2n+1}}{2}} \left(A_{2n}^{-\frac{t_{2n}}{2}} \dots \right. \right. \\ & \quad \left(A_2^{-\frac{t_2}{2}} \left(A_1^{\frac{t_1}{2}} B^{p_1} A_1^{\frac{t_1}{2}} \right)^{p_2} A_2^{-\frac{t_2}{2}} \right)^{p_3} \\ & \quad \left. \dots A_{2n}^{-\frac{t_{2n}}{2}} \right)^{p_{2n+1}} A_{2n+1}^{\frac{t_{2n+1}}{2}} \left. \right\}^{\frac{\beta(2n+1)}{\gamma(2n+1)}} \\ & \leq A_{2n+1}^{\beta(2n+1)}. \end{aligned}$$

Definition.

$$\beta(2n+1) = 1 + t_1 - t_2 + \dots + t_{2n+1}$$

$$\gamma(2n+1) = \{\dots((p_1 + t_1)p_2 - t_2)p_3 + \dots - t_{2n}\}p_{2n+1} + t_{2n+1}.$$

Theorem 3(KW).

ℓ : even natural number, $1 \leq p_1, \dots, p_{2n+\ell}$,

$0 \leq t_1, \dots, t_n, t_{n+1}, t_{n+3}, \dots, t_{n+\ell-1} \leq 1$,

$t_{n+1} \leq t_{n+2}, \dots, t_{n+\ell-1} \leq t_{n+\ell}$,

$0 \leq B \leq A_1 \leq A_2 \leq \dots \leq A_{2n+2}$ and $\exists A_1^{-1}$

$\implies$

$$\left\{ A_{2n+2}^{\frac{t_{n+\ell}}{2}} \left(A_{2n+2}^{-\frac{t_{n+\ell-1}}{2}} \dots \left(A_{2n+2}^{\frac{t_{n+2}}{2}} \left(A_{2n+1}^{-\frac{t_{n+1}}{2}} \left(A_{2n}^{\frac{t_n}{2}} \right. \right. \right. \right. \right. \right. \\ \left. \left. \left(A_{2n-1}^{-\frac{t_n}{2}} \dots \left(A_2^{\frac{t_1}{2}} \left(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}} \right)^{p_2} A_2^{\frac{t_1}{2}} \right)^{p_3} \dots A_{2n-1}^{-\frac{t_n}{2}} \right)^{p_{2n}} \right. \right. \right. \\ \left. \left. A_{2n}^{\frac{t_n}{2}} \right)^{p_{2n+1}} A_{2n+1}^{-\frac{t_{n+1}}{2}} \right)^{p_{2n+2}} A_{2n+2}^{\frac{t_{n+2}}{2}} \right)^{p_{2n+3}} \dots A_{2n+2}^{-\frac{t_{n+\ell-1}}{2}} \right)^{p_{2n+\ell}} A_{2n+2}^{\frac{t_{n+\ell}}{2}} \right\}^{\frac{\alpha'}{\psi'}}$$

$$\leq A_{2n+2}^{\alpha'},$$

where

$$\alpha' = 1 - t_{n+1} + t_{n+2} - \dots - t_{n+\ell-1} + t_{n+\ell}$$

$$\psi' = (\dots (((((p_1 - t_1)p_2 + t_1)p_3 - \dots - t_n)p_{2n} + t_n)p_{2n+1} \\ - t_{n+1})p_{2n+2} + \dots - t_{n+\ell-1})p_{2n+\ell} + t_{n+\ell}.$$

Corollary.

$$1 \leq p_1, \dots, p_{2n},$$

$$0 \leq t_{2k-1} \leq 1, \quad t_{2k-1} \leq t_{2k} \quad (k = 1, \dots, n),$$

$$0 \leq B \leq A \quad \text{and} \quad \exists A^{-1}$$

$\implies$

$$\begin{aligned} & \left\{ A^{\frac{t_{2n}}{2}} \left(A^{-\frac{t_{2n-1}}{2}} \dots \right. \right. \\ & \quad \left. \left(A^{\frac{t_2}{2}} \left(A^{-\frac{t_1}{2}} B^{p_1} A^{-\frac{t_1}{2}} \right)^{p_2} A^{\frac{t_2}{2}} \right)^{p_3} \right. \\ & \quad \left. \left. \dots A^{-\frac{t_{2n-1}}{2}} \right)^{p_{2n}} A^{\frac{t_{2n}}{2}} \right\}^{\frac{\alpha(2n)}{\psi(2n)}} \\ & \leq A^{\alpha(2n)}. \end{aligned}$$

Corollary.

$$1 \leq p_1, \dots, p_{2n+1},$$

$$0 \leq t_1, \quad 0 \leq t_{2k} \leq 1, \quad t_{2k} \leq t_{2k+1} \quad (k = 1, \dots, n),$$

$$0 \leq B \leq A \quad \text{and} \quad \exists A_1^{-1}$$

$\implies$

$$\begin{aligned} & \left\{ A^{\frac{t_{2n+1}}{2}} \left(A^{-\frac{t_{2n}}{2}} \dots \right. \right. \\ & \quad \left. \left(A^{-\frac{t_2}{2}} \left(A^{\frac{t_1}{2}} B^{p_1} A^{\frac{t_1}{2}} \right)^{p_2} A^{-\frac{t_2}{2}} \right)^{p_3} \right. \\ & \quad \left. \left. \dots A^{-\frac{t_{2n}}{2}} \right)^{p_{2n+1}} A^{\frac{t_{2n+1}}{2}} \right\}^{\frac{\beta(2n+1)}{\gamma(2n+1)}} \\ & \leq A^{\beta(2n+1)}. \end{aligned}$$

3. On range of parameters which make operator inequalities valid

Tanahashi showed the best possibility of the range in Furuta '87

$$p + r \leq (1 + r)q \quad \text{and} \quad 1 \leq q$$

as far as one considers positive parameters.

Theorem (Tanahashi '96).

Let $0 < p, q, r$. $(1 + r)q < p + r$ or $0 < q < 1$

$\implies \exists(A, B) : 0 < B \leq A,$

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1}{q}} \not\leq A^{\frac{p+r}{q}}.$$

Corollary.

Let $1 \leq p, 0 \leq r$. $1 < \alpha$

$\implies \exists(A, B) : 0 < B \leq A,$

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1+r}{p+r}\alpha} \not\leq A^{(1+r)\alpha}.$$

Corollary.

Let $0 < p < 1, 0 < r$.

$\implies \exists(A, B) : 0 < B \leq A,$

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1+r}{p+r}} \not\leq A^{1+r}.$$

Tanahashi also obtained the best possibility of the outer power in the grand Furuta inequality.

Theorem (Tanahashi '99).

Let $1 \leq p$, $1 \leq s$, $0 \leq t \leq 1$, $t \leq r$. $1 < \alpha$

$\Rightarrow \exists(A, B) : 0 < B \leq A$,

$$\left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}\alpha} \not\leq A^{(1-t+r)\alpha}.$$

Theorem 4(KW).

Let $1 < p_1$, $0 < p_j \leq 1$ ($j = 2, \dots, 2n$),

$1 \leq p_{2n+1}, p_{2n+2}$,

$0 \leq t_j \leq 1$ ($j = 1, \dots, n+1$), $t_{n+1} \leq t_{n+2}$ and

$$1 \leq (\dots((p_1 - t_1)p_2 + t_1)p_3 - \dots - t_n)p_{2n} + t_n.$$

Furthermore, if $1 < \alpha$

$\Rightarrow \exists(A, B) : 0 < B \leq A$,

$$\left\{ A^{\frac{t_{n+2}}{2}} \left(A^{-\frac{t_{n+1}}{2}} \left(A^{\frac{t_n}{2}} \left(A^{-\frac{t_n}{2}} \dots \right. \right. \right. \right. \\ \left. \left. \left(A^{\frac{t_1}{2}} \left(A^{-\frac{t_1}{2}} B^{p_1} A^{-\frac{t_1}{2}} \right)^{p_2} A^{\frac{t_1}{2}} \right)^{p_3} \right. \right. \\ \left. \left. \dots A^{-\frac{t_n}{2}} \right)^{p_{2n}} A^{\frac{t_n}{2}} \right)^{p_{2n+1}} A^{-\frac{t_{n+1}}{2}} \right)^{p_{2n+2}} A^{\frac{t_{n+2}}{2}} \right\}^{\frac{1-t_{n+1}+t_{n+2}}{\psi_1}\alpha} \\ \not\leq A^{(1-t_{n+1}+t_{n+2})\alpha},$$

where

$$\begin{aligned} \psi_1 = (((\cdots ((p_1 - t_1)p_2 + t_1)p_3 - \cdots - t_n)p_{2n} + t_n)p_{2n+1} \\ - t_{n+1})p_{2n+2} + t_{n+2}. \end{aligned}$$

Theorem 5(KW).

Let $0 < p$, $0 < s$, $0 < t \leq 1$, $t \leq r$.

Furthermore, we assume (i) or (ii) of the following :

$$(i) \quad t < p \quad \text{and} \quad \frac{1-t+r}{(p-t)s+r} \cdot sp < 1$$

$$(ii) \quad t = p < r \quad \text{and} \quad p < 1$$

$$\implies \exists(A, B) : 0 < B \leq A,$$

$$\left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}} \not\leq A^{1-t+r}.$$

Corollary.

Let $1 < p$, $0 < s \leq \frac{1}{p}$, $0 < t \leq 1$ and $t \leq r$.

$$\implies \exists(A, B) : 0 < B \leq A,$$

$$\left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}} \not\leq A^{1-t+r}.$$

References

- [1] T. Ando and F. Hiai, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl. 197/198 (1994), 113–131.
- [2] N. N. Chan and M. K. Kwong, *Hemitian matix inequalities and a conjecture*, Amer. Math. Monthly 92 (1985), 533–541.
- [3] M. Fujii, A. Matsumoto and R. Nakamoto, *A short proof of the best possibility for the grand Furuta inequality*, J. Inequal. Appl. 4 (1999), no. 4, 339–344.
- [4] T. Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1 + 2r)q \geq p + 2r$* , Proc. Amer. Math. Soc. 101 (1987), no. 1, 85–88.
- [5] T. Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl. 219 (1995), 139–155.
- [6] T. Furuta, *Invitation to linear operators*. Taylor & Francis, London, 2001.
- [7] T. Furuta, *A proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci. 78 (2002), no. 2, 26.
- [8] T. Furuta, *Further extension of an order preserving operator inequality*, J. Math. Inequal. 2 (2008), no. 4, 465–472.
- [9] E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. 124 (1951), 415–438.
- [10] T. Koizumi and K. Watanabe, *A remark on extension of order preserving operator inequality*, to appear in J. Math. Inequal.
- [11] T. Koizumi and K. Watanabe, *On best possibility of an extension of the Furuta inequality*, Int. J. Funct. Anal. Oper. Theory Appl. 3 (2011), 155–161.

- [12] T. Koizumi and K. Watanabe, *Another consequence of Tanahashi's argument on best possibility of the grand Furuta inequality*, preprint.
- [13] K. Löwner, *Über monotone Matrixfunktionen*, Math. Z. 38 (1934), 177-216.
- [14] K. Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc. 124 (1996), 141-146.
- [15] K. Tanahashi, *The best possibility of the grand Furuta inequality*, Proc. Amer. Math. Soc. 128 (1999), 511-519.
- [16] M. Uchiyama, *Criteria for monotonicity of operator means*, J. Math. Soc. Japan 55 (2003), no. 1, 197-207.
- [17] T. Yamazaki, *Simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality*, Math. Inequal. Appl. 2 (1999), 473-477.
- [18] C. Yang and Y. Wang, *Further extension of Furuta inequality*, J. Math. Inequal. 4 (2010), no. 3, 391-398.